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ORIGINAL RESEARCH

Study of Generalized Synchronization and Anti-synchronization Between Different Dimensional Fractional-Order Chaotic Systems with Uncertainties

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Abstract
In this article, generalized synchronization between different dimensional chaotic systems has been discussed. Some basic preliminaries and definitions related to generalized synchronization have also been explored. Since during synchronization presence of uncertainties produced huge irregularity so authors studied generalized synchronization between fractional-order chaotic systems with uncertainties. Further, generalized anti-synchronization between fractional-order chaotic systems with uncertainties also investigated. In order to show the effect of uncertainty and fractional-order on generalized synchronization; integer-order generalized synchronization between neural networks (Hopfield and cellular neural networks) without uncertainties also explained. Finally, numerical results agreed with theoretical results.

Keywords Fractional-order chaotic system · Generalized synchronization · Anti-synchronization · Uncertainties · Hopfield neural network · Cellular neural network

Introduction

Synchronization between chaotic systems has a broad spectrum in the branch of science, engineering and secure communication. It is more applicable due to comprising dynamic variables as well as static variables. Pecora and Carroll [1] first time analyzed the

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synchronization between chaotic systems. In the recent decades several types of synchronizations such as generalized synchronization [2], hybrid phase synchronization [3], multi-switching compound synchronization [4], difference synchronization [5] etc. have been examined. The concept of generalized synchronization has been explained via continuous differentiable function by Hunt et al. [6]. In 2007, Zhang et al. [7] discussed generalized synchronization between different dimensional chaotic systems. Abarbanel and Rulkov [8] also suggested a method to examine the generalized synchronization in chaotic systems experimentally. Further, Kocarev and Parlitz [2] established the connection between generalized synchronization and derived some conditions for coupled dynamical system. Hunt and Ott [6] also examined smooth function for the state of master and slave system. In addition, Yang and Duan [9] proposed a general method to apply generalized synchronization on chaotic system. Yang et al. [10] also presented a model to exemplify the switching schemes of chaotic system using generalized synchronization. Yang and Chua [11] studied a linear manifold scheme for generalized synchronization and achieved some necessary and sufficient condition for linear generalized synchronization. The relation between phase and generalized synchronization was investigated by Zheng and Hu [12] with assertion that generalized synchronization is stronger than phase synchronization. Further several researchers [13–19] studied generalized synchronization to explain communication strategy.

The application of fractional calculus has recently drawn the attention of several researchers in several fields. The various fractional-order differential systems including the Rossler system, the modified Duffing system and Chen system exhibit chaotic behaviour. Deng [20] discussed several procedure to obtain generalized synchronization between fractional-order systems and derived necessary conditions. Wu and Lu [21] applied Laplace theory-based technique to investigate generalized projective synchronization between fractional-order Chen systems. Wang et al. [22] studied the fractional-order hybrid synchronization and obtain a proper controller based on stability theorem. Wu et al. [23] also explored generalized synchronization for weighted fractional-order complex chaotic systems. Megherbi et al. [24] analyzed the impulsive synchronization and derived the sufficient conditions for fractional-order discrete chaotic system. Sayed and Radwan [25] also showed some generalization schemes for fractional-order chaotic systems based on secure communication of images.

In present time, neural network attracted the attention of several researchers. He et al. [26] constructed a new chaotic neural network with associative memory function and discussed pinning control method for the chaotic neural network. Further, they concluded that control strength of network is smaller at higher pinned density. He et al. [27] also discussed chaotic neural network with chaotic neurons and obtained the range of the threshold for control the neurons. Moreover. Wang and Li [28] investigated chaotic color image encryption algorithm for the Arnold mapping through a function transformation. Further, they generated the self-diffusion chaotic matrix for Hopfield chaotic neural network. Srivastava et al. [29] presented a novel algorithm to enhance the security of a hybrid model Hopfield neural network and shown that the security of transmitted data is better than traditional algorithms. Liu and Xiu [30] discussed model of single neuron with chaotic and hysteretic characteristics in viewpoint of optimization. Ichinose [31] developed a model of quasi-periodic chaotic neural networks and concluded that the chaotic domain can be identified by folding structure with invariant closed curve with restoration of images. Soleymanpour et al. [32] designed cellular manufacturing systems to minimize intercellular movements while maximizing utilization of machines. Zhou and Chen [33] also investigated a simple model of neuron from the viewpoint of optimization with property of temporal retrieval of stored patterns and developed chaotic annealing technique to search for the global minima. Zhao et al. [34] used Gauss wavelet to characterize local features for the chaotic neural network and explained the exponentially decaying dilation parameter enable the neural network to generate complex dynamics behavior.

The aforementioned conversations served as inspiration to authors for this research. To the best of the authors' knowledge this work has not been done before. The rest of the article is arranged as follows. The "Preliminaries and Basic Concepts", deals with the preliminaries, some definitions of fractional calculus and generalized synchronization. In "Study of Generalized Synchronization", generalized synchronization and generalized anti-synchronization have analyzed between different dimensional fractional-order chaotic systems and integer-order neural networks. "Numerical Results and Discussion" concerned the numerical results and discussion. A brief conclusion has incorporated in "Conclusion".

Preliminaries and Basic Concepts

Definition [35] The Caputo's fractional derivative is expressed as

$$
{}_{g}^{c}D_{t}^{\alpha}\eta(t) = \frac{d^{\alpha}\eta(t)}{dt^{\alpha}} = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_{g}^{t} \frac{\eta^{(m)}(\tau)d\tau}{(t-\tau)^{\alpha+1-m}}, & m-1 < \alpha < m \\ \frac{d^{m}\eta(t)}{dt^{m}}, & \alpha = m \end{cases}
$$

where $0 < \alpha \in R$, $m \in \mathrm{N}$, $\Gamma(\cdot)$ represents the Gamma function.

Some important properties and assumptions of fractional-order calculus in term of Caputo derivative are expressed as follow.

Property [36] *Consider a fractional-order system as*

$$
D_{t}^{\alpha}\eta = h(\eta), \tag{1}
$$

where $\eta \in R^{n}$ and $h(\eta)$ is continuous function and satisfies

$$
\|h(\eta) - h(\xi)\| < K\,\|\eta - \xi\|,
$$

where $\|\ .\ \|$ is a sup-norm and K is a positive constant.

Assumption Suppose the function $g(\eta)$ is bounded then there exist a constant $\zeta > 0$ satisfy following condition

$$
0 \le \frac{g(\eta) - g(\xi)}{\eta - \xi} \le \zeta, \quad \forall \eta,\ \xi \in R
$$

Here, the uncertain chaotic system is consider as master system which is expressed as

$$
\frac{d^{\alpha}\eta(t)}{dt^{\alpha}} = (\ell + \Delta\ell)\eta(t) + F(\eta(t)), \tag{2}
$$

and uncertain chaotic system is given as

$$\frac{d^\alpha \xi(t)}{dt^\alpha} = (m + \Delta m)\xi(t) + G(\xi(t)) + U(t), \tag{3}$$

where $\eta(t) = (\eta_1(t), \eta_2(t), ..., \eta_n(t))^T \in R^n$ *and* $\xi(t) = (\xi_1(t), \xi_2(t), ..., \xi_m(t))^T \in R^m$ are the state vectors. The ℓ and m are constant matrices of suitable order $\Delta\ell$ and Δm are uncertainties and satisfies $|\Delta\ell| \le \varepsilon_1$, $|\Delta m| \le \varepsilon_2$ and $\varepsilon_1, \varepsilon_2 > 0$. Further, F and G are nonlinear function and $U(t)$ represents control function.

The chaotic system (3) can be written as

$$\frac{d^\alpha \xi(t)}{dt^\alpha} = C\xi(t) + G(\xi), \tag{4}$$

where C is a constant matrix and $G(\xi)$ is the nonlinear term.

Definition 1 The generalized synchronization between systems given by (2) and (3) is achieved if $e(t) = \lim_{t \to \infty} \|\xi(t) - \chi(\eta(t))\| = 0$.

Theorem 1 *The generalized synchronization between system* (2) *and* (3) *will achieved if for* $\chi : R^n \to R^m$ *these exist* $U(\eta, \xi) \in R^m$ *such that*

$$U(\eta, \xi) = \rho(\chi(\eta) - \xi) - C\chi(\eta) - G(\xi) + D^\alpha \chi(\eta)((\ell + \Delta\ell)\eta(t) + F(\eta(t))) \tag{5}$$

where $\rho \in R^{m\times m}$ *is an unknown matrix to be determined and* $(C - \rho)^T + (C - \rho)$ *is negative definite matrix.*

Proof: In generalized anti-synchronization error is expressed as

$$\begin{aligned} \frac{d^\alpha e(t)}{dt^\alpha} &= \frac{d^\alpha \xi(t)}{dt^\alpha} - \frac{d^\alpha \chi(\eta)}{dt^\alpha} \\ &= C\xi(t) + G(\xi) + U(\eta, \xi) - D\chi(\eta)((\ell + \Delta\ell)\eta(t) + F(\eta(t))) \end{aligned} \tag{6}$$

where $D\chi(\eta)$ is known as Jacobian matrix.

Putting the value of $U(\eta, \xi)$ from Eq. (5) in (6), we get

$$\frac{d^\alpha e(t)}{dt^\alpha} = (C - \rho)e$$

The Lyapunov function V is defined as

$$V = e^T e$$

$$\begin{aligned} \frac{d^\alpha V}{dt^\alpha} &= \frac{d^\alpha e^T}{dt^\alpha} e + e^T \frac{d^\alpha e}{dt^\alpha}, \\ &= e^T \left[(C - \rho) + (C - \rho)^T \right] e < 0 \end{aligned}$$

If $(C - \rho)^T + (C - \rho)$ is negative definite matrix then the systems will generalize synchronized.

Definition 2 The generalized anti-synchronization between systems (2) and (3) will obtain if $\lim_{t\to\infty} \|\xi(t) + \chi(\eta(t))\| = 0$.

Theorem 2 *If for* $\chi : R^n \to R^m$, *these exist* $U(\eta,\xi) \in R^m$ *such that*

$$U(\eta,\xi) = \rho(\xi + \chi(\eta)) + C\chi(\eta) - G(\xi) - D\chi(\eta)[(\ell + \Delta\ell)\eta + F(\eta)] \tag{7}$$

then the generalized anti-synchronization has achieved. Where $\rho \in R^{m\times m}$ *is an unknown matrix to be determined and* $(C+\rho)^T + (C+\rho)$ *is negative definite matrix.*

Proof In generalized anti-synchronization error is expressed as

$$\begin{aligned}\frac{d^\alpha e(t)}{dt^\alpha} &= \frac{d^\alpha \xi(t)}{dt^\alpha} + \frac{d^\alpha \chi(\eta)}{dt^\alpha} \\ &= C\xi(t) + G(\xi) + U(\eta,\xi) + D\chi(\eta)[(\ell + \Delta\ell)\eta(t) + F(\eta(t))]\end{aligned} \tag{8}$$

Putting the value of $U(\eta,\xi)$ from Eq. (7) in (8), we get

$$\frac{d^\alpha e(t)}{dt^\alpha} = (C + \rho)e$$

Further, we construct Lyapunov function V which is defined as

$$V = e^T e$$

The derivative of V is expressed as

$$\begin{aligned}\frac{d^\alpha V}{dt^\alpha} &= \frac{d^\alpha e^T}{dt^\alpha} e + e^T \frac{d^\alpha e}{dt^\alpha}, \\ &= e^T\left[(C+\rho) + (C+\rho)^T\right] e < 0\end{aligned}$$

If $(C+\rho)^T + (C+\rho)$ is negative definite matrix then the system is generalize anti-synchronized.

Study of Generalized Synchronization

Generalized Synchronization Between Fractional-Order Uncertain Chaotic Systems

The fractional-order system [37] with uncertainties is expressed as

$$\begin{aligned}\frac{d^\alpha \eta_1(t)}{dt^\alpha} &= a(\eta_2 - \eta_1) + 0.05\,\eta_1 + k\eta_1\eta_3, \\ \frac{d^\alpha \eta_2(t)}{dt^\alpha} &= -\eta_1\eta_3 - c\eta_2 + 0.01\,\eta_2, \\ \frac{d^\alpha \eta_3(t)}{dt^\alpha} &= \eta_1\eta_2 - 0.03\,\eta_3 - b\end{aligned} \tag{9}$$

where a, b, c and k are constant parameters. The system (7) shows chaotic behavior for the value of parameters $a = 10$, $b = 100$, $c = 11.2, k = -0.2$.

The hyper-chaotic system [38] with uncertainties is given as

$$\begin{aligned}
\frac{d^\alpha \xi_1(t)}{dt^\alpha} &= \xi_4 + \alpha_1 (\xi_2 - \xi_1) + 0.02\xi_1, \\
\frac{d^\alpha \xi_2(t)}{dt^\alpha} &= \beta_1 \xi_1 + \xi_2 - \xi_1\xi_3 - 0.01\xi_2, \\
\frac{d^\alpha \xi_3(t)}{dt^\alpha} &= e^{\xi_1\xi_2} - \gamma_1\xi_3 + 0.01\xi_3, \\
\frac{d^\alpha \xi_4(t)}{dt^\alpha} &= \delta_1\xi_2\xi_3 - 0.03\xi_4
\end{aligned} \tag{10}$$

where α_1, β_1, γ_1 and δ_1 are parameters. The system (8) shows chaotic behavior for the value $\alpha_1 = 10$, $\beta_1 = 28$, $\gamma_1 = 8/3, \delta_1 = 0.01$.

Here, system (8) is considered as slave system and written as

$$\begin{aligned}
\frac{d^\alpha \xi_1(t)}{dt^\alpha} &= \xi_4 + \alpha_1 (\xi_2 - \xi_1) + 0.02\xi_1 + U_1(t), \\
\frac{d^\alpha \xi_2(t)}{dt^\alpha} &= \beta_1 \xi_1 + \xi_2 - \xi_1\xi_3 - 0.01\xi_2 + U_2(t), \\
\frac{d^\alpha \xi_3(t)}{dt^\alpha} &= e^{\xi_1\xi_2} - \gamma_1\xi_3 + 0.01\xi_3 + U_3(t), \\
\frac{d^\alpha \xi_4(t)}{dt^\alpha} &= \delta_1\xi_2\xi_3 - 0.03\xi_4 + U_4(t)
\end{aligned} \tag{11}$$

where $(U_1(t), U_2(t), U_3(t), U_4(t))^T$ is the controller. From above equation we obtain

$$C = \begin{bmatrix} -\alpha_1 + 0.02 & \alpha_1 & 0 & 1 \\ \beta_1 & 1 - 0.01 & 0 & 0 \\ 0 & 0 & -\gamma_1 + 0.01 & 0 \\ 0 & 0 & 0 & -0.03 \end{bmatrix}, \quad G(\xi) = \begin{bmatrix} 0 \\ -\xi_1\xi_3 \\ e^{\xi_1\xi_2} \\ \delta_1\xi_2\xi_3 \end{bmatrix} \tag{12}$$

Define differentiable map χ such that $\chi(\eta_1, \eta_2, \eta_3) = (\eta_1, \eta_2, \eta_3, \eta_1 + \eta_2)^T$. The Jacobian matrix of χ is given as

$$D\chi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Now choose a matrix ρ such type that $(C - \rho)^T + (C - \rho)$ is negative definite matrix

$$\rho = \begin{pmatrix} -9 & 10 & 0 & 1 \\ 28 & 2 & 0 & 0 \\ 0 & 0 & -5/3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In this manner, we obtain

$$C-\rho=\begin{pmatrix} -0.98 & 0 & 0 & 0 \\ 0 & -1.01 & 0 & 0 \\ 0 & 0 & -0.99 & 0 \\ 0 & 0 & 0 & -1.03 \end{pmatrix}$$

After putting the value of different parameters in (5), we get controller as

$$\begin{aligned} U_1 &= a(\eta_2-\eta_1)+0.96\eta_1+9\xi_1-10\xi_2-\xi_4+k\eta_1\eta_3\,, \\ U_2 &= -(c-0.01)\eta_1+2\eta_2-28.99\xi_1-2\xi_2-\eta_1\eta_3+\xi_2\xi_3, \\ U_3 &= 4.34\eta_3+\eta_1\eta_2-0.03\eta_3-e^{\xi_1\xi_2}-(5/3)\xi_3, \\ U_4 &= -(a-1.08)\eta_1-(c-1.04)\eta_2+(k-1)\eta_1\eta_3+a\xi_1-\delta_1\xi_2\xi_3-\xi_4 \end{aligned} \tag{13}$$

The graphical presentation of the generalized synchronization for chaotic systems with uncertainties is depicted through Fig. 4. The error dynamics $e_i(t)$ for master and slave system shows that the error becomes zero for all states.

Generalized Anti-synchronization Between Fractional-Order Uncertain Chaotic Systems

Here, we consider system (10) as master system and (9) as slave system. The system (9) with control function is written as

$$\begin{aligned} \frac{d^\alpha\xi_1(t)}{dt^\alpha} &= a(\xi_2-\xi_1)+0.05\,\xi_1+k\xi_1\xi_3+U_1(t), \\ \frac{d^\alpha\xi_2(t)}{dt^\alpha} &= -\xi_1\xi_3-c\xi_2+0.01\,\xi_2+U_2(t), \\ \frac{d^\alpha\xi_3(t)}{dt^\alpha} &= \xi_1\xi_2-0.03\,\xi_3-b+U_3(t) \end{aligned} \tag{14}$$

Here $(U_1(t),\ U_2(t),\ U_3(t))^T$ is the controller. From Eq. (14) we obtain C and $g(\xi)$ as

$$C=\begin{bmatrix} -a+0.05 & a & 0 \\ 0 & -c+0.01 & 0 \\ 0 & 0 & -0.03 \end{bmatrix},\quad G(\xi)=\begin{bmatrix} k\xi_1\xi_3 \\ -\xi_1\xi_3 \\ \xi_1\xi_2 \end{bmatrix}$$

Define another map χ such that $\chi(\eta_1, \eta_2, \eta_3, \eta_4)=(\eta_1, \eta_2, \eta_3+\eta_4)^T$ and Jacobian matrix $D\chi$ is written as

$$D\chi=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

We choose a new matrix ρ which satisfies $(C+\rho)+(C+\rho)^T$ and should be negative definite. Now, ρ is calculated as

$$\rho=\begin{bmatrix} 9 & -10 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

In this manner, we obtain $(C + \rho)$ as

$$C + \rho = \begin{bmatrix} -0.95 & 0 & 0 \\ 0 & -1.19 & 0 \\ 0 & 0 & -1.03 \end{bmatrix}$$

Finally, the controllers are obtained as

$$\begin{aligned} U_1 &= 9.03\eta_1 - 10\eta_2 - \eta_4 + 9\xi_1 - 10\xi_2 - k\xi_2\xi_3, \\ U_2 &= -\beta_1\eta_1 - 2.17\eta_2 + \eta_1\eta_3 + 10\xi_2 + \xi_1\xi_3, \\ U_3 &= -e^{\eta_1\eta_2} + 1.65\eta_3 - \eta_4 - \delta_1\eta_2\eta_3 - \xi_3 - \xi_1\xi_2 + b \end{aligned} \tag{15}$$

The graphical presentation of generalized anti-synchronization is shown in Fig. 5. Finally, the error dynamics $e_i(t)$ for master and slave system shows that the error becomes zero.

Generalized Synchronization Between Integer-Order Neural Networks

In this subsection, we discuss generalized synchronization between integer-order neural networks without uncertainties. Here, we take Hopfield neural network [39] as master system which is expressed as

$$\begin{aligned} \dot{\eta}_1(t) &= -\eta_1 + \tanh\left(\eta_1\right) - 3\tanh\left(\eta_3\right) + 0.5\tanh\left(\eta_2\right) - \tanh\left(\eta_4\right), \\ \dot{\eta}_2(t) &= -\eta_2 + 0.25\tanh\left(\eta_1\right) + 2\tanh\left(\eta_2\right) + 3\tanh\left(\eta_3\right), \\ \dot{\eta}_3(t) &= -\eta_3 - 3\tanh\left(\eta_2\right) + 3\tanh\left(\eta_1\right) + \tanh\left(\eta_3\right), \\ \dot{\eta}_4(t) &= -100\eta_4 + 170\tanh\left(\eta_4\right) + 100\tanh\left(\eta_1\right). \end{aligned} \tag{16}$$

This system represents chaotic behavior at I.C. (0.1, 0.1, 0.1, 0.1).

Further, cellular neural network [40] is supposed as slave system and expressed below

$$\begin{aligned} \dot{\xi}_1(t) &= -\xi_4 - \xi_3 + u_1(t), \\ \dot{\xi}_2(t) &= \xi_3 + 2\xi_2 + u_2(t), \\ \dot{\xi}_3(t) &= -14\xi_2 + 14\xi_1 + u_3(t). \\ \dot{\xi}_4(t) &= -100\xi_4 + 100\xi_1 + 100(|\xi_4 + 1| - |\xi_4 - 1|) + u_4(t), \\ \dot{\xi}_5(t) &= 18\xi_2 - \xi_5 + \xi_1 + u_5(t), \\ \dot{\xi}_6(t) &= -4\xi_6 + 4\xi_5 + 100\xi_2 + u_6(t). \end{aligned} \tag{17}$$

This system also represents chaotic behavior at I.C. $(0.1, 0.01, 0.015, 0.001, 0.02, 0.2)$.

Equation (17) can be expressed in form of $\dot{\xi} = C\xi + g(\xi)$, where

$$C = \begin{bmatrix} 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 14 & -14 & 0 & 0 & 0 & 0 \\ 100 & 0 & 0 & -100 & 0 & 0 \\ 1 & 18 & 0 & 0 & -1 & 0 \\ 0 & 100 & 0 & 0 & 4 & -4 \end{bmatrix}, \quad g(\xi) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 100(|\xi_4 + 1| - |\xi_4 - 1|) \\ 0 \\ 0 \end{bmatrix} \tag{18}$$

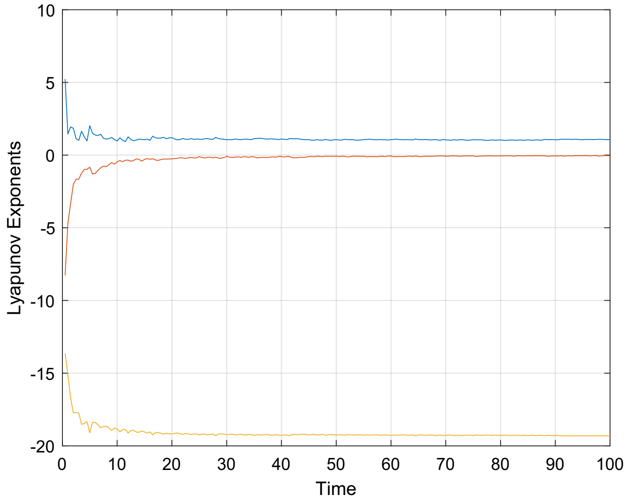

Fig. 1 Phase portraits of Lyapunov exponents of system (9)

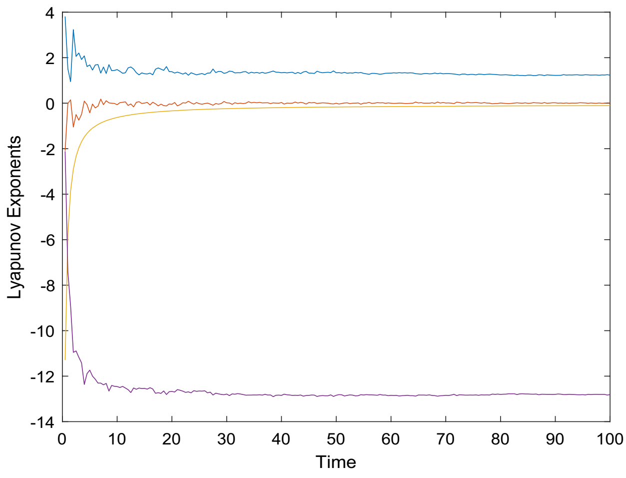

Fig. 2 Phase portraits of Lyapunov exponents of system (10)

Now define a map χ which is continuously differentiable and expressed as

$$\chi(\eta_1 , \eta_2 , \eta_3, \eta_4) = [\eta_1 , \eta_2 , \eta_3 , \eta_4 , \eta_2 + \eta_1, \eta_4 + \eta_3]^T .$$

The Jacobian matrix of above map is obtained as

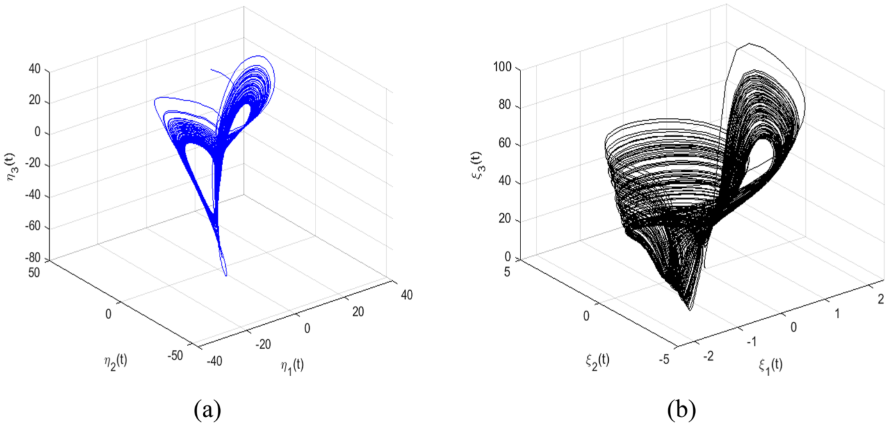

Fig. 3 Phase portraits of system (9) and (10) for $\alpha = 0.99$ in **a** $\eta_1 - \eta_2 - \eta_3$ space, **b** $\xi_1 - \xi_2 - \xi_3$ space respectively

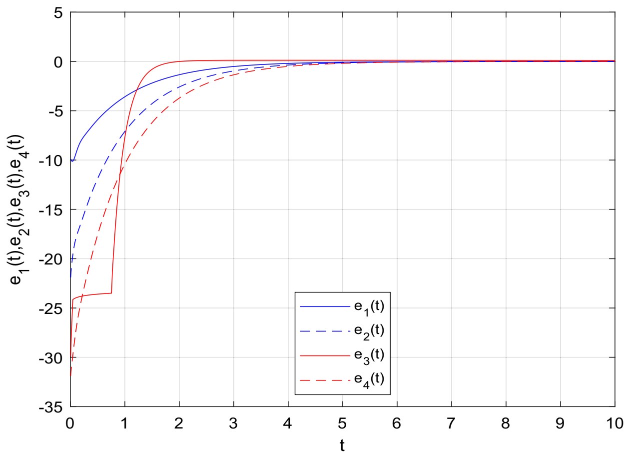

Fig. 4 The time evolution of state of errors for generalized synchronization

$$D\chi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \tag{19}$$

Now we assumed a new matrix ρ depending on (17) and expressed as

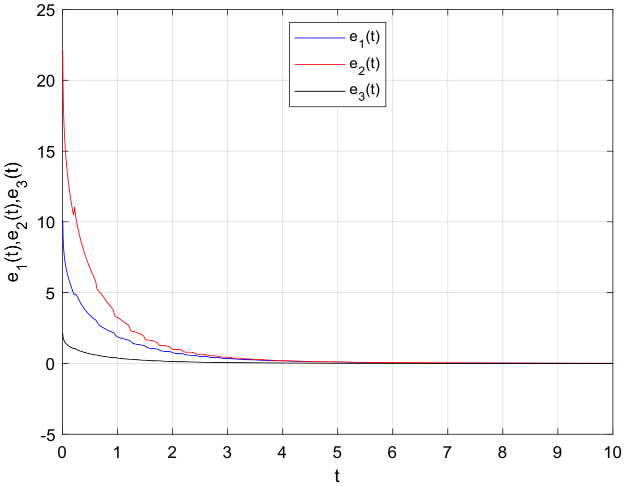

Fig. 5 The time evolution of error system for generalized anti-synchronization

$$\rho = \begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 12 & 1 & 0 & 0 & 0 \\ 14 & -14 & 6 & 0 & 0 & 0 \\ 100 & 0 & 0 & -96 & 0 & 0 \\ 1 & 18 & 0 & 0 & 3 & 0 \\ 0 & 100 & 0 & 0 & 4 & 9 \end{bmatrix} \tag{20}$$

In this manner, we obtain

$$C - \rho = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -10 & 0 & 0 & 0 & 0 \\ 0 & 0 & -6 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -5 \end{bmatrix} \tag{21}$$

Further, we get the control function as

$$\begin{aligned}
U_1 &= -\xi_1 + \xi_3 + \xi_4 - 3\tanh\left(\eta_3\right) + 0.5\tanh\left(\eta_2\right) - \tanh\left(\eta_4\right) + \tanh\left(\eta_1\right), \\
U_2 &= 9\eta_2 - 12\xi_2 - \xi_3 + 0.25\tanh\left(\eta_1\right) + 2\tanh\left(\eta_3\right) + 3\tanh\left(\eta_3\right), \\
U_3 &= 5\eta_3 - 14\xi_1 + 14\xi_2 - 6\xi_3 - 3\tanh\left(\eta_2\right) + 3\tanh\left(\eta_1\right) + \tanh\left(\eta_3\right), \\
U_4 &= -96\eta_4 - 100\xi_1 + 96\xi_4 - 100\left(|\xi_4 + 1| - |\xi_4 - 1|\right) + 100\tanh\left(\eta_1\right) + 170\tanh\left(\eta_4\right), \\
U_5 &= 3\eta_1 + 3\eta_2 - \xi_1 - 18\xi_2\tanh\left(\eta_1\right) - 3\tanh\left(\eta_3\right) + 0.5\tanh\left(\eta_2\right) - \tanh\left(\eta_4\right) + \\
&\quad 0.25\tanh\left(\eta_1\right) + 2\tanh\left(\eta_3\right) + 3\tanh\left(\eta_3\right), \\
U_6 &= 12\eta_3 - 87\eta_4 - 100\xi_2 - 4\xi_5 - 9\xi_6 + 3\tanh\left(\eta_1\right) + \tanh\left(\eta_3\right) - 3\tanh\left(\eta_2\right) + \\
&\quad 100\tanh\left(\eta_1\right) + 170\tanh\left(\eta_4\right),
\end{aligned} \tag{22}$$

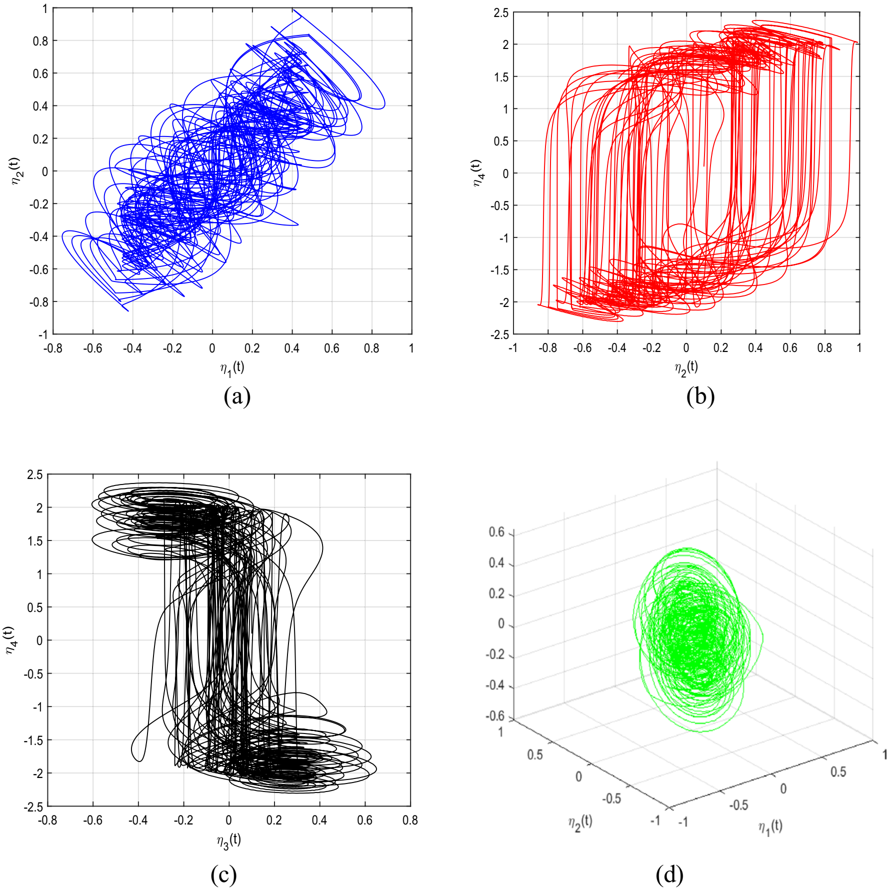

Fig. 6 Phase portraits of Hopfield neural network in **a** $\eta_1 - \eta_2$ plane, **b** $\eta_1 - \eta_3$ plane, **c** $\eta_2 - \eta_3$ plane, **d** $\eta_1 - \eta_2 - \eta_3$ space

Numerical Results and Discussion

In this section, the parametric values of system (9) and (10) are taken as $a = 10$, $b = 100$, $c = 11.2$, $k = -0.2$ and $\alpha_1 = 10$, $\beta_1 = 28$, $\gamma_1 = 8/3$, $\delta_1 = 0.01$ along with the initial conditions (10, 22, 30) and (0.1, 0.1, 0.1, 0.1) respectively. The phase portrait of Lyapunov exponents and systems for (9) and (10) have been revealed through Figs. 1, 2 and 3 respectively. The Figs. 4 and 5 reflects the error dynamics for generalized synchronization and anti-synchronization between different dimensional chaotic systems with uncertain terms respectively. The four dimensional Hopfield neural network and six dimensional cellular network have considered to explain the generalized synchronization. The initial conditions for Hopfield and cellular networks are (0.1, 0.1, 0.1, 0.1) and $(0.1, 0.01, 0.015, 0.001, 0.02, 0.2)$ respectively. The phase portraits of uncontrolled Hopfield and cellular neural networks are shown in Figs. 6 and 7. Finally, the error also converges to zero (see Fig. 8) in case of neural networks which implies that systems (16) and (17) achieve synchronization.

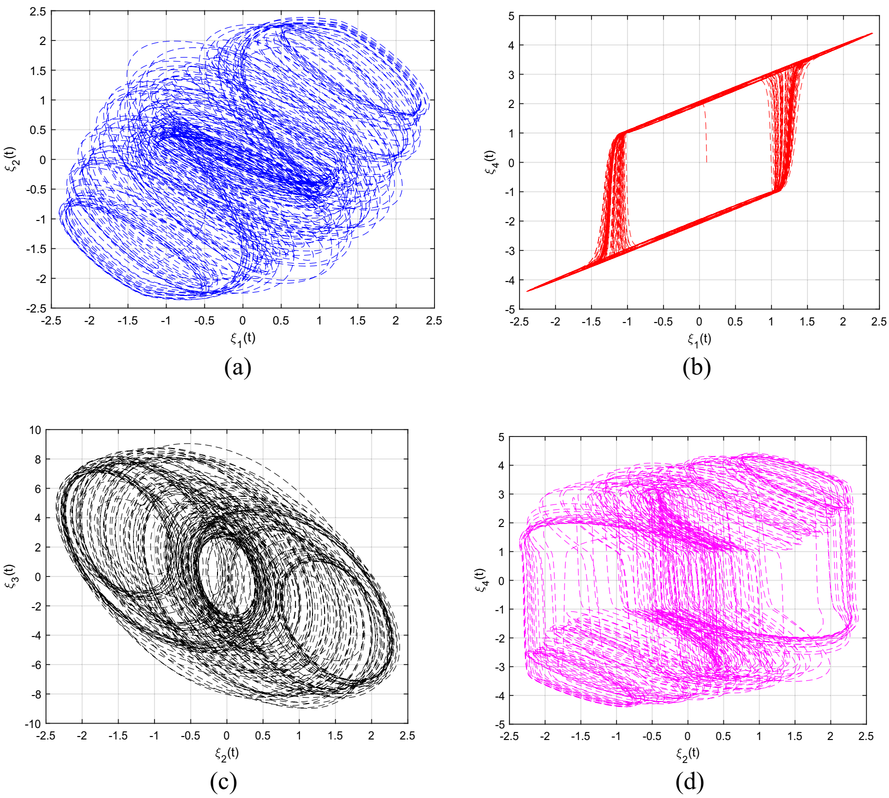

Fig. 7 Phase portraits of cellular neural network in **a** $\xi_1 - \xi_2$ plane, **b** $\xi_1 - \xi_4$ plane, **c** $\xi_2 - \xi_3$ plane, **d** $\xi_2 - \xi_4$ plane

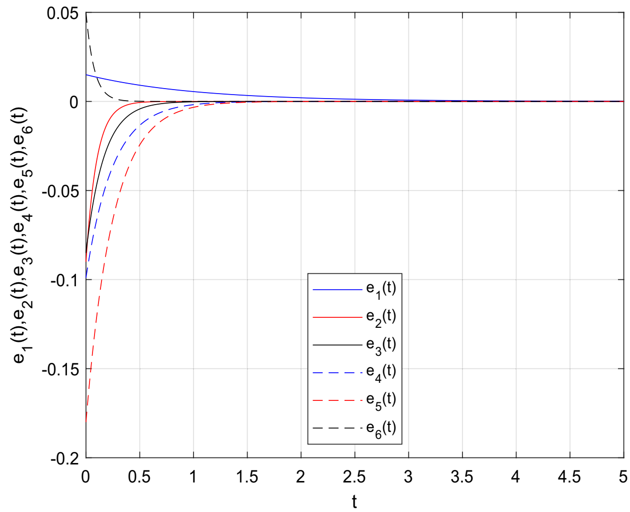

Fig. 8 Plot of error functions $e_1(t), e_2(t)$, $e_3(t), e_4(t),$ $e_5(t)$ *and* $e_6(t)$.

Conclusion

In the present study, the authors have successfully demonstrated the generalized synchronization and anti-synchronization between different dimensional fractional-order chaotic systems with uncertainties. The graphical presentation of the numerical outcomes verifies the authenticity of the proposed scheme. Further, the generalized synchronization between the integer-order Hopfield and cellular networks has also been analyzed. It is noteworthy that this scientific contribution will be more significant in the area of chaotic dynamical systems.

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Data availability Not applicable.

Declarations

Conflict of interest The authors declare no conflict of interest.

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