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Analysis of Caputo Sequential Fractional Differential Equations with Generalized Riemann–Liouville Boundary Conditions

Nallappan Gunasekaran [1], Murugesan Manigandan [2], Seralan Vinoth [3,*] and Rajarathinam Vadivel [4]

[1] Eastern Michigan Joint College of Engineering, Beibu Gulf University, Qinzhou 535011, China; gunasmaths@bbgu.edu.cn
[2] Centre for Computational Modeling, Chennai Institute of Technology, Chennai 600 069, Tamilnadu, India; manigandanm@citchennai.net
[3] Centre for Nonlinear Systems, Chennai Institute of Technology, Chennai 600 069, Tamilnadu, India
[4] Department of Mathematics, Faculty of Science and Technology, Phuket Rajabhat University, Phuket 83000, Thailand
* Correspondence: vinoth.s@citchennai.net

Abstract: This paper delves into a novel category of nonlocal boundary value problems concerning nonlinear sequential fractional differential equations, coupled with a unique form of generalized Riemann–Liouville fractional differential integral boundary conditions. For single-valued maps, we employ a transformation technique to convert the provided system into an equivalent fixed-point problem, which we then address using standard fixed-point theorems. Following this, we evaluate the stability of these solutions utilizing the Ulam–Hyres stability method. To elucidate the derived findings, we present constructed examples.

Keywords: Caputo fractional derivative; generalized Riemann–Liouville fractional integral; existence and uniqueness; nonlocal conditions; sequential derivatives; fixed-point theorem

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1. Introduction

Fractional-order differential equations (FDEs) generalize traditional differential equations that extend the notions of differentiation and integration to non-integer orders. With this generalization, researchers offer a powerful tool for modeling complex systems that display anomalous behaviors that integer-order models are unable to adequately reflect, such as memory effects and hereditary features. FDEs have found applications in diverse fields such as engineering, fluid mechanics, nonlinear optics, image processing, mathematical biology, and plasma physics [1–4]. These equations introduce advanced concepts that are valuable to researchers. The foundational work on fractional derivatives was formally introduced by Liouville and Riemann in the nineteenth century, although the idea dates back to Leibniz and L'Hospital in 1695. Several definitions of FDEs are now often used as alternatives to integer-order models; the introduction to fractional calculus is provided in [5], and a few applications in dynamic systems are examined in [6], which provides a wide range of uses for FDEs. A thorough framework for their theory and applications is discussed in [7], and the research work [8] emphasizes the significance of FDEs in mathematical physics.

The mathematical models have been greatly improved by the recent developments in FDEs, especially when considering different boundary conditions, including various forms of types such as classical, Riemann–Liouville, Hadamard, Erdélyi–Kober, and Katugampola, and used for nonlinear analytic techniques. For example, Ahmad et al. successfully synthesized and extended previous results [9,10] using fixed-point theory to show that unique solutions exist for nonlinear FDEs with nonlocal generalized fractional integral boundary conditions. In related work, the Mawhin continuation theorem was utilized to investigate the possibility of finding solutions for nonlinear fractional-order boundary value problems with generalized Riemann–Liouville integral boundary conditions and nonlocal Erdélyi–Kober boundary conditions [11]. FDEs are very versatile in reflecting a range of situations that extend numerous scientific and technical disciplines, as demonstrated by a thorough review that examines the significant contributions to the field [11,12]. Building on these foundations, recent work has introduced novel techniques and requirements for solving FDEs, including the use of fractional integral boundary conditions and Stieltjes. The ongoing studies being conducted to further the mathematical research and practical applications have the dual benefit of strengthening our theoretical understanding of FDEs and creating new opportunities for innovative applications in pressing real-world problems.

The generalized Riemann–Liouville or Katugampola fractional integral introduced by Katugampola [13] combines the Riemann–Liouville and Hadamard integrals into a single framework. The generalized fractional derivative associated with this integral was developed as a result of this novel technique [14]. The Lyapunov-type inequality for fractional boundary value problems using the Katugampola fractional derivative has been recently attempted to be derived using this framework [15]. Advances in the definition and use of nonlocal integral boundary conditions for fractional differential equations are also highlighted in the literature. For instance, thorough research has expanded the theoretical landscape by introducing more generalized nonlocal integral border conditions [16–19]. A prominent study analyzed such boundary conditions for systems and offered insights that guided further research [20]. A further study examined whether there are solutions to fractional differential equations with Liouville–Caputo types that included integral and multi-point boundary conditions in [21]. In [11], the authors applied the tool of the Mawhin continuation theorem to study the nonlinear fractional-order boundary value with nonlocal Erdélyi–Kober-type and generalized Riemann–Liouville-type integral boundary conditions. Recently, in [20], solutions were presented for the following coupled system of nonlinear fractional differential equations containing Caputo with a new kind of coupled boundary condition:

$$\begin{cases} {}^{\mathcal{C}}\mathcal{D}^{\alpha_1}\mathcal{O}(\mathfrak{t}) = \varrho_1(\mathfrak{t}, \mathcal{O}(\mathfrak{t}), \Xi(\mathfrak{t})), \mathfrak{t} \in \mathcal{Q}_1 := [0, \mathfrak{T}], \\ {}^{\mathcal{C}}\mathcal{D}^{\alpha_2}\mathcal{O}(\mathfrak{t}) = \mathcal{J}_1(\mathfrak{t}, \mathcal{O}(\mathfrak{t}), \Xi(\mathfrak{t})), \\ (\mathcal{O} + \Xi)(0) = -(\mathcal{O} + \Xi)(\mathfrak{T}), \\ \displaystyle\int_\eta^\xi (\mathcal{O} - \Xi)(s)ds = A. \end{cases}$$

Sequential fractional differential equations (SFDEs) provide the idea of fractional calculus to systems where derivatives are computed sequentially and have gained importance in recent years. Scholars like [7,22,23] have offered basic ideas and examples of such derivatives. SFDEs have attracted significant interest because of their versatility in simulating complicated events in various fields. Many researchers have investigated the existence and uniqueness of solutions for SFDEs under different boundary conditions [16,20,24,25] using continuation theorems and fixed-point theory. To further develop the theoretical framework, authors have recently addressed initial value issues using Riemann–Liouville SFDs in [26] and the periodic boundary value condition in [27]. Furthermore, the investigation of the existence of solutions for Caputo tripled fractional differential inclusions with integrals and multi-point boundary conditions was discussed in [24,28–30].

In Ref. [30], the authors investigated the existence of solutions for the nonlinear SFD system with coupled boundary conditions:

$$\begin{cases} ({}^{\mathcal{C}}\mathcal{D}^{\alpha_1} + \mathcal{K}_1 {}^{\mathcal{C}}\mathcal{D}^{\alpha_1 - 1})\mathcal{O}(\mathfrak{t}) = \varrho_1(\mathfrak{t}, \mathcal{O}(\mathfrak{t}), \Xi(\mathfrak{t})), \quad 0 < \mathfrak{t} < 1, \\ ({}^{\mathcal{C}}\mathcal{D}^{\alpha_1} + \mathcal{K}_1 {}^{\mathcal{C}}\mathcal{D}^{\alpha_1 - 1})\Xi(\mathfrak{t}) = \mathcal{L}_2(\mathfrak{t}, \mathcal{O}(\mathfrak{t}), \Xi(\mathfrak{t})), \quad 0 < \mathfrak{t} < 1, \\ \mathcal{O}(0) = \mathcal{O}'(0) = 0, \qquad \mathcal{O}(1) = a\Xi(\xi), \\ \Xi(0) = \Xi'(0) = 0, \qquad \Xi(1) = b\mathcal{O}(\eta), \end{cases}$$

where $\mathcal{K}_1$ is a parameter; $2 < \alpha_1, \alpha_2 \leq 3$, $({}^{\mathcal{C}}\mathcal{D}^{\alpha_1})$, $({}^{\mathcal{C}}\mathcal{D}^{\alpha_2})$ are the Caputo fractional derivatives; ξ, η satisfy $\xi, \eta \in (0,1)$; and the nonlinearity terms $\varrho_1, \mathcal{J}_1 : [0,1] \times \mathcal{R}_e \times \mathcal{R}_e \to \mathcal{R}_e$ represent the provided continuous function.

Furthermore, researchers have investigated both single-valued and multi-valued maps to comprehend the solutions of nonlinear coupled sequential fractional differential equations (SFDEs) with coupled boundary conditions [19,29,31]. UH stability, originally introduced by the authors in [32], and later refined by the authors in [33], then after, concerns the concept of stability. This stability concept has been further extended by various researchers, including [34]. Recently, many authors have been studying Ulam–Hyers stability and generalized stability in various research articles, such as [35–37], which have also employed UH stability criteria to investigate the stability of solutions in various fractional differential equations, highlighting its importance in contemporary mathematical research.

In this work, we explore a system of nonlinear coupled SFDEs of the Caputo type accompanied by a novel set of boundary conditions [21], namely

$$\begin{cases} ({}^{\mathcal{LC}}\mathcal{D}^{\alpha_1})\mathcal{O}(\mathfrak{t}) = \varrho_1(\mathfrak{t}, \mathcal{O}(\mathfrak{t}), \Xi(\mathfrak{t})), \mathfrak{t} \in \mathcal{Q}_1 := [0, \mathfrak{T}], \\ ({}^{\mathcal{LC}}\mathcal{D}^{\alpha_2})\Xi(\mathfrak{t}) = \mathcal{J}_1(\mathfrak{t}, \mathcal{O}(\mathfrak{t}), \Xi(\mathfrak{t})), \mathfrak{t} \in \mathcal{Q}_1 := [0, \mathfrak{T}], \\ (\mathcal{O} + \Xi)(0) = -(\mathcal{O} + \Xi)(\mathfrak{T}), \\ \int_{\varrho}^{\zeta} (\mathcal{O} - \Xi)(\varpi) d\varpi - \sum_{i=1}^{m} \mathfrak{q}_i (\mathcal{O} - \Xi)(\varphi_i) - \sum_{j=1}^{n} \mathfrak{u}_j (\mathcal{O} - \Xi)(\delta_j) = \mathcal{P}_1, \end{cases} \tag{1}$$

where ${}^{\mathcal{C}}\mathcal{D}^{\alpha_i}$ denotes the Caputo fractional derivative operator of order $\alpha_i; i = 1, 2$. $\alpha_1, \alpha_2 \in (1, 2]$, $0 < \varphi_i < \varrho < \zeta < \delta_j < \mathfrak{T}, i = 1, \cdots, m, j = 1, \cdots, n$, and $\varrho_1, \mathcal{J}_1 : \mathcal{Q}_1 \times \mathcal{R}_e \times \mathcal{R}_e \to \mathcal{R}_e$ are continuous functions.

With inspiration from the previously described work, we want to develop the literature on boundary value issues of order (1,2] using Caputo fractional differential equations. To be more specific, we look at and analyze the following nonlocal RL boundary value problem:

$$\begin{cases} ({}^{\mathcal{C}}\mathcal{D}^{\alpha_1} + \mathcal{K}_1 {}^{\mathcal{C}}\mathcal{D}^{\alpha_1 - 1})\mathcal{O}(\mathfrak{t}) = \varrho_1(\mathfrak{t}, \mathcal{O}(\mathfrak{t}), \Xi(\mathfrak{t})), \mathfrak{t} \in \mathcal{Q}_1 := [0, \mathfrak{T}], \\ ({}^{\mathcal{C}}\mathcal{D}^{\alpha_2} + \mathcal{K}_1 {}^{\mathcal{C}}\mathcal{D}^{\alpha_2 - 1})\Xi(\mathfrak{t}) = \mathcal{J}_1(\mathfrak{t}, \mathcal{O}(\mathfrak{t}), \Xi(\mathfrak{t})), \mathfrak{t} \in \mathcal{Q}_1 := [0, \mathfrak{T}], \\ (\mathcal{O} + \Xi)(0) = -(\mathcal{O} + \Xi)(\mathfrak{T}), \\ \beta \frac{\rho^{1-\mathfrak{q}}}{\Gamma(\mathfrak{q})} \int_0^{\zeta} \frac{\varpi^{\rho - 1}}{(\zeta^{\rho} - \varpi^{\rho})} (\mathcal{O} - \Xi)(\varpi) d\varpi = \mathcal{P}_1, \end{cases} \tag{2}$$

where ${}^{\mathcal{C}}\mathcal{D}^{\alpha_1}, {}^{\mathcal{C}}\mathcal{D}^{\alpha_2}$ represent the Caputo fractional derivative of order $1 < \alpha_1, \alpha_2 \leq 2$, $\varrho_1, \mathcal{J}_1 : [0, \mathfrak{T}] \times \mathcal{R}_e \times \mathcal{R}_e$ are a continuous function, ${}^{\rho}\mathcal{I}^{\mathfrak{q}}$ is the Generalized Riemann–Liouville fractional of order $\mathfrak{q} > 0, \rho > 0$, and $\mathcal{P}_1$ is a constant.

We use the Banach contraction mapping principle and Schaefer's fixed-point theorem in this article to explore the existence and uniqueness of solutions for system (2) under integral boundary conditions. Additionally, we analyze the system's stability using Ulam–Hyers (UH) stability. So, the novelty of our system (2) is provided by the existence of the integral term and different positive parameters in system (2), and by the RL boundary conditions. For new results obtained in recent years and for the applications of fractional calculus and fractional differential equations in various fields, we refer to the books [32–34] and their associated references [36–39] for the new findings in recent years.

The main contributions of this work are provided as follows:

1. Fundamental concepts and auxiliary lemmas related to the linear variant of problem (2), providing a solid theoretical foundation for our subsequent analysis.
2. The concerning problem (2), derived using standard fixed-point theorems. This section demonstrates the rigorous application of these mathematical tools to establish the existence and uniqueness of solutions.
3. We investigate the stability properties of the system of nonlinear coupled SFDEs of the Caputo type through UH stability analysis. This section highlights the robustness of

the solutions and their resilience to small perturbations, which is essential for practical implementations.

4. Finally, we provide detailed proofs and examples to further illustrate our findings. The numerical section ensures that the theoretical results are well-supported by concrete instances, enhancing the understanding and applicability of our work.

2. Preliminaries

In this section, we recall some basic concepts of fractional calculus [7] and present the known results that are needed in our forthcoming analysis.

Definition 1 ([40])**.** *The fractional integral of the RL type of order α_1 is defined by*

$$\mathcal{I}^{\alpha_1} h_1(\mathfrak{t}) = \frac{1}{\Gamma(\alpha_1)} \int_0^{\mathfrak{t}} \frac{h_1(\varpi)}{(\mathfrak{t}-\varpi)^{1-\alpha_1}} d\varpi, \ \ \alpha_1 > 0,$$

where $\Gamma(\cdot)$ is the gamma function.

Definition 2 ([40])**.** *The Caputo fractional derivative (CFD) of order $\alpha_1 > 0$ of the function $h_1(\mathfrak{t})$ is defined by*

$$^{c}\mathcal{D}^{\alpha_1} h_1(\mathfrak{t}) = \frac{1}{\Gamma(n-\alpha_1)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\varpi)^{n-\alpha_1-1} h_1^{(n)}(\varpi) d\varpi, \quad n-1 < \alpha_1 < n, \ n = \lfloor \alpha_1 \rfloor + 1.$$

Definition 3 ([40])**.** *The fractional integral of generalized RL type with order $\alpha_1 > 0$ and $\rho > 0$, of a function $\mathfrak{h}_1(\mathfrak{t})$, $\forall\, 0 < \mathfrak{t} < \infty$, is defined as*

$$^{\rho}\mathcal{I}^{\mathfrak{q}} \mathfrak{h}_1(\mathfrak{t}) = \frac{\rho^{1-\mathfrak{q}}}{\Gamma(\mathfrak{q})} \int_0^{\mathfrak{t}} \frac{\varpi^{\rho-1}}{(\mathfrak{t}^{\rho}-\varpi^{\rho})^{1-\mathfrak{q}}} \mathfrak{h}_1(\varpi) d\varpi,$$

provided the right-hand side is point-wise defined on (0,∞).

Lemma 1. *Let $\mathfrak{q} > 0$ and $\mathfrak{p} > 0$ be the given constants. Then,*

$$^{\rho}\mathcal{I}^{\mathfrak{q}} \mathfrak{t}^{\mathfrak{p}} = \frac{\Gamma\left(\frac{\mathfrak{p}+\rho}{\rho}\right)}{\Gamma\left(\frac{\mathfrak{p}+\rho\mathfrak{p}+\rho}{\rho}\right)} \frac{\mathfrak{t}^{\mathfrak{p}+\rho\mathfrak{q}}}{\rho^{\mathfrak{q}}}. \tag{3}$$

The following lemma aims to investigate the linear version of the problem described in Equation (2).

Lemma 2. *Let $\mathcal{H}_1, \mathcal{H}_2 \in \mathcal{C}(\mathcal{Q}_1, \mathcal{R}_e)$. The solution of the SFDEs,*

$$\begin{cases} ({}^{C}\mathcal{D}^{\alpha_1} + \mathcal{K}_1 {}^{C}\mathcal{D}^{\alpha_1-1})\mathcal{O}(\mathfrak{t}) = \mathcal{H}_1(\mathfrak{t}), \ \ \mathfrak{t} \in \mathcal{Q}_1 := [0, \mathfrak{T}], \\ ({}^{C}\mathcal{D}^{\alpha_2} + \mathcal{K}_1 {}^{C}\mathcal{D}^{\alpha_2-1})\Xi(\mathfrak{t}) = \mathcal{H}_2(\mathfrak{t}), \ \ \mathfrak{t} \in \mathcal{Q}_1 := [0, \mathfrak{T}], \\ (\mathcal{O}+\Xi)(0) = -(\mathcal{O}+\Xi)(\mathfrak{T}), \\ \beta \dfrac{\rho^{1-\mathfrak{q}}}{\Gamma(\mathfrak{q})} \displaystyle\int_0^{\zeta} \frac{\varpi^{\rho-1}}{(\zeta^{\rho}-\varpi^{\rho})} (\mathcal{O}-\Xi)(\varpi) d\varpi = \mathcal{P}_1, \end{cases} \tag{4}$$

is provided by

$$\begin{aligned} \mathcal{O}(\mathfrak{t}) =& \frac{e^{-\mathcal{K}_1 \mathfrak{t}}}{2} \Bigg[\Bigg\{ \frac{1}{\Delta_1} \Bigg(-\int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T}-\varpi)} \Bigg(\int_0^{\varpi} \frac{(\varpi-\mathfrak{u})^{\alpha_1-2}}{\Gamma(\alpha_1-1)} \mathcal{H}_1(\mathfrak{u}) d\mathfrak{u} \Bigg) d\varpi \\ & - \int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T}-\varpi)} \Bigg(\int_0^{\varpi} \frac{(\varpi-\mathfrak{u})^{\alpha_2-2}}{\Gamma(\alpha_1-1)} \mathcal{H}_2(\mathfrak{u}) d\mathfrak{u} \Bigg) d\varpi \Bigg) \end{aligned}$$

$$+\frac{1}{\Delta_2}\Bigg(\mathcal{P}_1-\beta\frac{\rho^{1-\mathfrak{q}}}{\Gamma(\mathfrak{q})}\int_0^\zeta\frac{\varpi^{\rho-1}}{(\zeta^\rho-\varpi^\rho)}\bigg(\int_0^\varpi e^{-\mathcal{K}_1(\varpi-\mathfrak{u})}\bigg(\int_0^\mathfrak{u}\frac{(\mathfrak{u}-\varphi)^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\varphi)d\varphi\bigg)d\mathfrak{u}\bigg)d\varpi$$
$$+\beta\frac{\rho^{1-\mathfrak{q}}}{\Gamma(\mathfrak{q})}\int_0^\zeta\frac{\varpi^{\rho-1}}{(\zeta^\rho-\varpi^\rho)}\bigg(\int_0^\varpi e^{-\mathcal{K}_1(\varpi-\mathfrak{u})}\bigg(\int_0^\mathfrak{u}\frac{(\mathfrak{u}-\varphi)^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\varphi)d\varphi\bigg)d\mathfrak{u}\bigg)d\varpi\bigg)\Bigg\}\Bigg]$$
$$+\int_0^\mathfrak{t} e^{-\mathcal{K}_1(\mathfrak{t}-\varpi)}\bigg(\int_0^\varpi\frac{(\varpi-\mathfrak{u})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathfrak{u})d\mathfrak{u}\bigg)d\varpi, \tag{5}$$

$$\Xi(\mathfrak{t})=\frac{e^{-\mathcal{K}_1\mathfrak{t}}}{2}\Bigg[\Bigg\{\frac{1}{\Delta_1}\bigg(-\int_0^\mathfrak{T} e^{-\mathcal{K}_1(\mathfrak{T}-\varpi)}\bigg(\int_0^\varpi\frac{(\varpi-\mathfrak{u})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathfrak{u})d\mathfrak{u}\bigg)d\varpi$$
$$-\int_0^\mathfrak{T} e^{-\mathcal{K}_1(\mathfrak{T}-\varpi)}\bigg(\int_0^\varpi\frac{(\varpi-\mathfrak{u})^{\alpha_2-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_2(\mathfrak{u})d\mathfrak{u}\bigg)d\varpi\bigg)$$
$$-\frac{1}{\Delta_2}\Bigg(\mathcal{P}_1-\beta\frac{\rho^{1-\mathfrak{q}}}{\Gamma(\mathfrak{q})}\int_0^\zeta\frac{\varpi^{\rho-1}}{(\zeta^\rho-\varpi^\rho)}\bigg(\int_0^\varpi e^{-\mathcal{K}_1(\varpi-\mathfrak{u})}\bigg(\int_0^\mathfrak{u}\frac{(\mathfrak{u}-\varphi)^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\varphi)d\varphi\bigg)d\mathfrak{u}\bigg)d\varpi$$
$$+\beta\frac{\rho^{1-\mathfrak{q}}}{\Gamma(\mathfrak{q})}\int_0^\zeta\frac{\varpi^{\rho-1}}{(\zeta^\rho-\varpi^\rho)}\bigg(\int_0^\varpi e^{-\mathcal{K}_1(\varpi-\mathfrak{u})}\bigg(\int_0^\mathfrak{u}\frac{(\mathfrak{u}-\varphi)^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\varphi)d\varphi\bigg)d\mathfrak{u}\bigg)d\varpi\bigg)\Bigg\}\Bigg]$$
$$+\int_0^\mathfrak{t} e^{-\mathcal{K}_1(\mathfrak{t}-\varpi)}\bigg(\int_0^\varpi\frac{(\varpi-\mathfrak{u})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathfrak{u})d\mathfrak{u}\bigg)d\varpi, \tag{6}$$

where

$$\Delta_1=(1+e^{-\mathcal{K}_1\mathfrak{T}}),\ \ \Delta_2=\beta^\rho\mathcal{I}^\mathfrak{p}e^{-\mathcal{K}_1(\zeta)}\neq 0. \tag{7}$$

and

$$\mathcal{I}_1=\bigg(-\int_0^\mathfrak{T} e^{-\mathcal{K}_1(\mathfrak{T}-\varpi)}\bigg(\int_0^\varpi\frac{(\varpi-\mathfrak{u})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathfrak{u})d\mathfrak{u}\bigg)d\varpi$$
$$-\int_0^\mathfrak{T} e^{-\mathcal{K}_1(\mathfrak{T}-\varpi)}\bigg(\int_0^\varpi\frac{(\varpi-\mathfrak{u})^{\alpha_2-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_2(\mathfrak{u})d\mathfrak{u}\bigg)d\varpi\bigg)$$

$$\mathcal{I}_2=\bigg(\mathcal{P}_1-\beta\frac{\rho^{1-\mathfrak{q}}}{\Gamma(\mathfrak{q})}\int_0^\zeta\frac{\varpi^{\rho-1}}{(\zeta^\rho-\varpi^\rho)}\bigg(\int_0^\varpi e^{-\mathcal{K}_1(\varpi-\mathfrak{u})}\bigg(\int_0^\mathfrak{u}\frac{(\mathfrak{u}-\varphi)^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\varphi)d\varphi\bigg)d\mathfrak{u}\bigg)d\varpi$$
$$+\beta\frac{\rho^{1-\mathfrak{q}}}{\Gamma(\mathfrak{q})}\int_0^\zeta\frac{\varpi^{\rho-1}}{(\zeta^\rho-\varpi^\rho)}\bigg(\int_0^\varpi e^{-\mathcal{K}_1(\varpi-\mathfrak{u})}\bigg(\int_0^\mathfrak{u}\frac{(\mathfrak{u}-\varphi)^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\varphi)d\varphi\bigg)d\mathfrak{u}\bigg)d\varpi\bigg).$$

Proof. Equation (4) can be equivalently written as

$$({}^{\mathcal{C}}\mathcal{D}^{\alpha_1}+\mathcal{K}_1{}^{\mathcal{C}}\mathcal{D}^{\alpha_1-1})\mathcal{O}(\mathfrak{t})=\mathcal{H}_1(\mathfrak{t}),$$
$$({}^{\mathcal{C}}\mathcal{D}^{\alpha_2}+\mathcal{K}_1{}^{\mathcal{C}}\mathcal{D}^{\alpha_2-1})\Xi(\mathfrak{t})=\mathcal{H}_2(\mathfrak{t}).$$

Rewriting Equation (4) as ${}^{\mathcal{C}}\mathcal{D}^{\alpha_1}(1+\mathcal{K}_1{}^{\mathcal{C}}\mathcal{D}^{-1})\mathcal{O}(\mathfrak{t})=\mathcal{H}_1(\mathfrak{t})$ and ${}^{\mathcal{C}}\mathcal{D}^{\alpha_2}(1+\mathcal{K}_2{}^{\mathcal{C}}\mathcal{D}^{-1})\Xi(\mathfrak{t})$ $=\mathcal{H}_2(\mathfrak{t})$, and then applying the integral operators $\mathcal{I}^{\alpha_1}_{0^+}$ and $\mathcal{I}^{\alpha_2}_{0^+}$, respectively, we obtain the following:

$$\mathcal{O}(\mathfrak{t})=\mathfrak{c}_0e^{-\mathcal{K}_1\mathfrak{t}}+\int_0^\mathfrak{t} e^{-\mathcal{K}_1(\mathfrak{t}-\varpi)}\bigg(\int_0^\mathfrak{t}\frac{(s-\mathfrak{a})^{(\alpha_1-2)}}{\Gamma(\alpha_1-1)}\mathcal{H}_1(\mathfrak{q})d\mathfrak{q}\bigg)d\varpi, \tag{8}$$

$$\Xi(\mathfrak{t})=\mathfrak{d}_0e^{-\mathcal{K}_1\mathfrak{t}}+\int_0^\mathfrak{t} e^{-\mathcal{K}_1(\mathfrak{t}-\varpi)}\bigg(\int_0^\mathfrak{t}\frac{(s-\mathfrak{a})^{(\alpha_2-2)}}{\Gamma(\alpha_2-1)}\mathcal{H}_2(\mathfrak{q})d\mathfrak{q}\bigg)d\varpi, \tag{9}$$

where $\mathfrak{c}_0,\mathfrak{d}_0$ are constants. Using boundary conditions (4) in (8) and (9), we obtain

$$\mathfrak{c}_0+\mathfrak{d}_0=\mathcal{I}_1 \tag{10}$$

$$\mathfrak{c}_0 - \mathfrak{d}_0 = \mathcal{I}_2. \tag{11}$$

Solving (10) and (11) for $\mathfrak{c}_0$ and $\mathfrak{d}_0$ yields

$$\begin{aligned}\mathfrak{c}_0 =& \frac{1}{2}\Bigg\{ \frac{1}{\Delta_1}\Bigg(-\int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T}-\varpi)} \Bigg(\int_0^{\varpi} \frac{(\varpi-\mathfrak{u})^{\alpha_1-2}}{\Gamma(\alpha_1-1)} \mathcal{H}_1(\mathfrak{u})d\mathfrak{u} \Bigg) d\varpi \\ &- \int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T}-\varpi)} \Bigg(\int_0^{\varpi} \frac{(\varpi-\mathfrak{u})^{\alpha_2-2}}{\Gamma(\alpha_1-1)} \mathcal{H}_2(\mathfrak{u})d\mathfrak{u} \Bigg) d\varpi \Bigg) \\ &+ \frac{1}{\Delta_2}\Bigg(\mathcal{P}_1 - \beta\frac{\rho^{1-\mathfrak{q}}}{\Gamma(\mathfrak{q})} \int_0^{\zeta} \frac{\varpi^{\rho-1}}{(\zeta^{\rho}-\varpi^{\rho})} \Bigg(\int_0^{\varpi} e^{-\mathcal{K}_1(\varpi-\mathfrak{u})} \Bigg(\int_0^{\mathfrak{u}} \frac{(\mathfrak{u}-\varphi)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} \mathcal{H}_1(\varphi)d\varphi \Bigg) d\mathfrak{u} \Bigg) d\varpi \\ &+ \beta\frac{\rho^{1-\mathfrak{q}}}{\Gamma(\mathfrak{q})} \int_0^{\zeta} \frac{\varpi^{\rho-1}}{(\zeta^{\rho}-\varpi^{\rho})} \Bigg(\int_0^{\varpi} e^{-\mathcal{K}_1(\varpi-\mathfrak{u})} \Bigg(\int_0^{\mathfrak{u}} \frac{(\mathfrak{u}-\varphi)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} \mathcal{H}_2(\varphi)d\varphi \Bigg) d\mathfrak{u} \Bigg) d\varpi \Bigg) \Bigg\},\end{aligned}$$

and

$$\begin{aligned}\mathfrak{d}_0 =& \frac{1}{2}\Bigg\{ \frac{1}{\Delta_1}\Bigg(-\int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T}-\varpi)} \Bigg(\int_0^{\varpi} \frac{(\varpi-\mathfrak{u})^{\alpha_1-2}}{\Gamma(\alpha_1-1)} \mathcal{H}_1(\mathfrak{u})d\mathfrak{u} \Bigg) d\varpi \\ &- \int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T}-\varpi)} \Bigg(\int_0^{\varpi} \frac{(\varpi-\mathfrak{u})^{\alpha_2-2}}{\Gamma(\alpha_1-1)} \mathcal{H}_2(\mathfrak{u})d\mathfrak{u} \Bigg) d\varpi \Bigg) \\ &- \frac{1}{\Delta_2}\Bigg(\mathcal{P}_1 - \beta\frac{\rho^{1-\mathfrak{q}}}{\Gamma(\mathfrak{q})} \int_0^{\zeta} \frac{\varpi^{\rho-1}}{(\zeta^{\rho}-\varpi^{\rho})} \Bigg(\int_0^{\varpi} e^{-\mathcal{K}_1(\varpi-\mathfrak{u})} \Bigg(\int_0^{\mathfrak{u}} \frac{(\mathfrak{u}-\varphi)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} \mathcal{H}_1(\varphi)d\varphi \Bigg) d\mathfrak{u} \Bigg) d\varpi \\ &+ \beta\frac{\rho^{1-\mathfrak{q}}}{\Gamma(\mathfrak{q})} \int_0^{\zeta} \frac{\varpi^{\rho-1}}{(\zeta^{\rho}-\varpi^{\rho})} \Bigg(\int_0^{\varpi} e^{-\mathcal{K}_1(\varpi-\mathfrak{u})} \Bigg(\int_0^{\mathfrak{u}} \frac{(\mathfrak{u}-\varphi)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} \mathcal{H}_2(\varphi)d\varphi \Bigg) d\mathfrak{u} \Bigg) d\varpi \Bigg) \Bigg\}.\end{aligned}$$

Solutions (5) and (6) are derived by substituting the values of $\mathfrak{c}_0$ and $\mathfrak{d}_0$ into Equation (8), respectively. □

3. Main Results

Let us introduce the space $\mathbb{E} = \mathcal{C}(\mathcal{Q}_1, \mathcal{R}_e) \times \mathcal{C}(\mathcal{Q}_1, \mathcal{R}_e)$ endowed with the norm $\|(\mathcal{O}, \Xi)\| = \sup_{\mathfrak{t}\in\mathcal{Q}_1} |\mathcal{O}(\mathfrak{t})| + \sup_{\mathfrak{t}\in\mathcal{Q}_1} |\Xi(\mathfrak{t})|$ for $(\mathcal{O}, \Xi) \in \mathbb{E}$. The product space is also a Banach space.

Based on Lemma 2, we define the operator $\Lambda : \mathbb{E} \to \mathbb{E}$ corresponding to system (2) as follows:

$$\Lambda(\mathcal{O}, \Xi)(\mathfrak{t}) = \begin{pmatrix} \Sigma_1(\mathcal{O}, \Xi)(\mathfrak{t}) \\ \Sigma_2(\mathcal{O}, \Xi)(\mathfrak{t}) \end{pmatrix}, \tag{12}$$

where

$$\begin{aligned}&\Sigma_1(\mathcal{O}, \Xi)(\mathfrak{t}) \\ &= \frac{e^{-\mathcal{K}_1\mathfrak{t}}}{2}\Bigg[\Bigg\{ \frac{1}{\Delta_1}\Bigg(-\int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T}-\varpi)} \Bigg(\int_0^{\varpi} \frac{(\varpi-\mathfrak{u})^{\alpha_1-2}}{\Gamma(\alpha_1-1)} \varrho_1(\mathfrak{u}, \mathcal{O}(\mathfrak{u}), \Xi(\mathfrak{u}))d\mathfrak{u} \Bigg) d\varpi \\ &- \int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T}-\varpi)} \Bigg(\int_0^{\varpi} \frac{(\varpi-\mathfrak{u})^{\alpha_2-2}}{\Gamma(\alpha_2-1)} \mathcal{J}_1(\mathfrak{u}, \mathcal{O}(\mathfrak{u}), \Xi(\mathfrak{u}))d\mathfrak{u} \Bigg) d\varpi \Bigg) \\ &+ \frac{1}{\Delta_2}\Bigg(\mathcal{P}_1 - \beta\frac{\rho^{1-\mathfrak{q}}}{\Gamma(\mathfrak{q})} \int_0^{\zeta} \frac{\varpi^{\rho-1}}{(\zeta^{\rho}-\varpi^{\rho})} \Bigg(\int_0^{\varpi} e^{-\mathcal{K}_1(\varpi-\mathfrak{u})} \Bigg(\int_0^{\mathfrak{u}} \frac{(\mathfrak{u}-\varphi)^{\alpha_1-2}}{\Gamma(\alpha_1-1)} \varrho_1(\varphi, \mathcal{O}(\varphi), \Xi(\varphi))d\varphi \Bigg) d\mathfrak{u} \Bigg) d\varpi \\ &+ \beta\frac{\rho^{1-\mathfrak{q}}}{\Gamma(\mathfrak{q})} \int_0^{\zeta} \frac{\varpi^{\rho-1}}{(\zeta^{\rho}-\varpi^{\rho})} \Bigg(\int_0^{\varpi} e^{-\mathcal{K}_1(\varpi-\mathfrak{u})} \Bigg(\int_0^{\mathfrak{u}} \frac{(\mathfrak{u}-\varphi)^{\alpha_2-2}}{\Gamma(\alpha_2-1)} \mathcal{J}_1(\varphi, \mathcal{O}(\varphi), \Xi(\varphi))d\varphi \Bigg) d\mathfrak{u} \Bigg) d\varpi \Bigg) \Bigg\}\Bigg]\end{aligned}$$

$$+\int_0^{\mathfrak{t}} e^{-\mathcal{K}_1(\mathfrak{t}-\varpi)}\left(\int_0^{\varpi}\frac{(\varpi-\mathfrak{u})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\varrho_1(\mathfrak{u},\mathcal{O}(\mathfrak{u}),\Xi(\mathfrak{u}))d\mathfrak{u}\right)d\varpi, \tag{13}$$

and

$$\begin{aligned}
&\Sigma_2(\mathcal{O},\Xi)(\mathfrak{t})\\
&=\frac{e^{-\mathcal{K}_1\mathfrak{t}}}{2}\Bigg[\Bigg\{\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T}-\varpi)}\left(\int_0^{\varpi}\frac{(\varpi-\mathfrak{u})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\varrho_1(\mathfrak{u},\mathcal{O}(\mathfrak{u}),\Xi(\mathfrak{u}))d\mathfrak{u}\right)d\varpi\\
&-\int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T}-\varpi)}\left(\int_0^{\varpi}\frac{(\varpi-\mathfrak{u})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{J}_1(\mathfrak{u},\mathcal{O}(\mathfrak{u}),\Xi(\mathfrak{u}))d\mathfrak{u}\right)d\varpi\Bigg)\\
&-\frac{1}{\Delta_2}\Bigg(\mathcal{P}_1-\beta\frac{\rho^{1-\mathfrak{q}}}{\Gamma(\mathfrak{q})}\int_0^{\zeta}\frac{\varpi^{\rho-1}}{(\zeta^{\rho}-\varpi^{\rho})}\left(\int_0^{\varpi}e^{-\mathcal{K}_1(\varpi-\mathfrak{u})}\left(\int_0^{\mathfrak{u}}\frac{(\mathfrak{u}-\varphi)^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\varrho_1(\varphi,\mathcal{O}(\varphi),\Xi(\varphi))d\varphi\right)d\mathfrak{u}\right)d\varpi\\
&+\beta\frac{\rho^{1-\mathfrak{q}}}{\Gamma(\mathfrak{q})}\int_0^{\zeta}\frac{\varpi^{\rho-1}}{(\zeta^{\rho}-\varpi^{\rho})}\left(\int_0^{\varpi}e^{-\mathcal{K}_1(\varpi-\mathfrak{u})}\left(\int_0^{\mathfrak{u}}\frac{(\mathfrak{u}-\varphi)^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{J}_1(\varphi,\mathcal{O}(\varphi),\Xi(\varphi))d\varphi\right)d\mathfrak{u}\right)d\varpi\Bigg)\Bigg\}\Bigg]\\
&+\int_0^{\mathfrak{t}} e^{-\mathcal{K}_1(\mathfrak{t}-\varpi)}\left(\int_0^{\varpi}\frac{(\varpi-\mathfrak{u})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{J}_1(\mathfrak{u},\mathcal{O}(\mathfrak{u}),\Xi(\mathfrak{u})d\mathfrak{u}\right)d\varpi.
\end{aligned} \tag{14}$$

Next, we establish hypotheses that form the basis for our primary findings.

$(\mathcal{W}_1)$ $\exists$ continuous non-negative functions Ψ_i and $k_i \in \mathcal{C}(\mathcal{Q}_1,\mathcal{R}_e^+)$ for $i=1,2,3$, such that

$$|\varrho_1(\mathfrak{t},\mathcal{O},\Xi)| \le \Psi_1(\mathfrak{t})+\Psi_2(\mathfrak{t})|\mathcal{O}|+\Psi_3|\Xi| \text{ forall } (\mathfrak{t},\mathcal{O},\Xi)\in\mathcal{Q}_1\times\mathcal{R}_e^2,$$
$$|\mathcal{J}_1(\mathfrak{t},\mathcal{O},\Xi)| \le k_1(\mathfrak{t})+k_2(\mathfrak{t})|\mathcal{O}|+k_3|\Xi| \text{ forall } (\mathfrak{t},\mathcal{O},\Xi)\in\mathcal{Q}_1\times\mathcal{R}_e^2.$$

$(\mathcal{W}_2)$ $\exists$ positive constants $\mathcal{S}_1,\mathcal{S}_2$, $\mathcal{Z}_1$, and $\mathcal{Z}_2$ such that $\forall$ $\mathfrak{t}\in\mathcal{Q}_1$, $\mathcal{O}_i,\Xi_i\in\mathcal{R}_e, i=1,2.$

$$|\varrho_1(\mathfrak{t},\mathcal{O}_1,\Xi_1)-\varrho_1(\mathfrak{t},\mathcal{O}_2,\Xi_2)| \le (\mathcal{S}_1|\mathcal{O}_1-\mathcal{O}_2|+\mathcal{S}_2|\Xi_1-\Xi_2|), \forall\ \mathfrak{t}\in\mathcal{Q}_1,$$
$$|\mathcal{J}_1(\mathfrak{t},\mathcal{O}_1,\Xi_1)-\mathcal{J}_1(\mathfrak{t},\mathcal{O}_2,\Xi_2)| \le (\mathcal{Z}_1|\mathcal{O}_1-\mathcal{O}_2|+\mathcal{Z}_2|\Xi_1-\Xi_2|), \forall\ \mathfrak{t}\in\mathcal{Q}_1.$$

To streamline calculations and improve computational efficiency, we introduce the following notation:

$$\begin{aligned}
\Omega_1 =&\frac{e^{-\mathcal{K}_1\mathfrak{t}}}{2}\left[\frac{1}{\Delta_1}\left\{\frac{\mathfrak{T}^{\alpha_1-1}}{\mathcal{K}_1\Gamma(\alpha_1)}(1-e^{\mathcal{K}_1\varpi})\right\}\right.\\
&\left.+\frac{1}{\Delta_2}\left\{\frac{\beta}{\mathcal{K}_1^2\Gamma(\alpha_1)}\frac{\zeta^{\alpha_1-1+\rho\mathfrak{q}}}{\rho^{\mathfrak{q}}}\frac{\Gamma\left(\frac{(\alpha_1-1+\rho)}{\rho}\right)}{\Gamma\left(\frac{\alpha_1-1+\rho\mathfrak{q}+\rho}{\rho}\right)}(\zeta\mathcal{K}_1+e^{-\mathcal{K}_1\zeta}-1)\right\}\right],
\end{aligned} \tag{15}$$

$$\begin{aligned}
\Omega_2 =&\frac{e^{-\mathcal{K}_1\mathfrak{t}}}{2}\left[\frac{1}{\Delta_1}\left\{\frac{\mathfrak{T}^{\alpha_2-1}}{\mathcal{K}_1\Gamma(\alpha_2)}(1-e^{\mathcal{K}_1\varpi})\right\}\right.\\
&\left.+\frac{1}{\Delta_2}\left\{\frac{\beta}{\mathcal{K}_1^2\Gamma(\alpha_2)}\frac{\zeta^{\alpha_2-1+\rho\mathfrak{q}}}{\rho^{\mathfrak{q}}}\frac{\Gamma\left(\frac{(\alpha_2-1+\rho)}{\rho}\right)}{\Gamma\left(\frac{\alpha_2-1+\rho\mathfrak{q}+\rho}{\rho}\right)}(\zeta\mathcal{K}_1+e^{-\mathcal{K}_1\zeta}-1)\right\}\right],
\end{aligned} \tag{16}$$

and

$$\Phi=\min\left\{1-\left[||\Psi_2||\left(2\Omega_1+\frac{\mathfrak{t}^{\alpha_1-1}}{\mathcal{K}_1\Gamma(\alpha_1)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)+||k_2||\left(2\Omega_2+\frac{\mathfrak{t}^{\alpha_2-1}}{\mathcal{K}_1\Gamma(\alpha_2)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)\right],\right.$$

$$1 - \left[||\Psi_3|| \left(2\Omega_1 + \frac{t^{\alpha_1 - 1}}{\mathcal{K}_1 \Gamma(\alpha_1)} (1 - e^{-\mathcal{K}_1 t}) \right) + ||k_3|| \left(2\Omega_2 + \frac{t^{\alpha_2 - 1}}{\mathcal{K}_1 \Gamma(\alpha_2)} (1 - e^{-\mathcal{K}_1 t}) \right) \right] \Bigg\}.$$

The initial existence result for problem (2) is derived from the following fixed-point theorem [40].

Theorem 1. *Assume that* $(\mathcal{W}_1)$ *holds. Furthermore, it is assumed that*

$$\begin{aligned} ||\Psi_2|| \left(2\Omega_1 + \frac{t^{\alpha_1 - 1}}{\mathcal{K}_1 \Gamma(\alpha_1)} (1 - e^{-\mathcal{K}_1 t}) \right) + ||k_2|| \left(2\Omega_2 + \frac{t^{\alpha_2 - 1}}{\mathcal{K}_1 \Gamma(\alpha_2)} (1 - e^{-\mathcal{K}_1 t}) \right) < 1, \\ ||\Psi_3|| \left(2\Omega_1 + \frac{t^{\alpha_1 - 1}}{\mathcal{K}_1 \Gamma(\alpha_1)} (1 - e^{-\mathcal{K}_1 t}) \right) + ||k_3|| \left(2\Omega_2 + \frac{t^{\alpha_2 - 1}}{\mathcal{K}_1 \Gamma(\alpha_2)} (1 - e^{-\mathcal{K}_1 t}) \right) < 1, \end{aligned} \tag{17}$$

where Ω_1, Ω_2 *are defined by (15) and (16). Then, problem (2) has at least one solution in* $\mathcal{Q}_1$.

Proof. In the first step, the operator $\Lambda : \mathbb{E} \to \mathbb{E}$ defined by (12) is completely continuous. The continuity of the functions ϱ_1 and $\mathcal{J}_1$ implies that the operator Λ is continuous and maps any bounded subset of $\mathbb{E}$. Let $\mathcal{Q}_1 \subset \mathbb{E}$ be a bounded set.

Next, the bounded set $\iota_{\bar{\mathfrak{r}}} \subset \mathbb{E}$. Consequently, $\exists$ positive constants $\mathcal{S}_{\varrho_1}$ and $\mathcal{S}_{\mathcal{J}_1}$ such that

$$\begin{aligned} |\varrho_1(t, \mathcal{O}(t), \Xi(t))| \le \mathcal{S}_{\varrho_1}, \\ |\mathcal{J}_1(t, \mathcal{O}(t), \Xi(t))| \le \mathcal{S}_{\mathcal{J}_1}, \end{aligned}$$

$\forall\ (\mathcal{O}, \Xi) \in \iota_{\bar{\mathfrak{r}}}$ and $t \in \mathcal{Q}_1$.

$$\begin{aligned}
&|\Sigma_1(\mathcal{O}, \Xi)(t)| \\
\le& \frac{e^{-\mathcal{K}_1 t}}{2} \Bigg[\Bigg\{ \frac{1}{\Delta_1} \Bigg(- \int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T} - \varpi)} \left(\int_0^{\varpi} \frac{(\varpi - u)^{\alpha_1 - 2}}{\Gamma(\alpha_1 - 1)} \varrho_1(u, \mathcal{O}(u), \Xi(u)) du \right) d\varpi \\
&+ \int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T} - \varpi)} \left(\int_0^{\varpi} \frac{(\varpi - u)^{\alpha_2 - 2}}{\Gamma(\alpha_2 - 1)} \mathcal{J}_1(u, \mathcal{O}(u), \Xi(u)) du \right) d\varpi \Bigg) \\
&+ \frac{1}{\Delta_2} \Bigg(\mathcal{P}_1 - \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^{\zeta} \frac{\varpi^{\rho - 1}}{(\zeta^{\rho} - \varpi^{\rho})} \left(\int_0^{\varpi} e^{-\mathcal{K}_1(\varpi - u)} \left(\int_0^{u} \frac{(u - \varphi)^{\alpha_1 - 2}}{\Gamma(\alpha_1 - 1)} \varrho_1(\varphi, \mathcal{O}(\varphi), \Xi(\varphi)) d\varphi \right) du \right) d\varpi \\
&+ \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^{\zeta} \frac{\varpi^{\rho - 1}}{(\zeta^{\rho} - \varpi^{\rho})} \left(\int_0^{\varpi} e^{-\mathcal{K}_1(\varpi - u)} \left(\int_0^{u} \frac{(u - \varphi)^{\alpha_2 - 2}}{\Gamma(\alpha_2 - 1)} \mathcal{J}_1(\varphi, \mathcal{O}(\varphi), \Xi(\varphi)) d\varphi \right) du \right) d\varpi \Bigg) \Bigg\} \Bigg] \\
&+ \int_0^{t} e^{-\mathcal{K}_1(t - \varpi)} \left(\int_0^{\varpi} \frac{(\varpi - u)^{\alpha_1 - 2}}{\Gamma(\alpha_1 - 1)} \varrho_1(u, \mathcal{O}(u), \Xi(u)) du \right) d\varpi, \\
\le& \mathcal{S}_{\varrho_1} \Bigg\{ \frac{e^{-\mathcal{K}_1 t}}{2} \Bigg[\frac{1}{\Delta_1} \left\{ \frac{\mathfrak{T}^{\alpha_1 - 1}}{\mathcal{K}_1 \Gamma(\alpha_1)} (1 - e^{\mathcal{K}_1 \varpi}) \right\} \\
&+ \frac{1}{\Delta_2} \left\{ \frac{\beta}{\mathcal{K}_1^2 \Gamma(\alpha_1)} \frac{\zeta^{\alpha_1 - 1 + \rho q}}{\rho^{q}} \frac{\Gamma\left(\frac{(\alpha_1 - 1 + \rho)}{\rho} \right)}{\Gamma\left(\frac{\alpha_1 - 1 + \rho q + \rho}{\rho} \right)} (\zeta \mathcal{K}_1 + e^{-\mathcal{K}_1 \zeta} - 1) \right\} \Bigg] \Bigg\} \\
&+ \mathcal{S}_{\mathcal{J}_1} \Bigg\{ \frac{e^{-\mathcal{K}_1 t}}{2} \Bigg[\frac{1}{\Delta_1} \left\{ \frac{\mathfrak{T}^{\alpha_2 - 1}}{\mathcal{K}_1 \Gamma(\alpha_2)} (1 - e^{\mathcal{K}_1 \varpi}) \right\} \\
&+ \frac{1}{\Delta_2} \left\{ \frac{\beta}{\mathcal{K}_1^2 \Gamma(\alpha_2)} \frac{\zeta^{\alpha_2 - 1 + \rho q}}{\rho^{q}} \frac{\Gamma\left(\frac{(\alpha_2 - 1 + \rho)}{\rho} \right)}{\Gamma\left(\frac{\alpha_2 - 1 + \rho q + \rho}{\rho} \right)} (\zeta \mathcal{K}_1 + e^{-\mathcal{K}_1 \zeta} - 1) \right\} \Bigg] \Bigg\}
\end{aligned}$$

$$+\mathcal{S}_{\varrho_1}\left(\frac{\mathfrak{t}^{\alpha_1-1}}{\mathcal{K}_1\Gamma(\alpha_1)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)+\frac{\mathcal{P}_1}{\Delta_2},$$
$$\leq\mathcal{S}_{\varrho_1}\left(\Omega_1+\frac{\mathfrak{t}^{\alpha_1-1}}{\mathcal{K}_1\Gamma(\alpha_1)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)+\mathcal{S}_{\mathcal{J}_1}\Omega_2+\frac{\mathcal{P}_1}{\Delta_2},$$

Hence,

$$||\Sigma_1(\mathcal{O},\Xi)||\leq\mathcal{S}_{\varrho_1}\left(\Omega_1+\frac{\mathfrak{t}^{\alpha_1-1}}{\mathcal{K}_1\Gamma(\alpha_1)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)+\mathcal{S}_{\mathcal{J}_1}\Omega_2+\frac{\mathcal{P}_1}{\Delta_2}, \tag{18}$$

and

$$|\Sigma_2(\mathcal{O},\Xi)(\mathfrak{t})|$$
$$\leq\frac{e^{-\mathcal{K}_1\mathfrak{t}}}{2}\Bigg[\Bigg\{\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathfrak{T}}e^{-\mathcal{K}_1(\mathfrak{T}-\varpi)}\left(\int_0^{\varpi}\frac{(\varpi-\mathfrak{u})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\varrho_1(\mathfrak{u},\mathcal{O}(\mathfrak{u}),\Xi(\mathfrak{u}))d\mathfrak{u}\right)d\varpi$$
$$+\int_0^{\mathfrak{T}}e^{-\mathcal{K}_1(\mathfrak{T}-\varpi)}\left(\int_0^{\varpi}\frac{(\varpi-\mathfrak{u})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{J}_1(\mathfrak{u},\mathcal{O}(\mathfrak{u}),\Xi(\mathfrak{u}))d\mathfrak{u}\right)d\varpi\Bigg)$$
$$+\frac{1}{\Delta_2}\Bigg(\mathcal{P}_1-\beta\frac{\rho^{1-\mathfrak{q}}}{\Gamma(\mathfrak{q})}\int_0^{\zeta}\frac{\varpi^{\rho-1}}{(\zeta^\rho-\varpi^\rho)}\left(\int_0^{\varpi}e^{-\mathcal{K}_1(\varpi-\mathfrak{u})}\left(\int_0^{\mathfrak{u}}\frac{(\mathfrak{u}-\varphi)^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\varrho_1(\varphi,\mathcal{O}(\varphi),\Xi(\varphi))d\varphi\right)d\mathfrak{u}\right)d\varpi$$
$$+\beta\frac{\rho^{1-\mathfrak{q}}}{\Gamma(\mathfrak{q})}\int_0^{\zeta}\frac{\varpi^{\rho-1}}{(\zeta^\rho-\varpi^\rho)}\left(\int_0^{\varpi}e^{-\mathcal{K}_1(\varpi-\mathfrak{u})}\left(\int_0^{\mathfrak{u}}\frac{(\mathfrak{u}-\varphi)^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{J}_1(\varphi,\mathcal{O}(\varphi),\Xi(\varphi))d\varphi\right)d\mathfrak{u}\right)d\varpi\Bigg)\Bigg\}\Bigg]$$
$$+\int_0^{\mathfrak{t}}e^{-\mathcal{K}_1(\mathfrak{t}-\varpi)}\left(\int_0^{\varpi}\frac{(\varpi-\mathfrak{u})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\varrho_1(\mathfrak{u},\mathcal{O}(\mathfrak{u}),\Xi(\mathfrak{u}))d\mathfrak{u}\right)d\varpi,$$
$$\leq\mathcal{S}_{\varrho_1}\Bigg\{\frac{e^{-\mathcal{K}_1\mathfrak{t}}}{2}\Bigg[\frac{1}{\Delta_1}\left\{\frac{\mathfrak{T}^{\alpha_1-1}}{\mathcal{K}_1\Gamma(\alpha_1)}(1-e^{\mathcal{K}_1\varpi})\right\}$$
$$+\frac{1}{\Delta_2}\left\{\frac{\beta}{\mathcal{K}_1^2\Gamma(\alpha_1)}\frac{\zeta^{\alpha_1-1+\rho\mathfrak{q}}}{\rho^{\mathfrak{q}}}\frac{\Gamma\left(\frac{(\alpha_1-1+\rho)}{\rho}\right)}{\Gamma\left(\frac{\alpha_1-1+\rho\mathfrak{q}+\rho}{\rho}\right)}(\zeta\mathcal{K}_1+e^{-\mathcal{K}_1\zeta}-1)\right\}\Bigg]\Bigg\}$$
$$+\mathcal{S}_{\mathcal{J}_1}\Bigg\{\frac{e^{-\mathcal{K}_1\mathfrak{t}}}{2}\Bigg[\frac{1}{\Delta_1}\left\{\frac{\mathfrak{T}^{\alpha_2-1}}{\mathcal{K}_1\Gamma(\alpha_2)}(1-e^{\mathcal{K}_1\varpi})\right\}$$
$$+\frac{1}{\Delta_2}\left\{\frac{\beta}{\mathcal{K}_1^2\Gamma(\alpha_2)}\frac{\zeta^{\alpha_2-1+\rho\mathfrak{q}}}{\rho^{\mathfrak{q}}}\frac{\Gamma\left(\frac{(\alpha_2-1+\rho)}{\rho}\right)}{\Gamma\left(\frac{\alpha_2-1+\rho\mathfrak{q}+\rho}{\rho}\right)}(\zeta\mathcal{K}_1+e^{-\mathcal{K}_1\zeta}-1)\right\}\Bigg]\Bigg\}$$
$$+\mathcal{S}_{\mathcal{J}_1}\left(\frac{\mathfrak{t}^{\alpha_1-1}}{\mathcal{K}_1\Gamma(\alpha_2)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)+\frac{\mathcal{P}_1}{\Delta_2},$$
$$\leq\mathcal{S}_{\varrho_1}\Omega_1+\mathcal{S}_{\mathcal{J}_1}\left(\Omega_2+\frac{\mathfrak{t}^{\alpha_2-1}}{\mathcal{K}_1\Gamma(\alpha_2)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)+\frac{\mathcal{P}_1}{\Delta_2}.$$

Hence,

$$||\Sigma_2(\mathcal{O},\Xi)||\leq\mathcal{S}_{\varrho_1}\Omega_1+\mathcal{S}_{\mathcal{J}_1}\left(\Omega_2+\frac{\mathfrak{t}^{\alpha_2-1}}{\mathcal{K}_1\Gamma(\alpha_2)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)+\frac{\mathcal{P}_1}{2\Delta_2}. \tag{19}$$

Thus, from Equations (18) and (19), we obtain

$$||\Lambda(\mathcal{O},\Xi)||=||\Sigma_1(\mathcal{O},\Xi)||+||\Sigma_2(\mathcal{O},\Xi)||\leq\mathcal{S}_{\varrho_1}\left(2\Omega_1+\frac{\mathfrak{t}^{\alpha_1-1}}{\mathcal{K}_1\Gamma(\alpha_1)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)$$

$$+\mathcal{S}_{\mathcal{J}_1}\left(2\Omega_2+\frac{\mathfrak{t}^{\alpha_2-1}}{\mathcal{K}_1\Gamma(\alpha_2)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)+\frac{2\mathcal{P}_1}{\Delta_2}.$$

Thus, it follows from the above inequalities that the operator Λ is uniformly bounded.

Next, we show that Λ is equicontinuous on $\mathbb{E}$.

Let $\mathfrak{t}_1, \mathfrak{t}_2 \in [0, \mathfrak{T}]$ with $\mathfrak{t}_1 < \mathfrak{t}_2$, and $(\mathcal{O}, \Xi) \in \iota_{\bar{\mathfrak{r}}}$. Then,

$$\begin{aligned}
&|\Sigma_1(\mathcal{O},\Xi)(\mathfrak{t}_2)-\Sigma_1(\mathcal{O},\Xi)(\mathfrak{t}_1)|\\
\leq&\left|\frac{e^{-\mathcal{K}_1\mathfrak{t}_2}-e^{-\mathcal{K}_1\mathfrak{t}_1}}{2}\left\{\left[\left\{\frac{1}{\Delta_1}\left(-\int_0^{\mathfrak{T}}e^{-\mathcal{K}_1(\mathfrak{T}-\varpi)}\left(\int_0^{\varpi}\frac{(\varpi-\mathfrak{u})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\varrho_1(\mathfrak{u},\mathcal{O}(\mathfrak{u}),\Xi(\mathfrak{u}))d\mathfrak{u}\right)d\varpi\right.\right.\right.\right.\\
&+\int_0^{\mathfrak{T}}e^{-\mathcal{K}_1(\mathfrak{T}-\varpi)}\left(\int_0^{\varpi}\frac{(\varpi-\mathfrak{u})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{J}_1(\mathfrak{u},\mathcal{O}(\mathfrak{u}),\Xi(\mathfrak{u}))d\mathfrak{u}\right)d\varpi\Bigg)\\
&+\frac{1}{\Delta_2}\left(\mathcal{P}_1-\beta\frac{\rho^{1-\mathfrak{q}}}{\Gamma(\mathfrak{q})}\int_0^{\zeta}\frac{\varpi^{\rho-1}}{(\zeta^{\rho}-\varpi^{\rho})}\left(\int_0^{\varpi}e^{-\mathcal{K}_1(\varpi-\mathfrak{u})}\left(\int_0^{\mathfrak{u}}\frac{(\mathfrak{u}-\varphi)^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\varrho_1(\varphi,\mathcal{O}(\varphi),\Xi(\varphi))d\varphi\right)d\mathfrak{u}\right)d\varpi\right.\\
&\left.\left.\left.\left.+\beta\frac{\rho^{1-\mathfrak{q}}}{\Gamma(\mathfrak{q})}\int_0^{\zeta}\frac{\varpi^{\rho-1}}{(\zeta^{\rho}-\varpi^{\rho})}\left(\int_0^{\varpi}e^{-\mathcal{K}_1(\varpi-\mathfrak{u})}\left(\int_0^{\mathfrak{u}}\frac{(\mathfrak{u}-\varphi)^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{J}_1(\varphi,\mathcal{O}(\varphi),\Xi(\varphi))d\varphi\right)d\mathfrak{u}\right)d\varpi\right)\right\}\right]\right\}\right|\\
&+\left|\int_0^{\mathfrak{t}_1}(e^{-\mathcal{K}_1(\mathfrak{t}_2-\varpi)}-e^{-\mathcal{K}_1(\mathfrak{t}_1-\varpi)})\left(\int_0^{\varpi}\frac{(\varpi-\mathfrak{u})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{S}_{\varrho_1}d\mathfrak{u}\right)d\varpi\right.\\
&\left.+\int_{\mathfrak{t}_1}^{\mathfrak{t}_2}e^{-\mathcal{K}_1(\mathfrak{t}_2-\varpi)}\left(\int_0^{\varpi}\frac{(\varpi-\mathfrak{u})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\mathcal{S}_{\varrho_1}d\mathfrak{u}\right)d\varpi\right|,\\
\leq&\frac{\mathcal{S}_{\varrho_1}}{\mathcal{K}_1\Gamma(\alpha_1+1)}(|\mathfrak{t}_1^{\alpha_1}-\mathfrak{t}_2^{\alpha_1}|+|\mathfrak{t}_1^{\alpha_1}e^{-\mathcal{K}_1\mathfrak{t}_1}-\mathfrak{t}_2^{\alpha_1}e^{-\mathcal{K}_1\mathfrak{t}_2}|)\to 0,
\end{aligned}$$

and

$$\begin{aligned}
&|\Sigma_2(\mathcal{O},\Xi)(\mathfrak{t}_2)-\Sigma_2(\mathcal{O},\Xi)(\mathfrak{t}_1)|\\
\leq&\left|\frac{e^{-\mathcal{K}_1\mathfrak{t}_2}-e^{-\mathcal{K}_1\mathfrak{t}_1}}{2}\left[\left\{\frac{1}{\Delta_1}\left(-\int_0^{\mathfrak{T}}e^{-\mathcal{K}_1(\mathfrak{T}-\varpi)}\left(\int_0^{\varpi}\frac{(\varpi-\mathfrak{u})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\varrho_1(\mathfrak{u},\mathcal{O}(\mathfrak{u}),\Xi(\mathfrak{u}))d\mathfrak{u}\right)d\varpi\right.\right.\right.\\
&+\int_0^{\mathfrak{T}}e^{-\mathcal{K}_1(\mathfrak{T}-\varpi)}\left(\int_0^{\varpi}\frac{(\varpi-\mathfrak{u})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{J}_1(\mathfrak{u},\mathcal{O}(\mathfrak{u}),\Xi(\mathfrak{u}))d\mathfrak{u}\right)d\varpi\Bigg)\\
&+\frac{1}{\Delta_2}\left(\mathcal{P}_1-\beta\frac{\rho^{1-\mathfrak{q}}}{\Gamma(\mathfrak{q})}\int_0^{\zeta}\frac{\varpi^{\rho-1}}{(\zeta^{\rho}-\varpi^{\rho})}\left(\int_0^{\varpi}e^{-\mathcal{K}_1(\varpi-\mathfrak{u})}\left(\int_0^{\mathfrak{u}}\frac{(\mathfrak{u}-\varphi)^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\varrho_1(\varphi,\mathcal{O}(\varphi),\Xi(\varphi))d\varphi\right)d\mathfrak{u}\right)d\varpi\right.\\
&\left.\left.\left.\left.+\beta\frac{\rho^{1-\mathfrak{q}}}{\Gamma(\mathfrak{q})}\int_0^{\zeta}\frac{\varpi^{\rho-1}}{(\zeta^{\rho}-\varpi^{\rho})}\left(\int_0^{\varpi}e^{-\mathcal{K}_1(\varpi-\mathfrak{u})}\left(\int_0^{\mathfrak{u}}\frac{(\mathfrak{u}-\varphi)^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{J}_1(\varphi,\mathcal{O}(\varphi),\Xi(\varphi))d\varphi\right)d\mathfrak{u}\right)d\varpi\right)\right\}\right]\right|\\
&+\left|\int_0^{\mathfrak{t}_1}(e^{-\mathcal{K}_1(\mathfrak{t}_2-\varpi)}-e^{-\mathcal{K}_1(\mathfrak{t}_1-\varpi)})\left(\int_0^{\varpi}\frac{(\varpi-\mathfrak{q})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{S}_{\mathcal{J}_1}d\mathfrak{u}\right)d\varpi\right.\\
&\left.+\int_{\mathfrak{t}_1}^{\mathfrak{t}_2}e^{-\mathcal{K}_1(\mathfrak{t}_2-\varpi)}\left(\int_0^{\varpi}\frac{(\varpi-\mathfrak{u})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{S}_{\mathcal{J}_1}d\mathfrak{u}\right)d\varpi\right|\\
\leq&\frac{\mathcal{S}_{\mathcal{J}_1}}{\mathcal{K}_1\Gamma(\alpha_2+1)}(|\mathfrak{t}_1^{\alpha_2}-\mathfrak{t}_2^{\alpha_2}|+|\mathfrak{t}_1^{\alpha_2}e^{-\mathcal{K}_1\mathfrak{t}_1}-\mathfrak{t}_2^{\alpha_2}e^{-\mathcal{K}_1\mathfrak{t}_2}|)\to 0.
\end{aligned}$$

As $\mathfrak{t}_1 \to \mathfrak{t}_2$, notice that the right-hand side of the above inequalities tends to zero, regardless of $(\mathcal{O}, \Xi) \in \iota_{\bar{\mathfrak{r}}}$.

This implies that $\Lambda(\mathcal{O}, \Xi)$ is equicontinuous. Applying the Arzelà–Ascoli theorem, we can conclude that the operator $\Lambda(\mathcal{O}, \Xi)$ is c.c.

Next, we consider the set $\bar{\Theta} = \{(\mathcal{O}, \Xi) \in \mathbb{E} \mid (\mathcal{O}, \Xi) = \Pi\Lambda(\mathcal{O}, \Xi), 0 < \Pi < 1\}$ and show that it is bounded. Let $(\mathcal{O}, \Xi) \in \bar{\Theta}$, and $(\mathcal{O}, \Xi) = \Pi\Lambda(\mathcal{O}, \Xi), 0 < \Pi < 1$. Therefore $\forall \mathfrak{t} \in \mathcal{Q}_1$, we have

$$\mathcal{O}(\mathfrak{t}) = \Pi\Sigma_1(\mathcal{O}, \Xi)(\mathfrak{t}), \;\; \Xi(\mathfrak{t}) = \Pi\Sigma_2(\mathcal{O}, \Xi)(\mathfrak{t}).$$

Using Ω_1 and Ω_2 provided by (15) and (16), we obtain

$$
\begin{aligned}
&|\mathcal{O}(\mathfrak{t})| = \Pi|\Sigma_1(\mathcal{O},\Xi)(\mathfrak{t})| \\
\leq &\frac{e^{-\mathcal{K}_1 \mathfrak{t}}}{2}\Bigg[\Bigg\{\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T}-\varpi)}\Bigg(\int_0^{\varpi}\frac{(\varpi-\mathfrak{u})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\varrho_1(|\Psi_1|+\Psi_2|\mathcal{O}(\mathfrak{u})|+\Psi_3|\Xi(\mathfrak{u})|)d\mathfrak{u}\Bigg)d\varpi \\
&+\int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T}-\varpi)}\Bigg(\int_0^{\varpi}\frac{(\varpi-\mathfrak{u})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{J}_1(|k_1|+k_2|\mathcal{O}(\mathfrak{u})|+k_3|\Xi(\mathfrak{u})|)d\mathfrak{u}\Bigg)d\varpi\Bigg) \\
&+\frac{1}{\Delta_2}\Bigg(\mathcal{P}_1-\beta\frac{\rho^{1-\mathfrak{q}}}{\Gamma(\mathfrak{q})}\int_0^{\zeta}\frac{\varpi^{\rho-1}}{(\zeta^{\rho}-\varpi^{\rho})} \\
&\times\Bigg(\int_0^{\varpi} e^{-\mathcal{K}_1(\varpi-\mathfrak{u})}\Bigg(\int_0^{\mathfrak{u}}\frac{(\mathfrak{u}-\varphi)^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\varrho_1(|\Psi_1|+\Psi_2|\mathcal{O}(\varphi)|+\Psi_3|\Xi(\varphi)|)d\varphi\Bigg)d\mathfrak{u}\Bigg)d\varpi \\
&+\beta\frac{\rho^{1-\mathfrak{q}}}{\Gamma(\mathfrak{q})}\int_0^{\zeta}\frac{\varpi^{\rho-1}}{(\zeta^{\rho}-\varpi^{\rho})} \\
&\times\Bigg(\int_0^{\varpi} e^{-\mathcal{K}_1(\varpi-\mathfrak{u})}\Bigg(\int_0^{\mathfrak{u}}\frac{(\mathfrak{u}-\varphi)^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{J}_1(|k_1|+k_2|\mathcal{O}(\varphi)|+k_3|\Xi(\varphi)|)d\varphi\Bigg)d\mathfrak{u}\Bigg)d\varpi\Bigg)\Bigg\}\Bigg] \\
&+\int_0^{\mathfrak{t}} e^{-\mathcal{K}_1(\mathfrak{t}-\varpi)}\Bigg(\int_0^{\varpi}\frac{(\varpi-\mathfrak{u})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\varrho_1(|\Psi_1|+\Psi_2|\mathcal{O}(\mathfrak{u})|+\Psi_3|\Xi(\mathfrak{u})|)d\mathfrak{u}\Bigg)d\varpi, \\
\leq &(||\Psi_1||+||\Psi_2||||\mathcal{O}||+||\Psi_3||||\Xi||)\Bigg\{\frac{e^{-\mathcal{K}_1 \mathfrak{t}}}{2}\Bigg[\frac{1}{\Delta_1}\Bigg\{\frac{\mathfrak{T}^{\alpha_1-1}}{\mathcal{K}_1\Gamma(\alpha_1)}(1-e^{\mathcal{K}_1\varpi})\Bigg\} \\
&+\frac{1}{\Delta_2}\Bigg\{\frac{\beta}{\mathcal{K}_1^2\Gamma(\alpha_1)}\frac{\zeta^{\alpha_1-1+\rho\mathfrak{q}}}{\rho^{\mathfrak{q}}}\frac{\Gamma\Big(\frac{(\alpha_1-1+\rho)}{\rho}\Big)}{\Gamma\Big(\frac{\alpha_1-1+\rho\mathfrak{q}+\rho}{\rho}\Big)}(\zeta\mathcal{K}_1+e^{-\mathcal{K}_1\zeta}-1)\Bigg\}\Bigg]+\Bigg\{\frac{\mathfrak{t}^{\alpha_1-1}}{\mathcal{K}_1\Gamma(\alpha_1)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\Bigg\}\Bigg\} \\
&+(||k_1||+||k_2||||\mathcal{O}||+||k_3||||\Xi||)\Bigg\{\frac{e^{-\mathcal{K}_1 \mathfrak{t}}}{2}\Bigg[\frac{1}{\Delta_1}\Bigg\{\frac{\mathfrak{T}^{\alpha_2-1}}{\mathcal{K}_1\Gamma(\alpha_2)}(1-e^{\mathcal{K}_1\varpi})\Bigg\} \\
&+\frac{1}{\Delta_2}\Bigg\{\frac{\beta}{\mathcal{K}_1^2\Gamma(\alpha_2)}\frac{\zeta^{\alpha_2-1+\rho\mathfrak{q}}}{\rho^{\mathfrak{q}}}\frac{\Gamma\Big(\frac{(\alpha_2-1+\rho)}{\rho}\Big)}{\Gamma\Big(\frac{\alpha_2-1+\rho\mathfrak{q}+\rho}{\rho}\Big)}(\zeta\mathcal{K}_1+e^{-\mathcal{K}_1\zeta}-1)\Bigg\}\Bigg]\Bigg\}+\frac{\mathcal{P}_1}{2\Delta_2}, \\
\leq &(||\Psi_1||+||\Psi_2||||\mathcal{O}||+||\Psi_3||||\Xi||)\Bigg(\Omega_1+\frac{\mathfrak{t}^{\alpha_1-1}}{\mathcal{K}_1\Gamma(\alpha_1)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\Bigg) \\
&+(||k_1||+||k_2||||\mathcal{O}||+||k_3||||\Xi||)\Omega_2+\frac{\mathcal{P}_1}{2\Delta_2},
\end{aligned}
$$

and

$$
\begin{aligned}
&|\Xi(\mathfrak{t})| = \Pi|\Sigma_2(\mathcal{O},\Xi)(\mathfrak{t})| \\
\leq &\frac{e^{-\mathcal{K}_1 \mathfrak{t}}}{2}\Bigg[\Bigg\{\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T}-\varpi)}\Bigg(\int_0^{\varpi}\frac{(\varpi-\mathfrak{u})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\varrho_1(|\Psi_1|+\Psi_2|\mathcal{O}(\mathfrak{u})|+\Psi_3|\Xi(\mathfrak{u})|)d\mathfrak{u}\Bigg)d\varpi \\
&+\int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T}-\varpi)}\Bigg(\int_0^{\varpi}\frac{(\varpi-\mathfrak{u})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{J}_1(|k_1|+k_2|\mathcal{O}(\mathfrak{u})|+k_3|\Xi(\mathfrak{u})|)d\mathfrak{u}\Bigg)d\varpi\Bigg) \\
&+\frac{1}{\Delta_2}\Bigg(\mathcal{P}_1-\beta\frac{\rho^{1-\mathfrak{q}}}{\Gamma(\mathfrak{q})}\int_0^{\zeta}\frac{\varpi^{\rho-1}}{(\zeta^{\rho}-\varpi^{\rho})} \\
&\times\Bigg(\int_0^{\varpi} e^{-\mathcal{K}_1(\varpi-\mathfrak{u})}\Bigg(\int_0^{\mathfrak{u}}\frac{(\mathfrak{u}-\varphi)^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\varrho_1(|\Psi_1|+\Psi_2|\mathcal{O}(\varphi)|+\Psi_3|\Xi(\varphi)|)d\varphi\Bigg)d\mathfrak{u}\Bigg)d\varpi
\end{aligned}
$$

$$
\begin{aligned}
&+\beta\frac{\rho^{1-\mathfrak{q}}}{\Gamma(\mathfrak{q})}\int_0^{\zeta}\frac{\varpi^{\rho-1}}{(\zeta^{\rho}-\varpi^{\rho})}\\
&\times\left(\int_0^{\varpi}e^{-\mathcal{K}_1(\varpi-\mathfrak{u})}\left(\int_0^{\mathfrak{u}}\frac{(\mathfrak{u}-\varphi)^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{J}_1(|k_1|+k_2|\mathcal{O}(\varphi)|+k_3|\Xi(\varphi)|)d\varphi\right)d\mathfrak{u}\right)d\varpi\Bigg)\Bigg\}\Bigg]\\
&+\int_0^{\mathfrak{t}}e^{-\mathcal{K}_1(\mathfrak{t}-\varpi)}\left(\int_0^{\varpi}\frac{(\varpi-\mathfrak{u})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}\mathcal{J}_1(|k_1|+k_2|\mathcal{O}(\mathfrak{u})|+k_3|\Xi(\mathfrak{u})|)d\mathfrak{u}\right)d\varpi,\\
\leq&(||\Psi_1||+||\Psi_2||||\mathcal{O}||+||\Psi_3||||\Xi||)\Omega_1\\
&+(||k_1||+||k_2||||\mathcal{O}||+||k_3||||\Xi||)\left(\Omega_2+\frac{\mathfrak{t}^{\alpha_2-1}}{\mathcal{K}_1\Gamma(\alpha_2)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)+\frac{\mathcal{P}_1}{2\Delta_2}.
\end{aligned}
$$

As a result, we obtain

$$
\begin{aligned}
&||\mathcal{O}||+||\Xi||\\
\leq&||\Psi_1||\left(2\Omega_1+\frac{\mathfrak{t}^{\alpha_1-1}}{\mathcal{K}_1\Gamma(\alpha_1)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)+||k_1||\left(2\Omega_2+\frac{\mathfrak{t}^{\alpha_2-1}}{\mathcal{K}_1\Gamma(\alpha_2)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)+\frac{\mathcal{P}_1}{\Delta_2}\\
&+\left[||\Psi_2||\left(2\Omega_1+\frac{\mathfrak{t}^{\alpha_1-1}}{\mathcal{K}_1\Gamma(\alpha_1)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)+||k_2||\left(2\Omega_2+\frac{\mathfrak{t}^{\alpha_2-1}}{\mathcal{K}_1\Gamma(\alpha_2)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)\right]||\mathcal{O}||\\
&+\left[||\Psi_3||\left(2\Omega_1+\frac{\mathfrak{t}^{\alpha_1-1}}{\mathcal{K}_1\Gamma(\alpha_1)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)+||k_3||\left(2\Omega_2+\frac{\mathfrak{t}^{\alpha_2-1}}{\mathcal{K}_1\Gamma(\alpha_2)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)\right]||\Xi||.
\end{aligned}
$$

By Equation (17), we can deduce that

$$
||(\mathcal{O},\Xi)||\leq\frac{||\Psi_1||\left(2\Omega_1+\frac{\mathfrak{t}^{\alpha_1-1}}{\mathcal{K}_1\Gamma(\alpha_1)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)+||k_1||\left(2\Omega_2+\frac{\mathfrak{t}^{\alpha_2-1}}{\mathcal{K}_1\Gamma(\alpha_2)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)+\frac{\mathcal{P}_1}{\Delta_2}}{\Phi}.
$$

Therefore, the set $\bar{\Theta}$ is bounded. This shows that $\|(\mathcal{O},\Xi)\|$ is bounded for $\mathfrak{t}$. Consequently, by applying Schaefer's fixed-point theorem, we can conclude that there is at least one fixed point. Thus, a solution to Equation (2) exists. □

The statement of Theorem 1 simplifies to the following special form by setting $\Psi_2(\mathfrak{t})=\Psi_3(\mathfrak{t})=0$ and $k_2(\mathfrak{t})=k_3(\mathfrak{t})=0$.

Remark 1. *$\exists$ non-negative functions $\Psi_1,k_1\in\mathcal{C}(\mathfrak{G},\mathcal{R}_e^+)$ and $\varrho_1,\mathcal{J}_1:\mathfrak{G}\times\mathcal{R}_e^2\to\mathcal{R}_e$ which are continuous functions, such that*

$$
|\varrho_1(\mathfrak{t},\mathcal{O},\Xi)|\leq\Psi_1(\mathfrak{t}),\ \ |\mathcal{J}_1(\mathfrak{t},\mathcal{O},\Xi)|\leq k_1(\mathfrak{t})\ \ \forall(\mathfrak{t},\mathcal{O},\Xi)\in\mathfrak{G}\times\mathcal{R}_e^2.
$$

Then, system (2) has at least one solution on $\mathfrak{G}$.

Remark 2. *Assume the Theorem 1, if $\Psi_i(\mathfrak{t})=\iota_i$ and $k_i(\mathfrak{t})=\varepsilon_i$ for $i=1,2,3$ (where ε_i and ι_i are non-negative constants), and if the conditions for the functions $\varrho_1,\mathcal{J}_1$ take the following form:*

$(\mathcal{O}_1')$ $\exists$ real constants $\iota_i,\varepsilon_i>0$ for $i=1,2,3$ such that

$$
\begin{aligned}
|\varrho_1(\mathfrak{t},\mathcal{O},\Xi)|&\leq\iota_1+\iota_2|\mathcal{O}|+\iota_3|\Xi|\ \ \forall(\mathfrak{t},\mathcal{O},\Xi)\in\mathfrak{G}\times\mathcal{R}_e^2,\\
|\mathcal{J}_1(\mathfrak{t},\mathcal{O},\Xi)|&\leq\varepsilon_1+\varepsilon_2|\mathcal{O}|+\varepsilon_3|\Xi|\ \ \forall(\mathfrak{t},\mathcal{O},\Xi)\in\mathfrak{G}\times\mathcal{R}_e^2,
\end{aligned}
$$

and then (17) becomes

$$
\begin{aligned}
\iota_2\left(2\Omega_1+\frac{\mathfrak{t}^{\alpha_1-1}}{\mathcal{K}_1\Gamma(\alpha_1)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)+\varepsilon_2\left(2\Omega_2+\frac{\mathfrak{t}^{\alpha_2-1}}{\mathcal{K}_1\Gamma(\alpha_2)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)<1,\\
\iota_3\left(2\Omega_1+\frac{\mathfrak{t}^{\alpha_1-1}}{\mathcal{K}_1\Gamma(\alpha_1)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)+\varepsilon_3\left(2\Omega_2+\frac{\mathfrak{t}^{\alpha_2-1}}{\mathcal{K}_1\Gamma(\alpha_2)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)<1.
\end{aligned}
$$

Now, we introduce our second result, which utilizes Banach's fixed-point theorem. This theorem guarantees the existence and uniqueness of solution (2).

Theorem 2. *Assume that* $(\mathcal{W}_2)$ *holds. Then,* (2) *has a unique solution on* $\mathbb{E}$ *if*

$$\mathcal{S}\left(2\Omega_1 + \frac{\mathfrak{t}^{\alpha_1-1}}{\mathcal{K}_1\Gamma(\alpha_1)}(1 - e^{-\mathcal{K}_1\mathfrak{t}})\right) + \mathcal{Z}\left(2\Omega_2 + \frac{\mathfrak{t}^{\alpha_2-1}}{\mathcal{K}_1\Gamma(\alpha_2)}(1 - e^{-\mathcal{K}_1\mathfrak{t}})\right) < 1, \tag{20}$$

where $\mathcal{S} = \max\{\mathcal{S}_1, \mathcal{S}_2\}$, $\mathcal{Z} = \max\{\mathcal{Z}_1, \mathcal{Z}_2\}$ *and* $\Omega_i, i = 1, 2$ *are provided by* (15) *and* (16).

Proof. Let us take $\mathcal{M}_1 = \sup_{\mathfrak{t}\in[0,\mathfrak{T}]} |\varrho_1(\mathfrak{t}, 0, 0)|$ and $\mathcal{M}_2 = \sup_{\mathfrak{t}\in[0,\mathfrak{T}]} |\mathcal{J}_1(\mathfrak{t}, 0, 0)|$ and fix

$$\mathfrak{r} > \frac{\mathcal{M}_1\left(2\Omega_1 + \frac{\mathfrak{t}^{\alpha_1-1}}{\mathcal{K}_1\Gamma(\alpha_1)}(1 - e^{-\mathcal{K}_1\mathfrak{t}})\right) + \mathcal{M}_2\left(2\Omega_2 + \frac{\mathfrak{t}^{\alpha_2-1}}{\mathcal{K}_1\Gamma(\alpha_2)}(1 - e^{-\mathcal{K}_1\mathfrak{t}})\right)}{1 - \left(\mathcal{S}\left(2\Omega_1 + \frac{\mathfrak{t}^{\alpha_1-1}}{\mathcal{K}_1\Gamma(\alpha_1)}(1 - e^{-\mathcal{K}_1\mathfrak{t}})\right) + \mathcal{Z}\left(2\Omega_2 + \frac{\mathfrak{t}^{\alpha_2-1}}{\mathcal{K}_1\Gamma(\alpha_2)}(1 - e^{-\mathcal{K}_1\mathfrak{t}})\right)\right)}.$$

Now, $\Lambda\mathcal{B}_\mathfrak{r} \subset \mathcal{B}_\mathfrak{r}$, where $\mathcal{B}_\mathfrak{r} = \{(\mathcal{O}, \Xi) \in \mathbb{E} : ||(\mathcal{O}, \Xi)|| \leq \mathfrak{r}\}$, and then

$$\begin{aligned}
&(\Sigma_1(\mathcal{O}, \Xi)) \\
&\leq \frac{e^{-\mathcal{K}_1\mathfrak{t}}}{2}\Bigg[\Bigg\{\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T}-\varpi)}\Bigg(\int_0^{\varpi} \frac{(\varpi-\mathfrak{u})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}[|\varrho_1(\mathfrak{u}, \mathcal{O}(\mathfrak{u}), \Xi(\mathfrak{u})) - \varrho_1(\mathfrak{u}, 0, 0)| + \mathcal{M}_1]d\mathfrak{u}\Bigg)d\varpi \\
&- \int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T}-\varpi)}\Bigg(\int_0^{\varpi} \frac{(\varpi-\mathfrak{u})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}[|\mathcal{J}_1(\mathfrak{u}, \mathcal{O}(\mathfrak{u}), \Xi(\mathfrak{u})) - \varrho_1(\varphi, 0, 0)| + \mathcal{M}_2]d\mathfrak{u}\Bigg)d\varpi\Bigg) \\
&+ \frac{1}{\Delta_2}\Bigg(\mathcal{P}_1 - \beta\frac{\rho^{1-\mathfrak{q}}}{\Gamma(\mathfrak{q})}\int_0^{\zeta} \frac{\varpi^{\rho-1}}{(\zeta^\rho - \varpi^\rho)} \\
&\times \Bigg(\int_0^{\varpi} e^{-\mathcal{K}_1(\varpi-\mathfrak{u})}\Bigg(\int_0^{\mathfrak{u}} \frac{(\mathfrak{u}-\varphi)^{\alpha_1-2}}{\Gamma(\alpha_1-1)}[|\varrho_1(\varphi, \mathcal{O}(\varphi), \Xi(\varphi)) - \varrho_1(\varphi, 0, 0)| + \mathcal{M}_1]d\varphi\Bigg)d\mathfrak{u}\Bigg)d\varpi \\
&+ \beta\frac{\rho^{1-\mathfrak{q}}}{\Gamma(\mathfrak{q})}\int_0^{\zeta} \frac{\varpi^{\rho-1}}{(\zeta^\rho - \varpi^\rho)} \\
&\times \Bigg(\int_0^{\varpi} e^{-\mathcal{K}_1(\varpi-\mathfrak{u})}\Bigg(\int_0^{\mathfrak{u}} \frac{(\mathfrak{u}-\varphi)^{\alpha_2-2}}{\Gamma(\alpha_2-1)}[|\mathcal{J}_1(\varphi, \mathcal{O}(\varphi), \Xi(\varphi)) - \varrho_1(\varphi, 0, 0)| + \mathcal{M}_2]d\varphi\Bigg)d\mathfrak{u}\Bigg)d\varpi\Bigg)\Bigg\}\Bigg] \\
&+ \int_0^{\mathfrak{t}} e^{-\mathcal{K}_1(\mathfrak{t}-\varpi)}\Bigg(\int_0^{\varpi} \frac{(\varpi-\mathfrak{u})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}[|\varrho_1(\mathfrak{u}, \mathcal{O}(\mathfrak{u}), \Xi(\mathfrak{u})) - \varrho_1(\mathfrak{u}, 0, 0)| + \mathcal{M}_1]d\mathfrak{u}\Bigg)d\varpi, \\
&\leq \left(\mathcal{S}\left(\Omega_1 + \frac{\mathfrak{t}^{\alpha_1-1}}{\mathcal{K}_1\Gamma(\alpha_1)}(1 - e^{-\mathcal{K}_1\mathfrak{t}})\right) + \mathcal{Z}\Omega_2\right)(||\mathcal{O}|| + ||\Xi||) \\
&+ \mathcal{M}_1\left(\Omega_1 + \frac{\mathfrak{t}^{\alpha_1-1}}{\mathcal{K}_1\Gamma(\alpha_1)}(1 - e^{-\mathcal{K}_1\mathfrak{t}})\right) + \mathcal{M}_2\Omega_2,
\end{aligned}$$

which, when the norm is applied to $\mathfrak{t}$, results in

$$\begin{aligned}
(\Sigma_1(\mathcal{O}, \Xi)) \leq &\left(\mathcal{S}\left(\Omega_1 + \frac{\mathfrak{t}^{\alpha_1-1}}{\mathcal{K}_1\Gamma(\alpha_1)}(1 - e^{-\mathcal{K}_1\mathfrak{t}})\right) + \mathcal{Z}\Omega_2\right)(||\mathcal{O}|| + ||\Xi||) \\
&+ \mathcal{M}_1\left(\Omega_1 + \frac{\mathfrak{t}^{\alpha_2-1}}{\mathcal{K}_1\Gamma(\alpha_2)}(1 - e^{-\mathcal{K}_1\mathfrak{t}})\right) + \mathcal{M}_2\Omega_2.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
(\Sigma_2(\mathcal{O}, \Xi)) \leq &\left(\mathcal{Z}\left(\Omega_2 + \frac{\mathfrak{t}^{\alpha_2-1}}{\mathcal{K}_1\Gamma(\alpha_2)}(1 - e^{-\mathcal{K}_1\mathfrak{t}})\right) + \mathcal{S}\Omega_1\right)(||\mathcal{O}|| + ||\Xi||) \\
&+ \mathcal{M}_2\left(\Omega_2 + \frac{\mathfrak{t}^{\alpha_2-1}}{\mathcal{K}_1\Gamma(\alpha_2)}(1 - e^{-\mathcal{K}_1\mathfrak{t}})\right) + \mathcal{M}_1\Omega_1.
\end{aligned}$$

Consequently, $\forall\ (\mathcal{O}, \Xi) \in \mathcal{B}_\mathfrak{r}$, we find

$$\begin{aligned}
&||(\Lambda(\mathcal{O}, \Xi))|| = ||(\Sigma_1(\mathcal{O}, \Xi))|| + ||(\Sigma_2(\mathcal{O}, \Xi))|| \\
&\leq \left(\mathcal{S}\left(2\Omega_1 + \frac{\mathfrak{t}^{\alpha_1-1}}{\mathcal{K}_1\Gamma(\alpha_1)}(1 - e^{-\mathcal{K}_1\mathfrak{t}})\right) + \mathcal{Z}\left(2\Omega_2 + \frac{\mathfrak{t}^{\alpha_1-1}}{\mathcal{K}_1\Gamma(\alpha_1)}(1 - e^{-\mathcal{K}_1\mathfrak{t}})\right)\right)(||\mathcal{O}|| + ||\Xi||)
\end{aligned}$$

$$+\mathcal{M}_1\left(2\Omega_1+\frac{\mathfrak{t}^{\alpha_1-1}}{\mathcal{K}_1\Gamma(\alpha_1)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)+\mathcal{M}_2\left(2\Omega_2+\frac{\mathfrak{t}^{\alpha_2-1}}{\mathcal{K}_1\Gamma(\alpha_2)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)<\mathfrak{r}.$$

This shows that Λ maps $\mathcal{B}_{\mathfrak{r}}$ into itself.

To demonstrate that the operator Λ is a contraction, let $(\mathcal{O}_1,\Xi_1),(\mathcal{O}_2,\Xi_2)\in\mathbb{E}$ and $\mathfrak{t}\in\mathcal{Q}_1$. By considering the relation $(\mathcal{W}_2)$, we obtain

$$\begin{aligned}
&|(\Sigma_1(\mathcal{O}_1,\Xi_1))-(\Sigma_1(\mathcal{O}_2,\Xi_2))|\\
\leq&\frac{e^{-\mathcal{K}_1\mathfrak{t}}}{2}\Bigg[\Bigg\{\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathfrak{T}}e^{-\mathcal{K}_1(\mathfrak{T}-\varpi)}\\
&\times\left(\int_0^{\varpi}\frac{(\varpi-\mathfrak{u})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}|\varrho_1(\mathfrak{u},\mathcal{O}_1(\mathfrak{u}),\Xi_1(\mathfrak{u}))-\varrho_1(\mathfrak{u},\mathcal{O}_2(\mathfrak{u}),\Xi_2(\mathfrak{u}))|d\mathfrak{u}\right)d\varpi\\
&+\int_0^{\mathfrak{T}}e^{-\mathcal{K}_1(\mathfrak{T}-\varpi)}\left(\int_0^{\varpi}\frac{(\varpi-\mathfrak{u})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}|\mathcal{J}_1(\mathfrak{u},\mathcal{O}_1(\mathfrak{u}),\Xi_1(\mathfrak{u}))-\mathcal{J}_1(\mathfrak{u},\mathcal{O}_2(\mathfrak{u}),\Xi_2(\mathfrak{u}))|d\mathfrak{u}\right)d\varpi\Bigg)\\
&+\frac{1}{\Delta_2}\Bigg(\mathcal{P}_1-\beta\frac{\rho^{1-\mathfrak{q}}}{\Gamma(\mathfrak{q})}\int_0^{\zeta}\frac{\varpi^{\rho-1}}{(\zeta^{\rho}-\varpi^{\rho})}\\
&\times\left(\int_0^{\varpi}e^{-\mathcal{K}_1(\varpi-\mathfrak{u})}\left(\int_0^{\mathfrak{u}}\frac{(\mathfrak{u}-\varphi)^{\alpha_1-2}}{\Gamma(\alpha_1-1)}|\varrho_1(\varphi,\mathcal{O}_1(\varphi),\Xi_1(\varphi))-\varrho_1(\varphi,\mathcal{O}_2(\varphi),\Xi_2(\varphi))|d\varphi\right)d\mathfrak{u}\right)d\varpi\\
&+\beta\frac{\rho^{1-\mathfrak{q}}}{\Gamma(\mathfrak{q})}\int_0^{\zeta}\frac{\varpi^{\rho-1}}{(\zeta^{\rho}-\varpi^{\rho})}\\
&\times\left(\int_0^{\varpi}e^{-\mathcal{K}_1(\varpi-\mathfrak{u})}\left(\int_0^{\mathfrak{u}}\frac{(\mathfrak{u}-\varphi)^{\alpha_2-2}}{\Gamma(\alpha_2-1)}|\mathcal{J}_1(\varphi,\mathcal{O}_1(\varphi),\Xi_1(\varphi))-\mathcal{J}_1(\varphi,\mathcal{O}_2(\varphi),\Xi_2(\varphi))|d\varphi\right)d\mathfrak{u}\right)d\varpi\Bigg)\Bigg\}\Bigg]\\
&+\int_0^{\mathfrak{t}}e^{-\mathcal{K}_1(\mathfrak{t}-\varpi)}\left(\int_0^{\varpi}\frac{(\varpi-\mathfrak{u})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}|\varrho_1(\mathfrak{u},\mathcal{O}_1(\mathfrak{u}),\Xi_1(\mathfrak{u}))-\varrho_1(\mathfrak{u},\mathcal{O}_2(\mathfrak{u}),\Xi_2(\mathfrak{u}))|d\mathfrak{u}\right)d\varpi,\\
\leq&\left(\mathcal{S}\left(\Omega_1+\frac{\mathfrak{t}^{\alpha_1-1}}{\mathcal{K}_1\Gamma(\alpha_1)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)+\mathcal{Z}\Omega_2\right)(||\mathcal{O}||+||\Xi||),
\end{aligned}$$

we have

$$|(\Sigma_1(\mathcal{O}_1,\Xi_1))-(\Sigma_1(\mathcal{O}_2,\Xi_2))|\leq\left(\mathcal{S}\left(\Omega_1+\frac{\mathfrak{t}^{\alpha_1-1}}{\mathcal{K}_1\Gamma(\alpha_1)}(1-e^{-\mathcal{K}_1\mathfrak{t}})\right)+\mathcal{Z}\Omega_2\right)(||\mathcal{O}||+||\Xi||).$$

and

$$\begin{aligned}
&|(\Sigma_2(\mathcal{O}_1,\Xi_1))-(\Sigma_2(\mathcal{O}_2,\Xi_2))|\\
\leq&\frac{e^{-\mathcal{K}_1\mathfrak{t}}}{2}\Bigg[\Bigg\{\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathfrak{T}}e^{-\mathcal{K}_1(\mathfrak{T}-\varpi)}\\
&\times\left(\int_0^{\varpi}\frac{(\varpi-\mathfrak{u})^{\alpha_1-2}}{\Gamma(\alpha_1-1)}|\varrho_1(\mathfrak{u},\mathcal{O}_1(\mathfrak{u}),\Xi_1(\mathfrak{u}))-\varrho_1(\mathfrak{u},\mathcal{O}_2(\mathfrak{u}),\Xi_2(\mathfrak{u}))|d\mathfrak{u}\right)d\varpi\\
&+\int_0^{\mathfrak{T}}e^{-\mathcal{K}_1(\mathfrak{T}-\varpi)}\\
&\times\left(\int_0^{\varpi}\frac{(\varpi-\mathfrak{u})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}|\mathcal{J}_1(\mathfrak{u},\mathcal{O}_1(\mathfrak{u}),\Xi_1(\mathfrak{u}))-\mathcal{J}_1(\mathfrak{u},\mathcal{O}_2(\mathfrak{u}),\Xi_2(\mathfrak{u}))|d\mathfrak{u}\right)d\varpi\Bigg)\\
&+\frac{1}{\Delta_2}\Bigg(\mathcal{P}_1-\beta\frac{\rho^{1-\mathfrak{q}}}{\Gamma(\mathfrak{q})}\int_0^{\zeta}\frac{\varpi^{\rho-1}}{(\zeta^{\rho}-\varpi^{\rho})}\\
&\times\left(\int_0^{\varpi}e^{-\mathcal{K}_1(\varpi-\mathfrak{u})}\left(\int_0^{\mathfrak{u}}\frac{(\mathfrak{u}-\varphi)^{\alpha_1-2}}{\Gamma(\alpha_1-1)}|\varrho_1(\varphi,\mathcal{O}_1(\varphi),\Xi_1(\varphi))-\varrho_1(\varphi,\mathcal{O}_2(\varphi),\Xi_2(\varphi))|d\varphi\right)d\mathfrak{u}\right)d\varpi\\
&+\beta\frac{\rho^{1-\mathfrak{q}}}{\Gamma(\mathfrak{q})}\int_0^{\zeta}\frac{\varpi^{\rho-1}}{(\zeta^{\rho}-\varpi^{\rho})}
\end{aligned}$$

$$\times\left(\int_0^{\varpi} e^{-\mathcal{K}_1(\varpi-\mathrm{u})}\left(\int_0^{\mathrm{u}} \frac{(\mathrm{u}-\varphi)^{\alpha_2-2}}{\Gamma(\alpha_2-1)}|\mathcal{J}_1(\varphi,\mathcal{O}_1(\varphi),\Xi_1(\varphi))-\mathcal{J}_1(\varphi,\mathcal{O}_2(\varphi),\Xi_2(\varphi))|d\varphi\right)d\mathrm{u}\right)d\varpi\right)\Bigg\}\Bigg]$$
$$+\int_0^{\mathrm{t}} e^{-\mathcal{K}_1(\mathrm{t}-\varpi)}\left(\int_0^{\varpi} \frac{(\varpi-\mathrm{u})^{\alpha_2-2}}{\Gamma(\alpha_2-1)}|\mathcal{J}_1(\mathrm{u},\mathcal{O}_1(\mathrm{u}),\Xi_1(\mathrm{u}))-\mathcal{J}_1(\mathrm{u},\mathcal{O}_2(\mathrm{u}),\Xi_2(\mathrm{u}))|d\mathrm{u}\right)d\varpi,$$
$$\leq\left(\mathcal{Z}\left(\Omega_2+\frac{\mathrm{t}^{\alpha_2-1}}{\mathcal{K}_1\Gamma(\alpha_2)}(1-e^{-\mathcal{K}_1\mathrm{t}})\right)+\mathcal{S}\Omega_1\right)(||\mathcal{O}||+||\Xi||).$$

In the like manner, we have

$$|(\Sigma_2(\mathcal{O}_1,\Xi_1))-(\Sigma_2(\mathcal{O}_2,\Xi_2))|\leq\left(\mathcal{Z}\left(\Omega_2+\frac{\mathrm{t}^{\alpha_2-1}}{\mathcal{K}_1\Gamma(\alpha_2)}(1-e^{-\mathcal{K}_1\mathrm{t}})\right)+\mathcal{S}\Omega_1\right)(||\mathcal{O}||+||\Xi||).$$

As a consequence of the preceding inequalities,

$$||(\Lambda(\mathcal{O}_1,\Xi_1))-(\Lambda(\mathcal{O}_2,\Xi_2))||=|(\Sigma_1(\mathcal{O}_1,\Xi_1))-(\Sigma_1(\mathcal{O}_2,\Xi_2))|+|(\Sigma_2(\mathcal{O}_1,\Xi_1))-(\Sigma_2(\mathcal{O}_2,\Xi_2))|$$
$$\leq\left(\mathcal{S}\left(2\Omega_1+\frac{\mathrm{t}^{\alpha_1-1}}{\mathcal{K}_1\Gamma(\alpha_1)}(1-e^{-\mathcal{K}_1\mathrm{t}})\right)+\mathcal{Z}\left(\Omega_2+\frac{\mathrm{t}^{\alpha_2-1}}{\mathcal{K}_1\Gamma(\alpha_2)}(1-e^{-\mathcal{K}_1\mathrm{t}})\right)\right)||(\mathcal{O}_1-\mathcal{O}_2,\Xi_1-\Xi_2)||.$$

Consequently, given (20), it follows that Λ is a contraction mapping. Therefore, by Banach's contraction mapping principle, Λ has a unique fixed point. This implies that the problem (2) has a unique solution on $\mathcal{Q}_1$. The proof is thus completed. □

4. Stability Results

In this section, we focus on the investigation of the stability of the coupled SFDEs represented by (2). We analyze the following inequality:

$$\begin{cases} |({}^{\mathcal{C}}\mathcal{D}^{\alpha_1}+\mathcal{K}_1{}^{\mathcal{C}}\mathcal{D}^{\alpha_1-1})\mathcal{O}(\mathrm{t})-\varrho_1(\mathrm{t},\mathcal{O}(\mathrm{t}),\Xi(\mathrm{t}))|\leq\epsilon_1 & \mathrm{t}\in\mathcal{Q}_1:=[0,\mathfrak{T}], \\ |({}^{\mathcal{C}}\mathcal{D}^{\alpha_2}+\mathcal{K}_1{}^{\mathcal{C}}\mathcal{D}^{\alpha_2-1})\Xi(\mathrm{t})-\mathcal{J}_1(\mathrm{t},\mathcal{O}(\mathrm{t}),\Xi(\mathrm{t}))|\leq\epsilon_2 & \mathrm{t}\in\mathcal{Q}_1:=[0,\mathfrak{T}]. \end{cases} \tag{21}$$

Here, $\varepsilon_1,\varepsilon_2$ are provided positive real numbers.

Definition 4. *[32] The problem (2) is UH-stable if $\exists\,\Omega_i>0$, $i=1,2$, such that, given $\varepsilon_1,\varepsilon_2>0$, and for each solution $(\mathcal{O},\Xi)\in\mathbb{E}([0,\mathfrak{T}]\times\mathcal{R}_e^2,\mathcal{R}_e)$ of the inequality (21), $\exists\,(\mathcal{O}^*,\Xi^*)\in\mathbb{E}([0,\mathfrak{T}]\times\mathcal{R}_e^2,\mathcal{R}_e)$ of system 2 with*

$$\begin{cases} |\mathcal{O}(\mathrm{t})-\mathcal{O}^*(\mathrm{t})|\leq\Omega_1\varepsilon_1+\Omega_2\varepsilon_2, & \mathrm{t}\in[0,\mathfrak{T}], \\ |\Xi(\mathrm{t})-\Xi^*(\mathrm{t})|\leq\Omega_1\varepsilon_1+\Omega_2\varepsilon_2, & \mathrm{t}\in[0,\mathfrak{T}]. \end{cases} \tag{22}$$

Remark 3. *$(\mathcal{O},\Xi)$ is a solution of inequality (21) if $\exists$ the functions $\mathcal{A}_i\in([0,\mathfrak{T}],\mathcal{R}_e)$, $i=1,2$, which depend upon $(\mathcal{O},\Xi)$, respectively, such that*

$$|\mathcal{A}_1(\mathrm{t})|\leq\varepsilon_1,\qquad|\mathcal{A}_2(\mathrm{t})|\leq\varepsilon_2,\qquad\mathrm{t}\in[0,\mathfrak{T}]. \tag{23}$$

$$\begin{cases} |({}^{\mathcal{C}}\mathcal{D}^{\alpha_1}+\mathcal{K}_1{}^{\mathcal{C}}\mathcal{D}^{\alpha_1-1})\mathcal{O}(\mathrm{t})=\varrho_1(\mathrm{t},\mathcal{O}(\mathrm{t}),\Xi(\mathrm{t}))|+\mathcal{A}_1(\mathrm{t}) & \mathrm{t}\in\mathcal{Q}_1:=[0,\mathfrak{T}], \\ |({}^{\mathcal{C}}\mathcal{D}^{\alpha_2}+\mathcal{K}_1{}^{\mathcal{C}}\mathcal{D}^{\alpha_2-1})\Xi(\mathrm{t})=\mathcal{J}_1(\mathrm{t},\mathcal{O}(\mathrm{t}),\Xi(\mathrm{t}))|+\mathcal{A}_2(\mathrm{t}) & \mathrm{t}\in\mathcal{Q}_1:=[0,\mathfrak{T}]. \end{cases} \tag{24}$$

Remark 4. *If $(\mathcal{O},\Xi)$ represent a solution of inequality (21), then $(\mathcal{O},\Xi)$ is a solution of following inequality*

$$\begin{cases} |\mathcal{O}(\mathrm{t})-\mathcal{O}^*(\mathrm{t})|\leq\Omega_1\varepsilon_1+\Omega_2\varepsilon_2, & \mathrm{t}\in[0,\mathfrak{T}], \\ |\Xi(\mathrm{t})-\Xi^*(\mathrm{t})|\leq\Omega_1\varepsilon_1+\Omega_2\varepsilon_2, & \mathrm{t}\in[0,\mathfrak{T}], \end{cases}$$

for all $(\mathcal{O},\Xi)\in\mathbb{E}([0,\mathfrak{T}],\mathcal{R}_e)$ of inequality.

Theorem 3. *Assume that $(\mathcal{W}_2)$ holds. Then, (2) is UH-stable.*

Proof. With the assistance of Definition 4 and Remark 3, we confirmed Remark 4 as demonstrated in the following lines.

$$\begin{cases} ({}^{\mathcal{C}}\mathcal{D}^{\alpha_1} + \mathcal{K}_1 {}^{\mathcal{C}}\mathcal{D}^{\alpha_1 - 1})\mathcal{O}(\mathfrak{t}) = \varrho_1(\mathfrak{t}, \mathcal{O}(\mathfrak{t}), \Xi(\mathfrak{t})) + \mathcal{A}_1(\mathfrak{t}) & \mathfrak{t} \in \mathcal{Q}_1 := [0, \mathfrak{T}], \\ ({}^{\mathcal{C}}\mathcal{D}^{\alpha_2} + \mathcal{K}_1 {}^{\mathcal{C}}\mathcal{D}^{\alpha_2 - 1})\Xi(\mathfrak{t}) = \mathcal{J}_1(\mathfrak{t}, \mathcal{O}(\mathfrak{t}), \Xi(\mathfrak{t})) + \mathcal{A}_2(\mathfrak{t}) & \mathfrak{t} \in \mathcal{Q}_1 := [0, \mathfrak{T}]. \end{cases}$$

Implying

$$\begin{aligned} \mathcal{O}(\mathfrak{t}) = \mathcal{O}^*(\mathfrak{t}) + \Bigg| \frac{e^{-\mathcal{K}_1 \mathfrak{t}}}{2} \Bigg[\Bigg\{ \frac{1}{\Delta_1} \Bigg(& - \int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T} - \varpi)} \Bigg(\int_0^{\varpi} \frac{(\varpi - \mathfrak{u})^{\alpha_1 - 2}}{\Gamma(\alpha_1 - 1)} \varrho_1(\mathfrak{u}, \mathcal{O}(\mathfrak{u}), \Xi(\mathfrak{u})) d\mathfrak{u} \Bigg) d\varpi \\ & + \int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T} - \varpi)} \Bigg(\int_0^{\varpi} \frac{(\varpi - \mathfrak{u})^{\alpha_2 - 2}}{\Gamma(\alpha_2 - 1)} \mathcal{J}_1(\mathfrak{u}, \mathcal{O}(\mathfrak{u}), \Xi(\mathfrak{u})) d\mathfrak{u} \Bigg) d\varpi \Bigg) \\ & + \frac{1}{\Delta_2} \Bigg(\mathcal{P}_1 - \beta \frac{\rho^{1-\mathfrak{q}}}{\Gamma(\mathfrak{q})} \int_0^{\zeta} \frac{\varpi^{\rho - 1}}{(\zeta^\rho - \varpi^\rho)} \\ & \times \Bigg(\int_0^{\varpi} e^{-\mathcal{K}_1(\varpi - \mathfrak{u})} \Bigg(\int_0^{\mathfrak{u}} \frac{(\mathfrak{u} - \varphi)^{\alpha_1 - 2}}{\Gamma(\alpha_1 - 1)} \varrho_1(\varphi, \mathcal{O}(\varphi), \Xi(\varphi)) d\varphi \Bigg) d\mathfrak{u} \Bigg) d\varpi \\ & + \beta \frac{\rho^{1-\mathfrak{q}}}{\Gamma(\mathfrak{q})} \int_0^{\zeta} \frac{\varpi^{\rho - 1}}{(\zeta^\rho - \varpi^\rho)} \Bigg(\int_0^{\varpi} e^{-\mathcal{K}_1(\varpi - \mathfrak{u})} \Bigg(\int_0^{\mathfrak{u}} \frac{(\mathfrak{u} - \varphi)^{\alpha_2 - 2}}{\Gamma(\alpha_2 - 1)} \mathcal{J}_1(\varphi, \mathcal{O}(\varphi), \Xi(\varphi)) d\varphi \Bigg) d\mathfrak{u} \Bigg) d\varpi \Bigg) \Bigg\} \Bigg] \\ & + \int_0^{\mathfrak{t}} e^{-\mathcal{K}_1(\mathfrak{t} - \varpi)} \Bigg(\int_0^{\varpi} \frac{(\varpi - \mathfrak{u})^{\alpha_1 - 2}}{\Gamma(\alpha_1 - 1)} \varrho_1(\mathfrak{u}, \mathcal{O}(\mathfrak{u}), \Xi(\mathfrak{u})) d\mathfrak{u} \Bigg) d\varpi \Bigg|. \end{aligned}$$

It follows that

$$\begin{aligned} |\mathcal{O}(\mathfrak{t}) - \mathcal{O}^*(\mathfrak{t})| \leq \frac{e^{-\mathcal{K}_1 \mathfrak{t}}}{2} \Bigg[\Bigg\{ \frac{1}{\Delta_1} \Bigg(& - \int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T} - \varpi)} \Bigg(\int_0^{\varpi} \frac{(\varpi - \mathfrak{u})^{\alpha_1 - 2}}{\Gamma(\alpha_1 - 1)} |\mathcal{A}_1(\mathfrak{u})| d\mathfrak{u} \Bigg) d\varpi \\ & - \int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T} - \varpi)} \Bigg(\int_0^{\varpi} \frac{(\varpi - \mathfrak{u})^{\alpha_2 - 2}}{\Gamma(\alpha_2 - 1)} |\mathcal{A}_2(\mathfrak{u})| d\mathfrak{u} \Bigg) d\varpi \Bigg) \\ & + \frac{1}{\Delta_2} \Bigg(\mathcal{P}_1 - \beta \frac{\rho^{1-\mathfrak{q}}}{\Gamma(\mathfrak{q})} \int_0^{\zeta} \frac{\varpi^{\rho - 1}}{(\zeta^\rho - \varpi^\rho)} \\ & \times \Bigg(\int_0^{\varpi} e^{-\mathcal{K}_1(\varpi - \mathfrak{u})} \Bigg(\int_0^{\mathfrak{u}} \frac{(\mathfrak{u} - \varphi)^{\alpha_1 - 2}}{\Gamma(\alpha_1 - 1)} |\mathcal{A}_1(\varphi)| d\varphi \Bigg) d\mathfrak{u} \Bigg) d\varpi \\ & + \beta \frac{\rho^{1-\mathfrak{q}}}{\Gamma(\mathfrak{q})} \int_0^{\zeta} \frac{\varpi^{\rho - 1}}{(\zeta^\rho - \varpi^\rho)} \Bigg(\int_0^{\varpi} e^{-\mathcal{K}_1(\varpi - \mathfrak{u})} \Bigg(\int_0^{\mathfrak{u}} \frac{(\mathfrak{u} - \varphi)^{\alpha_2 - 2}}{\Gamma(\alpha_2 - 1)} |\mathcal{A}_1(\varphi)| d\varphi \Bigg) d\mathfrak{u} \Bigg) d\varpi \Bigg) \Bigg\} \Bigg] \\ & + \int_0^{\mathfrak{t}} e^{-\mathcal{K}_1(\mathfrak{t} - \varpi)} \Bigg(\int_0^{\varpi} \frac{(\varpi - \mathfrak{u})^{\alpha_1 - 2}}{\Gamma(\alpha_1 - 1)} |\mathcal{A}_1(\mathfrak{u})| d\mathfrak{u} \Bigg) d\varpi, \\ \leq & \epsilon_1 \Bigg[\frac{e^{-\mathcal{K}_1 \mathfrak{t}}}{2} \Bigg[\frac{1}{\Delta_1} \Bigg\{ \frac{\mathfrak{T}^{\alpha_1 - 1}}{\mathcal{K}_1 \Gamma(\alpha_1)} (1 - e^{\mathcal{K}_1 \varpi}) \Bigg\} \\ & + \frac{1}{\Delta_2} \Bigg\{ \frac{\beta}{\mathcal{K}_1^2 \Gamma(\alpha_1)} \frac{\zeta^{\alpha_1 - 1 + \rho\mathfrak{q}}}{\rho^{\mathfrak{q}}} \frac{\Gamma\Big(\frac{(\alpha_1 - 1 + \rho)}{\rho}\Big)}{\Gamma\Big(\frac{\alpha_1 - 1 + \rho\mathfrak{q} + \rho}{\rho}\Big)} (\zeta \mathcal{K}_1 + e^{-\mathcal{K}_1 \zeta} - 1) \Bigg\} \Bigg] \Bigg] \\ & + \epsilon_2 \Bigg[\frac{e^{-\mathcal{K}_1 \mathfrak{t}}}{2} \Bigg[\frac{1}{\Delta_1} \Bigg\{ \frac{\mathfrak{T}^{\alpha_2 - 1}}{\mathcal{K}_1 \Gamma(\alpha_2)} (1 - e^{\mathcal{K}_1 \varpi}) \Bigg\} \\ & + \frac{1}{\Delta_2} \Bigg\{ \frac{\beta}{\mathcal{K}_1^2 \Gamma(\alpha_2)} \frac{\zeta^{\alpha_2 - 1 + \rho\mathfrak{q}}}{\rho^{\mathfrak{q}}} \frac{\Gamma\Big(\frac{(\alpha_2 - 1 + \rho)}{\rho}\Big)}{\Gamma\Big(\frac{\alpha_2 - 1 + \rho\mathfrak{q} + \rho}{\rho}\Big)} (\zeta \mathcal{K}_1 + e^{-\mathcal{K}_1 \zeta} - 1) \Bigg\} \Bigg] \Bigg] \end{aligned}$$

$$\leq \Omega_1 \varepsilon_1 + \Omega_2 \varepsilon_2. \tag{25}$$

Similarly,

$$
\begin{aligned}
&|\Xi(\zeta) - \Xi^*(\zeta)| \qquad (26)\\
\leq &\frac{e^{-\mathcal{K}_1 t}}{2}\Bigg[\Bigg\{\frac{1}{\Delta_1}\Bigg(-\int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T}-\varpi)}\Bigg(\int_0^{\varpi}\frac{(\varpi-u)^{\alpha_1-2}}{\Gamma(\alpha_1-1)}|\mathcal{A}_1(u)|du\Bigg)d\varpi\\
&-\int_0^{\mathfrak{T}} e^{-\mathcal{K}_1(\mathfrak{T}-\varpi)}\Bigg(\int_0^{\varpi}\frac{(\varpi-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)}|\mathcal{A}_2(u)|du\Bigg)d\varpi\Bigg)\\
&+\frac{1}{\Delta_2}\Bigg(\mathcal{P}_1 - \beta\frac{\rho^{1-q}}{\Gamma(q)}\int_0^{\zeta}\frac{\varpi^{\rho-1}}{(\zeta^{\rho}-\varpi^{\rho})}\\
&\times\Bigg(\int_0^{\varpi} e^{-\mathcal{K}_1(\varpi-u)}\Bigg(\int_0^{u}\frac{(u-\varphi)^{\alpha_1-2}}{\Gamma(\alpha_1-1)}|\mathcal{A}_1(\varphi)|d\varphi\Bigg)du\Bigg)d\varpi\\
&+\beta\frac{\rho^{1-q}}{\Gamma(q)}\int_0^{\zeta}\frac{\varpi^{\rho-1}}{(\zeta^{\rho}-\varpi^{\rho})}\Bigg(\int_0^{\varpi} e^{-\mathcal{K}_1(\varpi-u)}\Bigg(\int_0^{u}\frac{(u-\varphi)^{\alpha_2-2}}{\Gamma(\alpha_2-1)}|\mathcal{A}_1(\varphi)|d\varphi\Bigg)du\Bigg)d\varpi\Bigg)\Bigg\}\Bigg]\\
&+\int_0^{t} e^{-\mathcal{K}_1(t-\varpi)}\Bigg(\int_0^{\varpi}\frac{(\varpi-u)^{\alpha_2-2}}{\Gamma(\alpha_2-1)}|\mathcal{A}_2(u)|du\Bigg)d\varpi,\\
\leq &\Omega_1\varepsilon_1 + \Omega_2\varepsilon_2, \qquad (27)
\end{aligned}
$$

where Ω_1 and Ω_2 are defined in (15)–(20), respectively. Hence, problem (2) is UH-stable. □

5. Examples

In this section, we provide numerical examples to support the aforementioned analysis.

Example 1. *Consider the following system*

$$
\begin{cases}
({}^{\mathcal{C}}\mathcal{D}^{\alpha_1} + \mathcal{K}_1 {}^{\mathcal{C}}\mathcal{D}^{\alpha_1-1})\mathcal{O}(t) = \varrho_1(t,\mathcal{O}(t),\Xi(t)), t \in \mathcal{Q}_1 := [0,\mathfrak{T}],\\
({}^{\mathcal{C}}\mathcal{D}^{\alpha_2} + \mathcal{K}_1 {}^{\mathcal{C}}\mathcal{D}^{\alpha_2-1})\Xi(t) = \mathcal{J}_1(t,\mathcal{O}(t),\Xi(t)), t \in \mathcal{Q}_1 := [0,\mathfrak{T}],\\
(\mathcal{O}+\Xi)(0) = -(\mathcal{O}+\Xi)(\mathfrak{T}),\\
\beta\frac{\rho^{1-q}}{\Gamma(q)}\int_0^{\zeta}\frac{\varpi^{\rho-1}}{(\zeta^{\rho}-\varpi^{\rho})}(\mathcal{O}-\Xi)(\varpi)d\varpi = \mathcal{P}_1,
\end{cases} \tag{28}
$$

where $\alpha_1 = 3/2, \alpha_2 = 4/3, \rho = 2/3, \zeta = 3/4, \mathfrak{T} = 2, \mathcal{P}_1 = 1, \beta = 1/2, q = 3/2$. Utilizing the above data, we obtain $\Omega_1 = 0.26746$ $\Omega_2 = 0.18564$, where Ω_1 and Ω_2 are provided by (15) and (16). To illustrate Theorem 1, we use

$$\varrho_1(t,\mathcal{O}(t),\Xi(t)) = \frac{e^{-t}}{2\sqrt{900+t2}}(\mathcal{O}t + \sin\Xi + \cos t), \tag{29}$$

$$\mathcal{J}_1(t,\mathcal{O}(t),\Xi(t)) = \frac{1}{(3+t)^2}\left(\sin\mathcal{O} + \frac{\Xi}{2} + e^{-t}\right). \tag{30}$$

Next, ϱ_1 and $\mathcal{J}_1$ are continuous and accomplish hypothesis $(\mathcal{W}_1)$ with

$$\Psi_1(t) = \frac{e^{-t}\cos t}{2\sqrt{900+t^2}}, \Psi_2(t) = \frac{te^{-t}}{2\sqrt{900+t^2}}, \Psi_3(t) = \frac{e^{-t}}{2\sqrt{900+t^2}}, \tag{31}$$

$$k_1 = \frac{1}{(3+t)^2}, k_2 = \frac{e^{-t}}{(3+t)^2}, \text{ and } k_3 = \frac{1}{2(3+t)^2}. \tag{32}$$

Also,

$$||\Psi_2||\left(2\Omega_1+\frac{t^{\alpha_1-1}}{\mathcal{K}_1\Gamma(\alpha_1)}(1-e^{-\mathcal{K}_1 t})\right)+||k_2||\left(2\Omega_2+\frac{t^{\alpha_2-1}}{\mathcal{K}_1\Gamma(\alpha_2)}(1-e^{-\mathcal{K}_1 t})\right)\approx 0.067854 \tag{33}$$

and

$$||\Psi_3||\left(2\Omega_1+\frac{t^{\alpha_1-1}}{\mathcal{K}_1\Gamma(\alpha_1)}(1-e^{-\mathcal{K}_1 t})\right)+||k_3||\left(2\Omega_2+\frac{t^{\alpha_2-1}}{\mathcal{K}_1\Gamma(\alpha_2)}(1-e^{-\mathcal{K}_1 t})\right)\approx 0.034436. \tag{34}$$

Thus, by Theorem 1, ∃ a solution is achieved to problem (28) on $[0,2]$.

Example 2. *For the application of Theorem 2, we take into account*

$$\begin{aligned}\varrho_1(t,\mathcal{O}(t),\Xi(t)) &= \frac{1}{40(1+t^2)}\left(\frac{|\mathcal{O}|}{1+|\mathcal{O}|}+\tan^{-1}\Xi\right),\\ \mathcal{J}_1(t,\mathcal{O}(t),\Xi(t)) &= \frac{1}{\sqrt{900+t^2}}\left(\sin\Xi+2\tan^{-1}\mathcal{O}\right),\end{aligned} \tag{35}$$

here, ϱ_1 *and* $\mathcal{J}_1$ *are continuous and fulfil hypothesis* $(\mathcal{W}_2)$ *with* $\mathcal{S}_1=\mathcal{S}_2=1/40=\mathcal{S}$ *and* $\mathcal{Z}_1=1/15, \mathcal{Z}_2=1/30$, *and* $\mathcal{Z}=1/15$.

Further, we acquire

$$\left(\mathcal{S}\left(2\Omega_1+\frac{t^{\alpha_1-1}}{\mathcal{K}_1\Gamma(\alpha_1)}(1-e^{-\mathcal{K}_1 t})\right)+\mathcal{Z}\left(2\Omega_2+\frac{t^{\alpha_2-1}}{\mathcal{K}_1\Gamma(\alpha_2)}(1-e^{-\mathcal{K}_1 t})\right)\right)\approx 0.07457796<1. \tag{36}$$

Thus, all the conditions of Theorem 2 are satisfied.

Remark 5. *Our result improves the results in the theory of fractional-order systems and may encourage multidisciplinary research and provide ideas for novel avenues of investigation into a variety of subjects. Examining the use of fractional-order control in medical therapies and technologies, such as patient-specific therapy and medicine delivery systems, would be feasible with the help of medical specialists. Our techniques may be applied to tackle challenging control issues, perhaps resulting in advancements across several fields. Furthermore, by demonstrating the efficacy of fractional-order control techniques in chaotic systems, our work opens up new avenues for analysis and promotes further research into the application of fractional calculus in control theory.*

Remark 6. *Fundamental differences are presented between a traditional system and a fractional-order system by the ways in which they represent dynamic processes. In a conventional approach, the operations are represented by integrals and derivatives of integer order that are restricted to whole integers. This traditional non-integer model does not take memory effects or previous states into account, making it modeling for instantaneous behavior. For the non-integer values for differentiation and integration orders, allowing fractional orders offers a more flexible framework. The representation of complex behaviors requiring memory and nonlocal interactions is made feasible by this flexibility. Also, traditional models are well-understood in stimulating systems with integer-order behaviors, but they could have trouble exploring complex dynamics like viscoelasticity, anomalous diffusion, and characteristics like fractals [3,41]. In these kinds of situations, fractional-order systems perform well because they provide memory-driven responses and a broader variety of response patterns, such as power-law decays and growths. Therefore, the selection of these systems depends on the type of phenomenon under investigation; fractional-order systems provide higher levels of precision for systems exhibiting fractional-order dynamics, while traditional systems are still useful in situations exhibiting integer-order behaviors.*

6. Conclusions

In this paper, we establish the existence and uniqueness of solutions for coupled nonlinear Caputo sequential fractional differential equations with generalized Riemann–Liouville fractional integral boundary conditions. Our results are derived using Schaefer's fixed-point theorem and the Banach contraction mapping principle, which provide a rigorous framework for addressing these complex problems. Additionally, we conduct a UH stability analysis to assess the robustness of the solutions under perturbations, offering deeper insights into the stability characteristics of the system. These findings enhance our understanding of coupled fractional-order boundary value problems and offer new perspectives in this field. Numerical examples are provided to validate and illustrate the theoretical results, demonstrating the practical applicability of our approach. Our future research into various integral boundary conditions may be applied to coupled fractional differential equations, including the Hadamard, Caputo–Hadamard, and Hilfer types. Future investigations could benefit from employing Mönch's and Darbo's fixed-point theorems to explore the existence and uniqueness of solutions more extensively. Additionally, examining generalized Hyers–Ulam and Ulam–Hyers–Rassias stability criteria could provide a more comprehensive understanding of fractional differential equations. We also plan to extend this research by incorporating neural time delays into the system and exploring analogous results in that context.

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Abbreviations

The following abbreviations are used in this manuscript:

c.c	completely continuous
BVP	Boundary Value Problems
UH	Ulam–Hyers
FDEs	Fractional Differential Equations
SFD	Sequential Fractional Differential
CFDs	Caputo Fractional Derivatives
SFDEs	Sequential Fractional Differential Equations

References

1. Delavari, H.; Heydarinejad, H.; Baleanu, D. Adaptive fractional-order blood glucose regulator based on high-order sliding mode observer. *IET Syst. Biol.* **2019**, *13*, 43–54. [CrossRef]
2. Roohi, M.; Zhang, C.; Chen, Y. Adaptive model-free synchronization of different fractional-order neural networks with an application in cryptography. *Nonlinear Dyn.* **2020**, *100*, 3979–4001. [CrossRef]
3. Roohi, M.; Zhang, C.; Taheri, M.; Basse-O'Connor, A. Synchronization of Fractional-Order Delayed Neural Networks Using Dynamic-Free Adaptive Sliding Mode Control. *Fractal Fract.* **2023**, *7*, 682. [CrossRef]
4. Saad, K. Comparing the Caputo, Caputo-Fabrizio and Atangana-Baleanu derivative with fractional order: Fractional cubic isothermal auto-catalytic chemical system. *Eur. Phys. J. Plus* **2018**, *13*, 94. [CrossRef]
5. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006; Volume 204.
6. Shishkina, E.; Sitnik, S. *Transmutations, Singular and Fractional Differential Equations with Applications to Mathematical Physics*; Academic Press: Cambridge, MA, USA, 2020.
7. Miller, K.S.; Ross, B. *An Introduction to the Fractional Calculus and Fractional Differential Equations*; Wiley: New York, NY, USA, 1993.
8. Tarasov, V.E. *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media*; Springer Science and Business Media: Berlin/Heidelberg, Germany, 2011.
9. Ahmad, B.; Ntouyas, S.K.; Zhou, Y.; Alsaedi, A. A study of fractional differential equations and inclusions with nonlocal Erdélyi–Kober type integral boundary conditions. *Bull. Iran. Math. Soc.* **2018**, *44*, 1315–1328. [CrossRef]

10. Ahmad, B.; Alghanmi, M.; Ntouyas, S.K.; Alsaedi, A. Fractional differential equations involving generalized derivative with Stieltjes and fractional integral boundary conditions. *Appl. Math. Lett.* **2018**, *84*, 111–117. [CrossRef]
11. Sun, Q.; Meng, S.; Cui, Y. Existence results for fractional order differential equation with nonlocal Erdélyi–Kober and generalized Riemann–Liouville type integral boundary conditions at resonance. *Adv. Differ. Equations* **2018**, *2018*, 243. [CrossRef]
12. Agarwal, R.P.; Benchohra, M.; Hamani, S. A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions. *Acta Appl. Math.* **2010**, *109*, 973–1033. [CrossRef]
13. Katugampola, U.N. New approach to a generalized fractional integral. *Appl. Math. Comput.* **2011**, *218*, 860–865. [CrossRef]
14. Katugampola, U.N. A new approach to generalized fractional derivatives. *arXiv* **2011**, arXiv:1106.0965.
15. Łupińska, B.; Odzijewicz, T. A Lyapunov-type inequality with the Katugampola fractional derivative. *Math. Methods Appl. Sci.* 2018, *41*, 8985–8996. [CrossRef]
16. Zada, A.; Yar, M.; Li, T. Existence and stability analysis of the nonlinear sequential coupled system of Caputo fractional differential equations with integral boundary conditions. *Ann. Univ. Paedagog. Crac. Stud. Math.* **2018**, *17*, 103–125. [CrossRef]
17. Al-khateeb, A.; Zureigat, H.; Ala'yed, O.; Bawaneh, S. Ulam–Hyers Stability and Uniqueness for Nonlinear Sequential Fractional Differential Equations Involving Integral Boundary Conditions. *Fractal Fract.* **2021**, *5*, 235. [CrossRef]
18. Thakur, A.; Ali, J.; Rodríguez-López, R. Existence of Solutions to a Class of Nonlinear Arbitrary Order Differential Equations Subject to Integral Boundary Conditions. *Fractal Fract.* **2021**, *5*, 220. [CrossRef]
19. Ntouyas, S.K.; Al-Sulami, H.H. A study of coupled systems of mixed order fractional differential equations and inclusions with coupled integral fractional boundary conditions. *Adv. Differ. Equations* **2020**, *73*, 2020. [CrossRef]
20. Ahmad, B.; Alghanmi, M.; Alsaedi, A.; Nieto, J.J. Existence and uniqueness results for a nonlinear coupled system involving Caputo fractional derivatives with a new kind of coupled boundary conditions. *Appl. Math. Lett.* **2021**, *116*, 107018. [CrossRef]
21. Ahmad, B.; Alsaedi, A.; Alotaibi, F.M.; Alghanmi, M. Nonlinear coupled Liouville-Caputo Fractional differential equations with a new class of nonlocal boundary conditions. *Miskolc Math. Notes* **2023**, *24*, 31–46.. [CrossRef]
22. Fadhal, E.; Abuasbeh, K.; Manigandan, M.; Awadalla, M. Applicability of Mónch's Fixed Point Theorem on a System of (k, ψ)-Hilfer Type Fractional Differential Equations. *Symmetry* **2022**, *14*, 2572. [CrossRef]
23. Mohamed, H. Sequential fractional pantograph differential equations with nonlocal boundary conditions: Uniqueness and Ulam-Hyers-Rassias stability. *Results Nonlinear Anal.* **2022**, **5**, 29–41. [CrossRef]
24. Awadalla, M.; Manigandan, M. Existence results for Caputo tripled fractional differential inclusions with integral and multi-point boundary conditions. *Fractal Fract.* **2023**, *7*, 182. [CrossRef]
25. Murugesan, M.; Muthaiah, S.; Alzabut, J.; Gopal, T.N. Existence and HU stability of a tripled system of sequential fractional differential equations with multipoint boundary conditions. *Bound. Val. Prob.* **2023**, *2023*, 56. [CrossRef]
26. Wei, Z.; Li, Q.; Che, J. Initial value problems for fractional differential equations involving Riemann–Liouville sequential fractional derivative. *J. Math. Anal. Appl.* **2010**, *367*, 260–272. [CrossRef]
27. Wei, Z.; Dong, W. Periodic boundary value problems for Riemann-Liouville sequential fractional differential equations. *Electron. J. Qual. Theory Differ. Equations* **2011**, *2011*, 1–13. [CrossRef]
28. Subramanian, M.; Manigandan, M.; Tunç, C.; Gopal, T.; Alzabut, N.J. On the system of nonlinear coupled differential equations and inclusions involving Caputo-type sequential derivatives of fractional order. *J. Taibah Univ. Sci.* **2022**, *16*, 1–23. [CrossRef]
29. Manigandan, M.; Muthaiah, S.; Nandhagopal, T.; Vadivel, R.; Unyong, B.; Gunasekaran, N. Existence results for a coupled system of nonlinear differential equations and inclusions involving sequential derivatives of fractional order. *Aims Math.* **2022**, *7*, 723–755. [CrossRef]
30. Jiang, J.; Liu, L. Existence of solutions for a sequential fractional differential system with coupled boundary conditions. *Bound. Val. Prob.* **2016**, *1*, 159. . [CrossRef]
31. Saha, K.K.; Sukavanam, N.; Pan, S. Existence and uniqueness of solutions to fractional differential equations with fractional boundary conditions, *Alex. Eng. J.* **2023**, *72*, 147–155. [CrossRef]
32. Ulam, S.M. *A Collection of Mathematical Problems*; Interscience: New York, NY, USA, 1968.
33. Hyers, D.H. On the stability of the linear functional equation. *Proc. Natl. Acad. Sci. USA* **1941**, *27*, 222–224. [CrossRef]
34. Rassias, T.M. On the stability of linear mappings in Banach spaces. *Proc. Am. Math. Soc.* **1978**, *72*, 297–300. [CrossRef]
35. Rezapour, S.; Tellab, B.; Deressa, C.T.; Etemad, S.; Nonlaopon, K. HU-type stability and numerical solutions for a nonlinear model of the coupled systems of Navier BVPs via the generalized differential transform method. *Fractal Fract.* **2021**, *5*, 166. [CrossRef]
36. Murugesan, M.; Shanmugam, S.; Rhaima, M.; Ravi, R. Analysis of Non-Local Integro-Differential Equations with Hadamard Fractional Derivatives: Existence, Uniqueness, and Stability in the Context of RLC Models. *Fractal Fract.* **2024**, *8*, 409. [CrossRef]
37. Zada, A.; Alam, M.; Riaz, U. Analysis of q-fractional implicit boundary value problem having Stieltjes integral conditions. *Math. Methods Appl. Sci.* **2021**, *44*, 4381–4413. [CrossRef]
38. Deimling, K. *Nonlinear Functional Analysis*; Courier Corporation: Chelmsford, MA, USA, 2010.
39. Awadalla, M.; Subramanian, M.; Abuasbeh, k.; Manigandan, M. On the Generalized Liouville-Caputo Type Fractional Differential Equations Supplemented with Katugampola Integral Boundary Conditions. *Symmetry* **2022**, *14*, 2273. [CrossRef]

40. Smart, D.R. Fixed point theorems. In *Cambridge Tracts in Mathematics*; Cambridge University Press: London, UK; New York, NY, USA, 1974; Volume 66.
41. Roohi, M.; Mirzajani, S.; Basse-O'Connor, A. A no-chatter single-input finite-time PID sliding mode control technique for stabilization of a class of 4D chaotic fractional-order laser systems. *Mathematics* **2023**, *11*, 4463. [CrossRef]

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