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Analysis of Oscillatory Behavior of Second-Order Neutral Delay Difference Equations

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Abstract

The paper investigates the oscillation, zero-convergence, and solutions of second-order neutral delay difference equations containing three nonlinear delayed terms with different growth rates. By using positivity and monotonicity conditions on an auxiliary function along with divergence-type conditions on the coefficient sequences of the neutral and delayed terms, the paper establishes new criteria that guarantee oscillation or convergence of all solutions. These novel findings extend and enhance several of the existing oscillation criteria established by the literature. Suggestions for further investigation are included with illustrative examples.

Keywords: oscillation behavior; difference equation; second-order equation; neutral delay; simulation

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1. Introduction

Difference equations constitute a fundamental tool for modelling discrete dynamical systems and serve as the natural discrete analogue of differential equations. They find widespread applications in population biology, economics, numerical analysis, control theory, and many other fields. The inclusion of delay arguments allows these equations to capture memory effects and historical dependence, making delay difference equations particularly suitable for systems where past states significantly influence the current evolution [1–3]. Neutral delay difference equations, in which the delay appears inside the principal difference operator, exhibit even richer dynamics and have therefore attracted considerable attention in recent years [4,5].

The present paper is devoted to the oscillatory behavior of all solutions of the second-order linear neutral delay difference equation

$$\Delta\Big(m_l\big(\Delta(\Upsilon_l + n_l\Upsilon(\eta_l))\big)^{\sigma}\Big) + p_l\Upsilon^{\beta}(\gamma_l) + q_l\Upsilon^{\mu}(\tau_l) = 0, \tag{1}$$

where σ, β, and μ are ratios of odd positive integers, the sequences m_l, p_l, q_l, and n_l are positive real numbers with $|n_l| < \infty$, and the delay (and advanced) arguments satisfy

$\eta_l \leq l$, $\gamma_l \leq l$, $\tau_l \leq l$ with $\lim_{l\to\infty} \eta_l = \lim_{l\to\infty} \gamma_l = \lim_{l\to\infty} \tau_l = \infty$. Our main objective is to derive new sufficient conditions ensuring that *every* solution of (1) oscillates, without resorting to comparison principles. The results are established under either of the structural assumptions (G_1) $\sum_{l=0}^{\infty} \frac{1}{m_l} = \infty$ or (G_2) $\sum_{l=0}^{\infty} \frac{1}{m_l} < \infty$ and hold for all admissible bounded neutral coefficients n_l.

Difference equations play a fundamental role in the analysis of discrete dynamical systems, providing a natural framework for modeling processes that evolve at distinct time steps. They serve as the discrete counterpart of differential equations and appear in diverse applications, including population biology, economics, numerical simulations, and control theory. When delays are incorporated, difference equations capture memory effects by linking the present state of the system to its past values [1–3]. In particular, neutral difference equations, where delayed terms occur inside the principal difference operator, exhibit richer and more intricate dynamics. Understanding the oscillatory behavior of their solutions is essential for characterizing stability, cyclic patterns, and long-term system behavior. This motivates the development of new criteria that ensure oscillation or convergence of solutions under general structural and coefficient conditions [4,5].

Delay difference equations (DDEs) are unique in that they account for the system's past states. By including these historical effects, DDEs allow for more accurate and effective predictions about the future behavior of the system, especially in cases where past events play a critical role in current dynamics [6–8]. The study of oscillation in neutral difference equations has attracted considerable attention due to their ability to model dynamic processes in which the future state depends not only on delayed values but also on delayed differences of the unknown function. Unlike ordinary or simple delay difference equations, neutral difference equations include terms where the delay appears inside the main difference operator, making their qualitative behavior more complex [9–11]. In recent years, researchers have focused on developing oscillation criteria for neutral difference equations under various structural assumptions. Many existing results treat cases with a single nonlinear delayed term or assume that all exponents appearing in the nonlinear terms are equal. However, real-world systems often involve several delayed components with different growth rates. This motivates the study of neutral difference equations with multiple nonlinear terms and distinct exponents, requiring more refined analytical tools and conditions to determine whether solutions oscillate or tend to zero [12,13]. The present work contributes to this direction by establishing new oscillation criteria for second-order neutral difference equations under general exponent settings and coefficient conditions. These results unify and extend previous findings for special cases of neutral difference equations and provide a broader framework applicable to systems with more complex nonlinear interactions [14–16].

In the qualitative theory of difference equations, oscillatory behavior is of central importance [17,18]. A nontrivial solution is said to be oscillatory if it is neither eventually positive nor eventually negative. This behavior arises naturally in numerous physical systems, including vibrating mechanical systems, electrical circuits, and interacting particle systems governed by electromagnetic forces with finite propagation speed [3–5,9,10,19–21]. When propagation delays are taken into account, and the governing physical laws are symmetric under time reversal, both delayed (retarded) and advanced terms appear in equations [11,16,22–25]. This delayed–advanced interaction has been recognized since the early work of Tetrode, Fokker, Wheeler, Feynman, and later Driver and Hoag. For general background on difference equations, refer to the monographs [6,7,12,13] and the numerous recent contributions on oscillation and non-oscillation of neutral delay difference equations, among which we particularly mention [9–11,22–24] and the references therein.

2. Main Result

Sufficient Conditions for Oscillation Criteria

Throughout the discussion, this notation is used

$$x_l = \Upsilon_l + n_l \Upsilon(\eta_l). \tag{2}$$

Lemma 1 ([26]). *Let $n_l \geq 0$. Suppose that (G_1) holds. If Υ_l is an eventually positive solution of (1), then the corresponding function $x_l = \Upsilon_l + n_l\Upsilon(\eta_l)$ satisfies $x_l > 0$, $\Delta(x_l) > 0$, $\Delta(m_l(\Delta x_l)^\sigma) \leq 0$, for any large value of l, where σ is a quotient of odd positive integers.*

Lemma 2 ([26]). *Assume that*

$$P \geq 0,\ Q \geq 0 \text{ and } \theta \geq 1. \text{ Then we get}$$

$$(P+Q)^\theta \leq 2^{\theta-1}(P^\theta + Q^\theta). \tag{3}$$

If $0 < \theta \leq 1$, then

$$(P+Q)^\theta \leq (P^\theta + Q^\theta).$$

Now, suppose that $0 \leq n_l \leq g < \infty$, $\eta(\gamma_l) = \gamma(\eta_l)$ and $\eta(\tau_l) = \tau(\eta_l)$ for all $l > 0$.

Theorem 1. *Let $\sigma = \beta = \mu$ and (G_1) hold. Suppose that*

$$\sum_{l=l_0}^{\infty} 2^{1-\beta} R^\mu \left[B_l + A_l \right] = \infty, \tag{4}$$

where

$$B_l = min\{p_l, p(\eta_l)\},\ A_l = min\{q_l, q(\eta_l)\}.$$

Then every solution of (1) oscillates.

Proof. By contradiction, assume that Υ_l is a non-oscillatory solution of (1). Without loss of generality, we can assume that $\Upsilon_l > 0$ for $l \geq l_0$. Consequently there exists $l_1 \geq l_0$ such that $\Upsilon_l > 0$, $\Upsilon(\eta_l) > 0$, $\Upsilon(\gamma_l) > 0$, and $\Upsilon(\tau_l) > 0$ for $l \geq l_1$. By (1), it becomes

$$\Delta(m_l(\Delta x_l)^\sigma) = -p_l \Upsilon^\beta(\gamma_l) - q_l \Upsilon^\mu(\tau_l) \leq 0, \text{ not identically zero for } l \geq l_1. \tag{5}$$

Thus, $m_l(\Delta x_l)^\sigma$ is non-increasing that is, either $\Delta x_l > 0$ or $\Delta x_l < 0$. According to Lemma 1, it results that $\Delta x_l > 0$ for $l \geq l_2$. From (1), it is easy to see that

$$\Delta(m_l(\Delta x_l)^\sigma) + g^\beta \Delta(m(\eta_l))(\Delta x(\eta_l)^\sigma) + p_l \Upsilon^\beta(\gamma_l) + g^\beta p(\eta_l) \Upsilon^\beta(\gamma(\eta_l)) + q_l \Upsilon^\mu(\tau_l) + g^\beta q(\eta_l) \Upsilon^\mu(\tau(\eta_l)) = 0 \text{ for } l \geq l_1. (\text{since } \sigma = \beta = \mu). \tag{6}$$

By Lemma 2, (6) implies that

$$\Delta(m_l(\Delta x_l)^\sigma) + g^\beta \Delta(m(\eta_l))(\Delta x(\eta_l)^\sigma) + 2^{1-\beta} B_l x^\mu(\gamma_l) + 2^{1-\beta} A_l x^\mu(\tau_l) \leq 0. \tag{7}$$

By Lemma 2, it follows that x_l is non-decreasing and hence $x_l \geq R$ for $l \geq l_2 > l_1$. Hence, (7) implies that

$$\Delta(m_l(\Delta x_l)^\sigma) + g^\beta \Delta(m(\eta_l))(\Delta x(\eta_l)^\sigma) + 2^{1-\beta} B_l R^\mu + 2^{1-\beta} A_l R^\mu \leq 0.$$

Then there exists $l_3 > l_2$ such that

$$2^{1-\beta}B_lR^\mu + 2^{1-\beta}A_lR^\mu \le -\Delta(m_l(\Delta x_l)^\sigma) - g^\beta\Delta(m(\eta_l)(\Delta x(\eta_l)^\sigma)), \tag{8}$$

for $l \ge l_3$.
Summing (8) from l_3 *to* ∞, we obtain a contradiction to (4). This concludes the proof of the theorem. □

Theorem 2. *Let $\sigma > \beta > \mu$. If (G_1) and (4) hold, then every solution of (1) oscillates.*

Proof. By contradiction, assume that Y_l is a non-oscillatory solution of (1). Without loss of generality, we may assume that $Y_l > 0$ for $l \ge l_0$. So there exists $l_1 > l_0$ such that $Y_l > 0$, $Y(\eta_l) > 0$, $Y(\gamma_l) > 0$ and $Y(\tau_l) > 0$ for $l \ge l_1$. By using (1), it becomes

$$\Delta(m_l(\Delta x_l)^\sigma) = -p_lY^\beta(\gamma_l) - q_lY^\mu(\tau_l) \le 0, \text{ not identically zero for } l \ge l_1.$$

Thus $m_l(\Delta x_l)^\sigma$ is non-increasing that is, either $\Delta x_l > 0$ or $\Delta x_l < 0$. According to Lemma 1, it follows that $\Delta x_l > 0$ for $l \ge l_2$. From (1), it is easy to see that

$$\begin{aligned}\Delta(m_l(\Delta x_l)^\sigma) + g^\beta\Delta(m(\eta_l))(\Delta x(\eta_l)^\sigma) + p_lY^\beta(\gamma_l) + g^\beta p(\eta_l)Y^\beta(\gamma(\eta_l)) + q_lY^\mu(\tau_l)\\ + g^\beta q(\eta_l)Y^\mu(\tau(\eta_l)) = 0 \; for \; l \ge l_1. \; (since \; \sigma > \beta > \mu).\end{aligned} \tag{9}$$

By Lemma 2, (9) implies that

$$\Delta(m_l(\Delta x_l)^\sigma) + g^\beta\Delta(m(\eta_l))(\Delta x(\eta_l)^\sigma) + 2^{1-\beta}B_lx^\mu(\gamma_l) + 2^{1-\beta}A_lx^\mu(\tau_l) \le 0. \tag{10}$$

By Lemma 2, it follows that x_l is non-decreasing and hence $x_l \ge R$ for $l \ge l_2 > l_1$. Hence (10) implies that

$$\Delta(m_l(\Delta x_l)^\sigma) + g^\beta\Delta(m(\eta_l))(\Delta x(\eta_l)^\sigma) + 2^{1-\beta}B_lR^\mu + 2^{1-\beta}A_lR^\mu \le 0.$$

Hence there exists $l_3 > l_2$ such that

$$2^{1-\beta}B_lR^\mu + 2^{1-\beta}A_lR^\mu \le -\Delta(m_l(\Delta x_l)^\sigma) - g^\beta\Delta(m(\eta_l)(\Delta x(\eta_l)^\sigma)), \tag{11}$$

for $l \ge l_3$. Summing (11) from l_3 *to* ∞, we obtain a contradiction to (4). This concludes the proof of the theorem. □

Theorem 3. *Let $\sigma < \beta < \mu$. Suppose that (G_1) holds and*

$$\sum_{l=l_0}^{\infty} 2^{1-\mu}R^\beta\Big[B_l + A_l\Big] = \infty \tag{12}$$

holds. Then every solution of (1) oscillates.

Proof. By contradiction, assume that Y_l is a non-oscillatory solution of (1). Without loss of generality, we may assume that $Y_l > 0$ for $l \ge l_0$. So there exits $l_1 > l_0$ such that $Y_l > 0$, $Y(\eta_l) > 0$, $Y(\gamma_l) > 0$ and $Y(\tau_l) > 0$ for $l \ge l_1$.
By using (1), it becomes

$$\Delta(m_l(\Delta x_l)^\sigma) = -p_lY^\beta(\gamma_l) - q_lY^\mu(\tau_l) \le 0, \text{ not identically zero for } l \ge l_1.$$

Thus $m_l(\Delta x_l)^\sigma$ is non-increasing; that is, either $\Delta x_l > 0$ or $\Delta x_l < 0$. According to Lemma 1, it follows that $\Delta x_l > 0$ for $l \geq l_2$. From (1), it is easy to see that

$$\Delta(m_l(\Delta x_l)^\sigma) + g^\mu \Delta(m(\eta_l))(\Delta x(\eta_l)^\beta) + p_l \Upsilon^\beta(\gamma_l) + g^\beta p(\eta_l)\Upsilon^\beta(\gamma(\eta_l)) + q_l \Upsilon^\mu(\tau_l) + g^\beta q(\eta_l)\Upsilon^\mu(\tau(\eta_l)) = 0 \; for \; l \geq l_1. \; (since \; \sigma > \beta > \mu). \tag{13}$$

By Lemma 2, (13) implies that

$$\Delta(m_l(\Delta x_l)^\sigma) + g^\beta \Delta(m(\eta_l))(\Delta x(\eta_l)^\sigma) + 2^{1-\beta} B_l x^\mu(\gamma_l) + 2^{1-\beta} A_l x^\mu(\tau_l) \leq 0. \tag{14}$$

By Lemma 2, it follows that x_l is non-decreasing and hence $x_l \geq R$ for $l \geq l_2 > l_1$. Hence, (10) implies that

$$\Delta(m_l(\Delta x_l)^\sigma) + g^\beta \Delta(m(\eta_l))(\Delta x(\eta_l)^\sigma) + 2^{1-\beta} B_l R^\mu + 2^{1-\beta} A_l R^\mu \leq 0.$$

Hence, there exists $l_3 > l_2$ such that

$$2^{1-\beta} B_l R^\mu + 2^{1-\beta} A_l R^\mu \leq -\Delta(m_l(\Delta x_l)^\sigma) - g^\beta \Delta(m(\eta_l)(\Delta x(\eta_l)^\sigma)), \tag{15}$$

for $l \geq l_3$. Summing (15) from l_3 *to* ∞, we obtain a contradiction to (4). This concludes the proof of the theorem. □

Moreover, assume that $-1 < g \leq n_l \leq 0$.

Theorem 4. *Let $\sigma > \beta > \mu$. Assume that (G_1) holds.*

$$\sum_{s=l_0}^{\infty} \left[p_s + q_s \right] = \infty \tag{16}$$

and

$$\sum_{s=l_0}^{\infty} \left[p(\eta_s) + q(\eta_s) \right] = \infty. \tag{17}$$

Then every solution of (1) oscillates or tends to zero as $l \to \infty$.

Proof. By contradiction, assume that Υ_l is a non-oscillatory solution of (1). Without loss of generality, we can assume that $\Upsilon_l > 0$ for $l \geq l_0$. Consequently there exists $l_1 \geq l_0$ such that $\Upsilon_l > 0$, $\Upsilon(\eta_l) > 0$, $\Upsilon(\gamma_l) > 0$ and $\Upsilon(\tau_l) > 0$ for $l \geq l_1$. By (1), it becomes

$$\Delta(m_l(\Delta x_l)^\sigma) = -p_l \Upsilon^\beta(\gamma_l) - q_l \Upsilon^\mu(\tau_l) \leq 0, \; not \; identically \; zero \; for \; l \geq l_1. \tag{18}$$

Suppose that x_l and Δx_l are monotonic functions. If $\Delta x_l < 0$ for $l \geq l_2$, then $\Delta(m_l(\Delta x_l)^\sigma) \leq 0$ implies that

$$m_l(\Delta x_l)^\sigma \leq m_{l_2}(\Delta x_{l_2})^\sigma,$$

That is,

$$(\Delta x_l)^\sigma \leq \frac{m_{l_2}(\Delta x_{l_2})^\sigma}{m_l}, \tag{19}$$

Summing (18) from l_1 *to* $l-1$,

$$x_l \leq x_{l_1} + m_{l_2}^{\frac{1}{\sigma}} \Delta(x_{l_2}) \sum_{s=l_1}^{l-1} \frac{1}{m_s^{\frac{1}{\sigma}}} \to -\infty \; as \; l \to \infty,$$

which is absurd given the fact that $x_l > 0$. Moreover, the contradiction mentioned above is true if $x_l < 0$. Indeed Υ_l is bounded when $x_l < 0$. Assume that this does not hold. Then there exists a sequence of points $\{t_r\}$ such that $t_r \to \infty$ as $r \to \infty$ and $\Upsilon(t_r) = max\{\Upsilon_l : l_3 \leq l \leq t_r\}$. Given that $\gamma(t_r) \leq t_r$, $\Upsilon(\gamma(t_r)) \leq \Upsilon(t_r)$ implies that

$$\begin{aligned} x(t_r) &\geq \Upsilon(t_r) + g\Upsilon(\gamma(t_r)) \\ &= (1+g)\Upsilon(t_r) \ \ (since\ 1+g > 0) \\ &\to +\infty\ as\ r \to \infty, \end{aligned}$$

which is a contradiction. Consequently, Υ_l is bounded and $\lim_{l\to\infty} x_l$ exists. As a consequence, we address the following two cases,

$$(i)\ x_l > 0, \Delta x_l > 0, (ii)\ x_l < 0, \Delta x_l > 0.$$

Case (i): In this case, $x_l \leq \Upsilon_l$ and $\lim_{l\to\infty} m_l \Delta x_l$ exists. Thus, (1) becomes

$$\Delta(m_l(\Delta x_l)^\sigma) + p_l \Upsilon^\beta(\gamma_l) + q_l \Upsilon^\mu(\tau_l) \leq 0 \tag{20}$$

for $l \geq l_3 > l_2$, since $\sigma > \beta > \mu$. Using Lemma 2, we obtain $x_l \geq R$. Therefore (19) gives that

$$\begin{aligned} \Delta(m_l(\Delta x_l)^\sigma) + p_l R^\mu + q_l R^\mu &\leq 0, \\ R^\mu \left[p_l + q_l \right] &\leq -\Delta(m_l(\Delta x_l)^\sigma). \end{aligned} \tag{21}$$

Summing (20) from l_3 *to* $+\infty$, we have a contradiction to (16).
Case (ii): Let $\lim_{l\to\infty} x_l$ and $\lim_{l\to\infty} x_l = \rho$ exist we say that $\rho = 0$. If not, then there exists $t < 0$ and $l_3 > l_2$ such that

$$x(\gamma_l) \leq x_l < t, x(\tau_l) \leq x_l < t\ for\ l \geq l_3.$$

By (1), it follows that $x_l > g\Upsilon(\eta_l)$ and hence $\Upsilon(\eta(\gamma_l)) > \frac{1}{g} x(\gamma_l)$;
that is,

$$\Upsilon(\gamma(\eta_l)) > (\frac{t}{g})\ for\ l \geq l_3.$$

Additionally, $\Upsilon(\tau(\eta_l)) > \left(\frac{t}{g}\right)$ for $l \geq l_3$. According to (1), it can be written as

$$\Delta(m(\eta_l))(\Delta x(\eta_l)^\sigma) + p(\eta_l)\Upsilon^\beta(\gamma(\eta_l)) + q(\eta_l)\Upsilon^\mu(\tau(\eta_l)) = 0$$

for $l \geq l_3$, we obtain

$$\Delta(m(\eta_l))(\Delta x(\eta_l)^\sigma) + \left(\frac{t}{g}\right)^\mu p(\eta_l) + \left(\frac{t}{g}\right)^\mu q(\eta_l) \leq 0.\ (since\ \sigma > \beta > \mu). \tag{22}$$

Summing (22) from l_3 *to* $l-1$, we get

$$\begin{aligned} \left(\frac{t}{g}\right)^\mu \left[\sum_{s=l_3}^{l-1} p(\eta_s) + \sum_{s=l_3}^{l-1} q(\eta_s) \right] &\leq -[\Delta(m(\eta_l)(\Delta x(\eta_l)^\sigma)) - \Delta(m(\eta(l_3))(\Delta x(\eta(l_3))^\sigma))] \\ &< -\Delta(m(\eta_l)(\Delta x(\eta_l)^\sigma)) < \infty\ as\ l \to \infty, \end{aligned}$$

which is a contradiction to (17). Hence $\rho = 0$. Applying the same reason as before, it is easy to show that Y_l is bounded. Therefore,

$$\begin{aligned} 0 \geq \lim_{l\to\infty} x_l &= \lim_{l\to\infty} sup\, x_l \\ &\geq \lim_{l\to\infty} sup(\ (Y_l) + gY(\eta_l)) \\ &\geq \lim_{l\to\infty} sup\ (Y_l) + \lim_{l\to\infty} inf\ (gY(\eta_l)) \\ &= \lim_{l\to\infty} sup\ Y_l + g \lim_{l\to\infty} sup\ (Y(\eta_l)) \\ &= (1+g) \lim_{l\to\infty} sup\ Y_l, \end{aligned}$$

which implies that $\lim_{l\to\infty} sup\ Y_l = 0$; that is, $\lim_{l\to\infty} Y_l = 0$. This completes the proof of the theorem. □

Theorem 5. *Let $\sigma < \beta < \mu$. If (G_1), (16) and (17) hold, then every solution of (1) oscillates or tends to zero as $l \to \infty$.*

Proof. By contradiction, assume that Y_l is a non-oscillatory solution of (1). Without loss of generality, we can assume that $Y_l > 0$ for $l \geq l_0$. Consequently there is $l_1 \geq l_0$ such that $Y_l > 0$, $Y(\eta_l) > 0$, $Y(\gamma_l) > 0$ and $Y(\tau_l) > 0$ for $l \geq l_1$. By (1), it becomes

$$\Delta(m_l(\Delta x_l)^\sigma) = -p_l Y^\beta(\gamma_l) - q_l Y^\mu(\tau_l) \leq 0,\ not\ identically\ zero\ for\ l \geq l_1. \tag{23}$$

Suppose that x_l and Δx_l are monotonic functions. We consider the following four possible cases:

$$(i)\ x_l > 0, \Delta x_l > 0, (ii)\ x_l < 0, \Delta x_l > 0,$$

$$(iii)\ x_l > 0, \Delta x_l < 0, (iv)\ x_l < 0, \Delta x_l < 0.$$

Case (i): In this case, $x_l \leq Y_l$ and $\lim_{l\to\infty} m_l \Delta x_l$ exists. Thus, (1) becomes

$$\Delta(m_l(\Delta x_l)^\sigma) + p_l x^\beta(\gamma_l) + q_l x^\mu(\tau_l) \leq 0.\ (since\ \sigma < \beta < \mu) \tag{24}$$

for $l \geq l_3 > l_2$. Using Lemma 2, we obtain $x_l \geq R$. Therefore (22) results in

$$\Delta(m_l(\Delta x_l)^\sigma) + p_l R^\beta + q_l R^\beta \leq 0,$$

$$R^\beta \left[p_l + q_l \right] \leq -\Delta(m_l(\Delta x_l)^\sigma). \tag{25}$$

Summing (23) from l_3 *to* $+\infty$, we have a contradiction to (16).
Case (ii): Let $\lim_{l\to\infty} x_l$ exist. Let $\lim_{l\to\infty} x_l = \rho$ we say that $\rho = 0$. If not, then there exists $t < 0$ and $l_3 > l_2$ such that

$$x(\gamma_l) \leq x_l < t, x(\tau_l) \leq x_l < t\ for\ l \geq l_3.$$

By (1), it follows that $x_l > gY(\eta_l)$ and hence $Y(\eta(\gamma_l)) > \frac{1}{g} x(\gamma_l)$; that is,

$$Y(\gamma(\eta_l)) > (\frac{t}{g})\ for\ l \geq l_3.$$

Additionally, $\Upsilon(\tau(\eta_l)) > \left(\frac{t}{g}\right)$ for $l \geq l_3$. According to (1), it can be written as

$$\Delta(m(\eta_l))(\Delta x(\eta_l)^{\sigma}) + p(\eta_l)\Upsilon^{\beta}(\gamma(\eta_l)) + q(\eta_l)\Upsilon^{\mu}(\tau(\eta_l)) = 0$$

For $l \geq l_3$, we obtain

$$\Delta(m(\eta_l))(\Delta x(\eta_l)^{\sigma}) + \left(\frac{t}{g}\right)^{\mu} p(\eta_l) + \left(\frac{t}{g}\right)^{\mu} q(\eta_l) \leq 0. \ (since\ \sigma > \beta > \mu). \tag{26}$$

Summing (26) from l_3 *to* $l-1$, we get

$$\left(\frac{t}{g}\right)^{\mu}\left[\sum_{s=l_3}^{l-1} p(\eta_s) + \sum_{s=l_3}^{l-1} q(\eta_s)\right] \leq -[\Delta(m(\eta_l)(\Delta x(\eta_l)^{\sigma})) - \Delta(m(\eta(l_3))(\Delta x(\eta(l_3))^{\sigma}))]$$

$$< -\Delta(m(\eta_l)(\Delta x(\eta_l)^{\sigma})) < \infty \ as\ l \to \infty,$$

which is a contradiction to (17). Hence $\rho = 0$. Applying the same reason as before, it is easy to show that y_l is bounded. Therefore,

$$\begin{aligned} 0 \geq \lim_{l\to\infty} x_l &= \lim_{l\to\infty} sup\ x_l \\ &\geq \lim_{l\to\infty} sup\ ((\Upsilon_l) + g\Upsilon(\eta_l)) \\ &\geq \lim_{l\to\infty} sup\ (\Upsilon_l) + \lim_{l\to\infty} inf\ (g\Upsilon(\eta_l)) \\ &= \lim_{l\to\infty} sup\ \Upsilon_l + g \lim_{l\to\infty} sup\ (\Upsilon(\eta_l)) \\ &= (1+g) \lim_{l\to\infty} sup\ \Upsilon_l, \end{aligned}$$

which implies that $\lim_{l\to\infty} sup\ \Upsilon_l = 0$ that is, $\lim_{l\to\infty} \Upsilon_l = 0$. This completes the proof of the theorem. □

Theorem 6. *Let $\beta < \sigma < \mu$. If (G_1), (16), and (17) hold, then as $l \to \infty$, all solutions to (1) oscillate or tend to zero.*

Proof. By contradiction, assume that Υ_l is a non-oscillatory solution of (1). Without loss of generality, we can assume that $\Upsilon_l > 0$ for $l \geq l_0$. Consequently there exists $l_1 \geq l_0$ such that $\Upsilon_l > 0$, $\Upsilon(\eta_l) > 0$, $\Upsilon(\gamma_l) > 0$ and $\Upsilon(\tau_l) > 0$ for $l \geq l_1$. Therefore, by (1), it becomes

$$\Delta(m_l(\Delta x_l)^{\sigma}) = -p_l\Upsilon^{\beta}(\gamma_l) - q_l\Upsilon^{\mu}(\tau_l) \leq 0,\ not\ identically\ zero for\ l \geq l_1. \tag{27}$$

Suppose that x_l and Δx_l are monotonic functions. We consider the following four possible cases:

$$(i)\ x_l > 0, \Delta x_l > 0, (ii)\ x_l < 0, \Delta x_l > 0,$$

$$(iii)\ x_l > 0, \Delta x_l < 0, (iv)\ x_l < 0, \Delta x_l < 0.$$

Case (i): In this case, $x_l \leq \Upsilon_l$ and $\lim_{l\to\infty} m_l\Delta x_l$ exists. Thus, (1) becomes

$$\Delta(m_l(\Delta x_l)^{\sigma}) + p_l x^{\beta}(\gamma_l) + q_l x^{\mu}(\tau_l) \leq 0. \ (since\ \sigma < \beta < \mu) \tag{28}$$

for $l \geq l_3 > l_2$. Using Lemma 2, we obtain $x_l \geq R$. Therefore (22) gives that

$$\Delta(m_l(\Delta x_l)^{\sigma}) + p_l R^{\beta} + q_l R^{\beta} \leq 0,$$

$$R^{\beta}\left[p_l + q_l\right] \leq -\Delta(m_l(\Delta x_l)^{\sigma}). \tag{29}$$

For case (ii), Y_l is finally bounded, as shown in Theorem 4. This implies that as $l \to \infty$, Y_l converges to zero. Thus, in the case of unbounded Y_l, case (ii) does not conclude. □

Theorem 7. *Assume that for all the conditions of Theorem 5, as $l \to \infty$, any bounded solution of (1) oscillates or tends to zero.*

Proof. By contradiction, assume that Y_l is a non-oscillatory solution of (1). Without loss of generality, we assume that $Y_l > 0$ for $l \geq l_0$. Consequently there is $l_1 \geq l_0$ such that $Y_l > 0$, $Y(\eta_l) > 0$, $Y(\gamma_l) > 0$ and $Y(\tau_l) > 0$ for $l \geq l_1$. By (1), it becomes

$$\Delta(m_l(\Delta x_l)^{\sigma}) = -p_l Y^{\beta}(\gamma_l) - q_l Y^{\mu}(\tau_l) \leq 0, \; not\ identically\ zero\ for\ l \geq l_1. \tag{30}$$

Suppose that x_l and Δx_l are monotonic functions. If $\Delta x_l < 0$ for $l \geq l_2$, then $\Delta(m_l(\Delta x_l)^{\sigma}) \leq 0$ implies that

$$m_l(\Delta x_l)^{\sigma} \leq m_{l_2}(\Delta x_{l_2})^{\sigma},$$

That is,

$$(\Delta x_l)^{\sigma} \leq \frac{m_{l_2}(\Delta x_{l_2})^{\sigma}}{m_l}, \tag{31}$$

Summing (18) from l_1 *to* $l-1$,

$$x_l \leq x_{l_1} + m_{l_2}^{\frac{1}{\sigma}}\Delta(x_{l_2}) \sum_{s=l_1}^{l-1} \frac{1}{m_s^{\frac{1}{\sigma}}} \to -\infty \ as\ l \to \infty,$$

which is contrary to the fact that $x_l > 0$. Moreover, the contradiction mentioned above is true if $x_l < 0$. Indeed Y_l is bounded when $x_l < 0$. Assume that this does not hold. Then there exists a sequence of points $\{t_r\}$ such that $t_r \to \infty$ as $r \to \infty$ and $Y(t_r) = max\{Y(l) : l_3 \leq l \leq t_r\}$.
Given that $\gamma(t_r) \leq t_r$, $Y(\gamma(t_r)) \leq Y(t_r)$ implies that

$$\begin{aligned} x(t_r) &\geq Y(t_r) + gY(\gamma(t_r)) \\ &= (1+g)Y(t_r) \;\; (since\ 1+g > 0) \\ &\to +\infty \ as\ r \to \infty, \end{aligned}$$

which is a contradiction. Consequently, Y_l is bounded and $\lim_{l\to\infty} x_l$ exists. As a consequence, we deal with the following two cases,

$$(i)\ x_l > 0, \Delta x_l > 0, (ii)\ x_l < 0, \Delta x_l > 0.$$

Case (i): In this case, $x_l \leq Y_l$ and $\lim_{l\to\infty} m_l \Delta x_l$ exists. Thus, (1) becomes

$$\Delta(m_l(\Delta x_l)^{\sigma}) + p_l Y^{\beta}(\gamma_l) + q_l Y^{\mu}(\tau_l) \leq 0, \tag{32}$$

for $l \geq l_3 > l_2$, since $\sigma > \beta > \mu$. Using Lemma 2, we obtain $x_l \geq R$. Therefore (19) results in

$$\Delta(m_l(\Delta x_l)^{\sigma}) + p_l R^{\mu} + q_l R^{\mu} \leq 0,$$

$$R^{\mu}\left[p_l + q_l\right] \leq -\Delta(m_l(\Delta x_l)^{\sigma}). \tag{33}$$

Summing (20) from l_3 *to* $+\infty$, we have a contradiction to (16). In the case $\lim_{l\to\infty} x_l = 0$, we make use of the fact that Υ_l is bounded and it occurs that

$$\begin{aligned} 0 = \lim_{l\to\infty} x_l &= \lim_{l\to\infty} inf\ x_l \\ &\le \lim_{l\to\infty} inf\ (\Upsilon_l + h\Upsilon(\eta_l)) \\ &\le \lim_{l\to\infty} sup\ \Upsilon_l + \lim_{l\to\infty} inf\ (h\Upsilon(\eta_l)) \\ &= \lim_{l\to\infty} sup\ \Upsilon_l + h \lim_{l\to\infty} sup\ (\Upsilon(\eta_l)) \\ &= (1+h) \lim_{l\to\infty} sup\ \Upsilon_l, \end{aligned}$$

which implies that $\lim_{l\to\infty} sup\ \Upsilon_l = 0$ that is, $\lim_{l\to\infty} \Upsilon_l = 0$. This completes Theorem 7 proof. □

3. Oscillation Criteria with (G_2)

Lemma 3 ([26]). *Suppose that* (G_2) *holds. Let* u_l *be an eventually positive,* $l_0 \ge 0$ *with* $\Delta(m_l\Delta(u_l^\sigma)) \le 0$, not identically zero *for large* l. *Then the following hold:*

(i) *If* $\Delta u_l > 0$, *then there exists a constant* $R^\sigma > 0$ *such that* $u_l > R^\sigma D_{\sigma l}$ *for large* l.

(ii) *If* $\Delta u_l < 0$, *then* $u_l \ge -(m_l(\Delta u_l)^\sigma)^{\frac{1}{\sigma}} D_{\sigma l}$, *where*

$$D_{\sigma l} = \sum_{s=l}^{\infty} \left(\frac{1}{m_s}\right)^{\frac{1}{\sigma}}.$$

Assume that $0 \le n_l \le g < \infty$, $\eta(\gamma_l) = \gamma(\eta_l)$ and $\eta(\tau_l) = \tau(\eta_l)$ for all $l > 0$.

Theorem 8. *Let* $\sigma = \beta = \mu$ *and* G_2 *hold. Suppose that*

$$\sum_{s=l_0}^{\infty} \left[D_\sigma^\mu(\gamma_s)B_s + D_\sigma^\mu(\tau_s)A_s\right] = \infty \tag{34}$$

and

$$\sum_{l=0}^{\infty} \left[\frac{1}{m_l} \sum_{s=T}^{l-1} \left[D_\sigma^\mu(\gamma_s)B_s + D_\sigma^\mu(\tau_s)A_s\right]\right]^{\frac{1}{\mu}} = \infty, T > 0, \tag{35}$$

where $B_t = min\{p_l, p(\eta_l)\}$, $A_l = min(q_l, q(\eta_l))$. *Then every solution of* (1) *oscillates.*

Proof. By contradiction, assume that Υ_l is a non-oscillatory solution of (1). Without loss of generality, we assume that $\Upsilon_l > 0$ for $l \ge l_0$. Consequently there is $l_1 \ge l_0$ such that $\Upsilon_l > 0$, $\Upsilon(\eta_l) > 0$, $\Upsilon(\gamma_l) > 0$ and $\Upsilon(\tau_l)$ for $l \ge l_1$. By (1), it becomes

$$\Delta(m_l(\Delta x_l)^\sigma) = -p_l\Upsilon^\beta(\gamma_l) - q_l\Upsilon^\mu(\tau_l) \le 0, \text{ not identically zero for l} \ge \text{l}_1. \tag{36}$$

Thus, $m_l\Delta x_l^\sigma$ is non-increasing that is, either $\Delta x_l > 0$ or $\Delta x_l < 0$. According to Lemma 1, it follows that $\Delta x_l > 0$ for $l \ge l_2$. From (1),

$$\begin{aligned} &\Delta(m_l(\Delta x_l)^\sigma) + g^\beta\Delta(m(\eta_l))(\Delta x(\eta_l)^\sigma) + p_l\Upsilon^\beta(\gamma_l) + g^\beta p(\eta_l)\Upsilon^\beta(\gamma(\eta_l)) + q(l)\Upsilon^\mu(\tau_l) \\ &\quad + g^\beta q(\eta_l)\Upsilon^\mu(\tau(\eta_l)) = 0\ for\ l \ge l_1.\ (since\ \sigma = \beta = \mu). \end{aligned} \tag{37}$$

By Lemma 2, (37) implies that

$$\Delta(m_l(\Delta x_l)^\sigma) + g^\beta\Delta(m(\eta_l))(\Delta x(\eta_l)^\sigma) + 2^{1-\beta}B_l x^\mu(\gamma_l) + 2^{1-\beta}A_l x^\mu(\tau_l) \le 0. \tag{38}$$

By Lemma 3, it follows that x_l is non-decreasing and hence

$$x_l \geq R^{\sigma} D_{\sigma l} \; for \; l \geq l_2 > l_1.$$

Hence, (38) implies that

$$\Delta(m_l(\Delta x_l)^{\sigma}) + g^{\beta}\Delta(m(\eta_l))(\Delta x(\eta_l)^{\sigma}) + 2^{1-\beta} B_l R^{\mu} D^{\mu}_{\sigma}(\gamma_l) + 2^{1-\beta} A_l R^{\mu} D^{\mu}_{\sigma}(\tau_l) \leq 0.$$

Hence there exists $l_3 > l_2$ such that

$$2^{1-\beta} B_l R^{\mu} D^{\mu}_{\sigma}(\gamma_l) + 2^{1-\beta} A_l R^{\mu} D^{\mu}_{\sigma}(\tau_l) \leq -\Delta(m_l(\Delta x_l)^{\sigma}) - g^{\beta}\Delta(m(\eta_l)(\Delta x(\eta_l)^{\sigma})), \tag{39}$$

for $l \geq l_3$. Summing (39) from $l_3 \; to \; \infty$, we obtain a contradiction to (34). By Lemma 3, we get that

$$x_l \geq -D_{\sigma l} m_l^{\frac{1}{\sigma}} \Delta x_l \; for \; l \geq l_2 > l_1.$$

Since $m_l(\Delta(x_l))^{\sigma}$ is non-increasing, we can find a constant $R^{\sigma} > 0$ and $l_3 > l_2$ such that $m_l \Delta(x_l)^{\sigma} \leq -R^{\sigma}$ and $x_l \geq R D_{\sigma l}$ for $l \geq l_3$. Summing (39) from $l_3 \; to \; l-1$, we get

$$\frac{2^{1-\beta} R^{\mu}}{m_l} \sum_{s=l_3}^{l-1} \left[D^{\mu}_{\sigma}(\gamma_s) B_s + D^{\mu}_{\sigma}(\tau_s) A_s \right] \leq -\Delta(m_l(\Delta x_l)^{\sigma}) - g^{\beta}\Delta(m(\eta_l)(\Delta x(\eta_l)^{\sigma})),$$

due to non-increasing $m_l(\Delta x_l)^{\sigma}$. Hence

$$\frac{2^{1-\beta} R^{\mu}}{m_l} \sum_{s=l_3}^{l-1} \left[D^{\mu}_{\sigma}(\gamma_s) B_s + D^{\mu}_{\sigma}(\tau_s) A_s \right] \leq -(1+g^{\beta})(\Delta x_l)^{\sigma}.$$

Since $\lim\limits_{l\to\infty} x_l$ exists, it is evident from the inequality above that

$$\left[\frac{2^{1-\beta} R^{\mu}}{(1+g^{\beta})}\right]^{\frac{1}{\mu}} \sum_{l=l_3}^{\infty} \left[\frac{1}{m_l} \sum_{s=l_3}^{l-1} \left[D^{\mu}_{\sigma}(\gamma_s) B_s + D^{\mu}_{\sigma}(\tau_s) A_s \right] \right]^{\frac{1}{\mu}} < \infty,$$

which is contrary to (35). □

Theorem 9. *Let $\sigma < \beta < \mu$. Suppose that (G_2) holds;*

$$\sum_{l=l_0}^{\infty} \left[D^{\mu}_{\sigma}(\gamma_l) B_l + D^{\mu}_{\sigma}(\tau_l) A_l \right] = \infty, \tag{40}$$

and

$$\sum_{l=0}^{\infty} \left[\frac{1}{m_l} \sum_{s=T}^{l-1} \left[D^{\mu}_{\sigma}(\gamma_s) B_s + D^{\mu}_{\sigma}(\tau_s) A_s \right] \right]^{\frac{1}{\beta}} = \infty, T > 0. \tag{41}$$

Then every solution of (1) oscillates.

Proof. By contradiction, assume that Y_l is a non-oscillatory solution of (1). Without loss of generality, we assume that $Y_l > 0$ for $l \geq l_0$. Consequently there is $l_1 > l_0$ such that $Y_l > 0$, $Y(\eta_l) > 0$, $Y(\gamma_l) > 0$ and $Y(\tau_l) > 0$ for $l \geq l_1$. By (1), it becomes

$$\Delta(m_l(\Delta x_l)^{\sigma}) = -p_l Y^{\beta}(\gamma_l) - q(l) Y^{\mu}(\tau_l) \leq 0, \text{ not identically zero for } \mathrm{l} \geq \mathrm{l}_1. \tag{42}$$

Thus, $m_l \Delta x_l^{\sigma}$ is non-increasing that is, either $\Delta(x_l) > 0$, or $\Delta x_l < 0$. According to Lemma 1, it follows that $\Delta x_l > 0$ for $l \geq l_2$. From (1),

$$\Delta(m_l(\Delta x_l)^{\sigma}) + g^{\mu}\Delta(m(\eta_l))(\Delta x(\eta_l)^{\sigma}) + p_l \Upsilon^{\beta}(\gamma_l) + g^{\beta} p(\eta_l)\Upsilon^{\beta}(\gamma(\eta_l)) + q(l)\Upsilon^{\mu}(\tau_l) \\ + g^{\mu} q(\eta_l)\Upsilon^{\mu}(\tau(\eta_l)) = 0 \; for \; l \geq l_1 . (since \; \sigma < \beta < \mu). \tag{43}$$

By Lemma 2, (43) implies that

$$\Delta(m_l(\Delta x_l)^{\sigma}) + g^{\beta}\Delta(m(\eta_l))(\Delta x(\eta_l)^{\sigma}) + 2^{1-\mu} B_l x^{\mu}(\gamma_l) + 2^{1-\mu} A_l x^{\mu}(\tau_l) \leq 0. \tag{44}$$

By Lemma 3, it follows that x_l is non-decreasing and hence $x_l \geq R^{\sigma} D_{\sigma l}$ for $l \geq l_2 > l_1$. Hence (38) implies that

$$\Delta(m_l(\Delta x_l)^{\sigma}) + g^{\beta}\Delta(m(\eta_l))(\Delta x(\eta_l)^{\sigma}) + 2^{1-\mu} B_l R^{\mu} D_{\sigma}^{\mu}(\gamma_l) + 2^{1-\mu} A_l R^{\mu} D_{\sigma}^{\mu}(\tau_l) \leq 0.$$

Hence there exists $l_3 > l_2$ such that

$$2^{1-\mu} B_l R^{\mu} D_{\sigma}^{\mu}(\gamma_l) + 2^{1-\mu} A_l R^{\mu} D_{\sigma}^{\mu}(\tau_l) \leq -\Delta(m_l(\Delta x_l)^{\sigma}) - g^{\beta}\Delta(m(\eta_l)(\Delta x(\eta_l)^{\sigma})), \tag{45}$$

for $l \geq l_3$. Summing (45) from $l_3 \; to \; \infty$, we obtain a contradiction to (40).
By Lemma 3, $x_l \geq -D_{\sigma l} m_l^{\frac{1}{\sigma}} \Delta x_l$ for $l \geq l_2 > l_1$.
Since $m_l(\Delta(x_l))^{\sigma}$ is decreasing, we have constant $R^{\sigma} > 0$ and $l_3 > l_2$ such that

$$m_l \Delta(x_l)^{\sigma} \leq -R^{\sigma} \; and \; x_l \geq R D_{\sigma}(l) \; for \; l \geq l_3.$$

Summing (45) from $l_3 \; to \; l-1$, we get

$$\frac{2^{1-\beta} R^{\beta}}{m_l} \sum_{s=l_3}^{l-1} \left[D_{\sigma}^{\beta}(\gamma_s) B_s + D_{\sigma}^{\beta}(\tau_s) A_s \right] \leq -\Delta(m_l(\Delta x_l)^{\sigma}) - g^{\beta}\Delta(m(\eta_l)(\Delta x(\eta_l)^{\sigma})),$$

due to non-increasing $m_l(\Delta x_l)^{\sigma}$. Hence

$$\frac{2^{1-\beta} R^{\beta}}{m_l} \sum_{s=l_3}^{l-1} \left[D_{\sigma}^{\beta}(\gamma_s) B_s + D_{\sigma}^{\beta}(\tau_s) A_s \right] \leq -(1 + g^{\beta})(\Delta x_l)^{\sigma},$$

Since $\lim_{l \to \infty} x_l$ exists, it is evident from the inequality above that

$$\left[\frac{2^{1-\beta} R^{\beta}}{(1 + g^{\beta})} \right]^{\frac{1}{\beta}} \sum_{l=l_3}^{\infty} \left[\frac{1}{m_l} \sum_{s=l_3}^{l-1} \left[D_{\sigma}^{\beta}(\gamma_s) B_s + D_{\sigma}^{\beta}(\tau_s) A_s \right] \right]^{\frac{1}{\beta}} < \infty,$$

which is contrary to (41). □

Theorem 10. *Assume that $\sigma > \beta > \mu$ and (G_2) holds. If*

$$\sum_{s=l_0}^{\infty} \left[p_s D_{\sigma}^{\mu} \gamma_s + q_s D_{\sigma}^{\mu} \eta_s \right] = \infty, \tag{46}$$

and

$$\sum_{\lambda=T_1}^{\infty} \left[\frac{1}{m_{\lambda}} \sum_{s=l_0}^{\lambda-1} \left[p_s D_{\sigma}^{\mu} \gamma_s + q_s D_{\sigma}^{\mu} \eta_s \right] \right]^{\frac{1}{\mu}} = \infty, T_1 > 0, \tag{47}$$

then every solution of (1) oscillates or tends to zero as $l \to \infty$.

Proof. By contradiction, assume that Y_l is a non-oscillatory solution of (1). Without loss of generality, we may assume that $Y_l > 0$ for $l \geq l_0$. So there is $l_1 > l_0$ such that $Y_l > 0$, $Y(\eta_l) > 0$, $Y(\gamma_l) > 0$ and $Y(\tau_l) > 0$ for $l \geq l_1$.
By using (1), it becomes

$$\Delta(m_l(\Delta x_l)^\sigma) = -p_l Y^\beta(\gamma_l) - q_l Y^\mu(\tau_l) \leq 0, \text{ not identically zero for } l \geq l_1.$$

Suppose that x_l and Δx_l are monotonic functions. We consider the following four possible cases:

$$(i)\ x_l > 0, \Delta x_l > 0, (ii)\ x_l < 0, \Delta x_l > 0,$$

$$(iii)\ x_l > 0, \Delta x_l < 0, (iv)\ x_l < 0, \Delta x_l < 0.$$

Case (i): In this case, $x_l \leq Y_l$ and $\lim_{l\to\infty} m_l \Delta x_l$ exists. Thus, (1) becomes

$$\Delta(m_l(\Delta x_l)^\sigma) + p_l x^\mu(\gamma_l) + q_l x^\mu(\tau_l) \leq 0. \ (since\ \sigma > \beta > \mu) \tag{48}$$

for $l \geq l_3 > l_2$. Using Lemma 3, we obtain $x_l \geq R^\sigma D_{\sigma l}$. Therefore (48)

$$\Delta(m_l(\Delta x_l)^\sigma) + p_l R^\mu D^\mu_\sigma(\gamma_l) + q_l R^\mu D^\mu_\sigma(\tau_l) \leq 0,$$

$$p_l R^\mu D^\mu_\sigma(\gamma_l) + q_l R^\mu D^\mu_\sigma(\tau_l) \leq -\Delta(m_l(\Delta x_l)^\sigma). \tag{49}$$

Summing (49) from l_3 *to* $+\infty$, we have a contradiction to (46).

$$\frac{R^\mu}{m_l} \sum_{s=l_3}^{l-1} \left[p_s D^\mu_\sigma(\gamma_s) + q_s D^\mu_\sigma(\eta_s)\right] \leq -\Delta(m_l(\Delta x_l)^\sigma),$$

due to non-increasing $m_l(\Delta x_l)^\sigma$. Hence

$$\frac{R^\mu}{m_l} \sum_{s=l_3}^{l-1} \left[p_s D^\mu_\sigma(\gamma_s) + q_s D^\mu_\sigma(\eta_s)\right] \leq -(\Delta x_l)^\sigma.$$

Since $\lim_{l\to\infty} x_l$ exists, it is evident from the inequality above that

$$\left[R^\mu\right]^{\frac{1}{\mu}} \sum_{l=l_3}^{\infty} \left[\frac{1}{m_l} \sum_{s=l_3}^{l-1} \left[p_s D^\mu_\sigma(\gamma_s) + q_s D^\mu_\sigma(\eta_s)\right]\right]^{\frac{1}{\mu}} < \infty,$$

which is contrary to (47). □

Theorem 11. *Assume that $\sigma < \beta < \mu$. Assume that (G_2) holds:*

$$\sum_{s=l_0}^{\infty} \left[p_s D^\beta_\sigma \gamma_s + q_s D^\beta_\sigma \tau_s\right] = \infty, \tag{50}$$

and

$$\sum_{\lambda=T_1}^{\infty} \left[\frac{1}{m_\lambda} \sum_{s=l_0}^{\lambda-1} \left[p_s D^\beta_\sigma \gamma_s + q_s D^\beta_\sigma \tau_s\right]\right]^{\frac{1}{\mu}} = \infty, T_1 > 0, \tag{51}$$

hold. Then every solution of (1) *oscillates or tends to zero as $l \to \infty$.*

Proof. By contradiction, assume that Υ_l is a non-oscillatory solution of (1). Without loss of generality, we may assume that $\Upsilon_l > 0$ for $l \geq l_0$. So there is $l_1 > l_0$ such that $\Upsilon_l > 0$, $\Upsilon(\eta_l) > 0$, $\Upsilon(\gamma_l) > 0$ and $\Upsilon(\tau_l) > 0$ for $l \geq l_1$. By using (1), it becomes

$$\Delta(m_l(\Delta x_l)^\sigma) = -p_l\Upsilon^\beta(\gamma_l) - q_l\Upsilon^\mu(\tau_l) \leq 0, \text{ not identically zero for l} \geq \text{l}_1.$$

Suppose that x_l and Δx_l are monotonic functions. We consider the following four possible cases:

$$(i)\ x_l > 0, \Delta x_l > 0, (ii)\ x_l < 0, \Delta x_l > 0,$$

$$(iii)\ x_l > 0, \Delta x_l < 0, (iv)\ x_l < 0, \Delta x_l < 0.$$

Case (i): In this case, $x_l \leq \Upsilon_l$ and $\lim_{l\to\infty} m_l \Delta x_l$ exists. Thus, (1) becomes

$$\Delta(m_l(\Delta x_l)^\sigma) + p_l x^\beta(\gamma_l) + q_l x^\beta(\tau_l) \leq 0, \tag{52}$$

for $l \geq l_3 > l_2$. Since $\sigma < \beta < \mu$, using Lemma 3, we obtain $x_l \geq R^\sigma D_{\sigma l}$. Therefore (49) gives that

$$\Delta(m_l(\Delta x_l)^\sigma) + p_l R^\beta D_\sigma^\beta(\gamma_l) + q_l R^\beta D_\sigma^\beta(\tau_l) \leq 0,$$

$$p_l R^\beta D_\sigma^\beta(\gamma_l) + q_l R^\beta D_\sigma^\beta(\tau_l) \leq -\Delta(m_l(\Delta x_l)^\sigma). \tag{53}$$

Summing (53) from l_3 *to* $+\infty$, we have a contradiction to (50).

$$\frac{R^\beta}{m_l}\sum_{s=l_3}^{l-1}\left[p_s D_\sigma^\beta(\gamma_s) + q_s D_\sigma^\beta(\tau_s)\right] \leq -\Delta(m_l(\Delta x_l)^\sigma),$$

due to non-increasing $m_l(\Delta x_l)^\sigma$. Hence

$$\frac{R^\beta}{m_l}\sum_{s=l_3}^{l-1}\left[p_s D_\sigma^\beta(\gamma_s) + q_s D_\sigma^\beta(\tau_s)\right] \leq -(\Delta x_l)^\sigma.$$

Since $\lim_{l\to\infty} x_l$ exists, it is evident from the inequality above that

$$\left[R^\beta\right]^{\frac{1}{\mu}}\sum_{l=l_3}^{\infty}\left[\frac{1}{m_l}\sum_{s=l_3}^{l-1}\left[p_s D_\sigma^\beta(\gamma_s) + q_s D_\sigma^\beta(\tau_s)\right]\right]^{\frac{1}{\mu}} < \infty,$$

which is contrary to (51). □

4. Numerical Examples

In this section, two examples demonstrate the efficacy and efficiency of our theoretical findings with simulation results.

Example 1. *Consider a neutral delay difference equation of second order, which has the form*

$$\Delta\left[l\left\{\Delta\Big(\Upsilon_l + \Upsilon_{l-2}\Big)\right\}^{\frac{5}{3}}\right] + 4^{\frac{5}{3}}(l+1)\Upsilon_{l-1}^{\frac{5}{3}} + 4^{\frac{5}{3}} l\Upsilon_{l-1}^{\frac{5}{3}} = 0, \tag{54}$$

where $m_l = l, n_l = 1, \sigma = \beta = \mu = \frac{5}{3}$.

$$\sum_{l=0}^{\infty}\frac{1}{m_l} = \infty,$$

where

$$B_l = min\{p_l, p(\eta_l)\} = 4^{\frac{5}{3}}(l-1), A_l = min\{q_l, q(\eta_l)\} = 4^{\frac{5}{3}}(l-1).$$

$$\sum_{l=l_0}^{\infty} 2^{1-\beta} R^{\mu}\Big[B_l + A_l\Big]$$

$$\sum_{l=l_0}^{\infty} 2^{1-\frac{5}{3}}(-2(-1)^{\frac{5l}{3}})\left[4^{\frac{5}{3}}(l-1) + 4^{\frac{5}{3}}(l-1)\right] = \infty,$$

Then all the conditions of Theorem 1 are satisfied for (1). Hence, every solution of (1) oscillates. $\Upsilon_l = (-1)^{l+1}$ *is one such solution of (1). Figure 1 illustrates the oscillatory response of the nonlinear neutral delay difference equation characterized by a power of 5/3 nonlinearity. The solution* $\Upsilon_l = (-1)^{l+1}$ *displays a flawless alternating sequence of +1 and* -1 *within the index range* $l \in [0, 20]$*, indicating that the suggested analytical solution adheres to the fractional powers and neutral delay terms, while the steady amplitude preservation corroborates the stability of the oscillatory behavior in the nonlinear domain.*

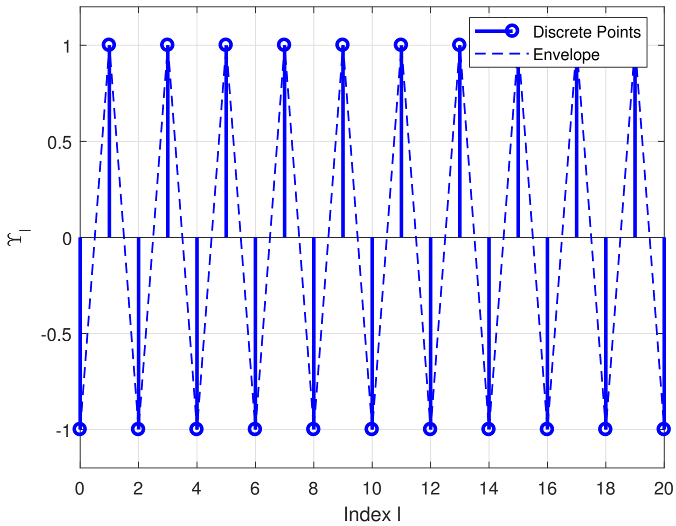

Figure 1. Oscillatory response of neutral delay difference equations in Example 1.

Example 2. *Consider a neutral delay difference equation of second order, which has the form*

$$\Delta\left[l^2\left\{\Delta\Big(\Upsilon_l + \Upsilon_{l-2}\Big)\right\}\right] + 4(l+1)^2\Upsilon_{l-1} + 4l^2\Upsilon_{l-1} = 0, \tag{55}$$

where $m_l = l^2, n_l = 1, \sigma = \beta = \mu = 1$, *where* $B_t = min\{p_l, p(\eta_l)\} = 4(l^2 - 2l + 4)$, $A_l = min(q_l, q(\eta_l)) = 4(l^2 - 2l + 4)$.

$$\sum_{l=0}^{\infty} \frac{1}{m_l} < \infty,$$

$$\sum_{s=l_0}^{\infty}\left[D^{\mu}_{\sigma}(\gamma_s)B_s + D^{\mu}_{\sigma}(\tau_s)A_s\right]$$

$$\sum_{s=l_0}^{\infty}\left[\frac{1}{(s-1)^2}4(l^2 - 2l + 4) + \frac{1}{(s-1)^2}4(l^2 - 2l + 4)\right] = \infty,$$

and

$$\sum_{l=0}^{\infty}\left[\frac{1}{m_l}\sum_{s=T}^{l-1}\left[D^{\mu}_{\sigma}(\gamma_s)B_s + D^{\mu}_{\sigma}(\tau_s)A_s\right]\right]^{\frac{1}{\mu}}$$

$$\sum_{l=0}^{\infty}\left[\frac{1}{s^2}\sum_{s=T}^{l-1}\left[\frac{1}{(s-1)^2}4(l^2-2l+4)+\frac{1}{s^2}4(l^2-2l+4)\right]\right]^{\frac{1}{\mu}}=\infty, T>0.$$

Then all the conditions of Theorem 8 are satisfied for (1). Hence, every solution of (1) oscillates. $\Upsilon_l = (-1)^{l+1}$ *is one such solution of (1). Figure 2 illustrates the response of the linear neutral delay difference equation, exhibiting a similar oscillating pattern* $\Upsilon_l = (-1)^{l+1}$. *The linear case, defined by quadratic coefficients and first-order differences, possesses the same fundamental solution as its nonlinear equivalent. This visual confirmation emphasizes the universality of the oscillatory mode across all problem types and demonstrates the resilience of the analytical method in addressing both linear and nonlinear neutral delay difference equations with uniform delay structures.*

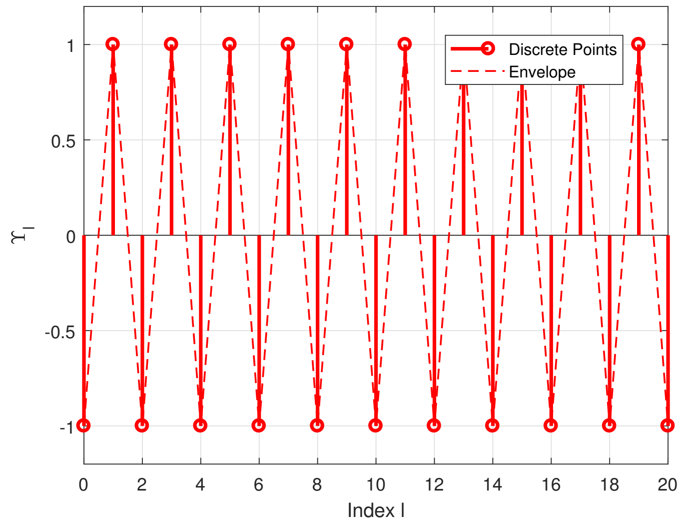

Figure 2. Oscillatory response of neutral delay difference equations in Example 2.

5. Conclusions

This paper analyzes the oscillation criteria for second-order neutral delay difference equations of the form

$$\Delta\left(m_l\left(\Delta(\Upsilon_l+n_l\Upsilon(\eta_l))\right)^{\sigma}\right)+p_l\Upsilon^{\beta}(\gamma_l)+q_l\Upsilon^{\mu}(\tau_l)=0.$$

We assume that $\sum_{l=0}^{\infty}\left(\frac{1}{m_l}\right)^{\frac{1}{\sigma}}=\infty$ or $\sum_{l=0}^{\infty}\left(\frac{1}{m_l}\right)^{\frac{1}{\sigma}}<\infty$ for various ranges of n_l, where σ is a quotient of odd positive integers. For the solution of second-order neutral delay difference equations, we develop new oscillation conditions. Some existing results for difference equations are extended and unified by our findings. Future work will focus on extending the present oscillation criteria to stochastic and probabilistic neutral difference equations, as well as developing control-based approaches for regulating oscillatory behavior in discrete-time systems, and also apply comparison techniques with second-order equations and explore oscillation conditions under more general nonlinearities and time-varying delays.

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