





Research Article

A New Generalized Class of Exponential Factor-Type Estimators for Population Distribution Function Using Two Auxiliary Variables

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Received 29 April 2022; Accepted 27 June 2022; Published 8 August 2022

Academic Editor: Tahir Mehmood

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In this article, we propose a generalized class of exponential factor type estimators for estimation of the finite population distribution function (PDF) using an auxiliary variable in the form of the mean and rank of the auxiliary variable exist. The expressions of the bias and mean square error of the estimators are computed up to the first order approximation. The proposed estimators provide minimum mean square error as compared to all other considered estimators. Three real data sets are used to check the performance of the proposed estimators. Moreover, simulation studies are also carried out to observe the performances of the proposed estimators. The proposed estimators confirmed their superiority numerically as well as theoretically by producing efficient results as compared to all other competing estimators.

1. Introduction

In survey sampling, it is well-known fact that the suitable use of auxiliary information improves the precision of estimators by taking advantage of the correlation between the study variable and the auxiliary variable. Several estimators exist in the literature for estimating various population parameters including mean, median, and total, but little attention has been paid to estimate the distribution function (DF). Some important references to the population mean using the auxiliary information include [1–21]. In their work, several authors have suggested improved ratio, product, and regression type estimators for estimating the finite population mean.

The problem of estimating a finite population (DF) arises when we are interested to find out the proportion of certain values that are less than or equal to the threshold value. There

are situations where estimating the cumulative distribution function is assessed as essential. For example, for a nutritionist, it is interesting to know the proportion of the population that consumed 25 percent or more of the calorie intake from saturated fat. Similarly, a soil scientist may be interested in knowing the proportion of people living in a developing country below the poverty line. In certain situations, the need of a cumulative distribution function is much more important. Some important work in the field of distribution function (DF) includes [22], which suggested an estimator for estimating the DF that requires information both on the study variable and the auxiliary variable. On similar lines, [23] proposed ratio and difference-type estimators for estimating the DF using the auxiliary information. Ahmad and Abu-Dayyeh [24] estimated the DF using the information on multiple auxiliary variables. Rueda et al. [25] used a calibration approach for estimating the

DF. Singh et al. [26] considered the problem of estimating the DF and quantiles with the use of auxiliary information at the estimation stage, [27] considered a generalized class of estimators for estimating the DF in the presence of nonresponse, [28] suggested finite population distribution function estimation with dual use of auxiliary information under simple and stratified random sampling. Moreover, two new estimators were proposed for estimating the DF in simple and stratified sampling using the auxiliary variable and rank of the auxiliary variable.

The rest of the article is composed as follows: in Section 2, some notations and symbols are given. In Section 3, the existing estimators for estimating the DF are given. In Section 4, we define two new generalized exponential factor type estimators. Section 6 discusses the numerical study of the proposed class of estimators. We also conduct a simulation study for the support of our proposed generalized family of estimators in Section 7. Section 8 gives the concluding remarks.

2. Notations and Symbols

A finite population $\Omega = \{1, 2, \dots, N\}$ of N distinct and identified units is considered. To estimate the finite population distribution function (DF), a sample of size n units is drawn from a population using simple random sampling without replacement (SRSWOR). Let Y_i , X_i and R_x be the values of the study variable Y , the auxiliary variable X and R_x rank of the auxiliary variable X , respectively. Let $I(Y \leq t_y)$ and $I(X \leq t_x)$ be the indicator variables based on Y and X , respectively.

Let $\hat{F}(t_y) = \sum_{i=1}^n I(Y_i \leq t_y)/n$ and $\hat{F}(t_x) = \sum_{i=1}^n I(X_i \leq t_x)/n$ be the sampled distribution functions corresponding to the population distribution function $F(t_y) = \sum_{i=1}^N I(Y_i \leq t_y)/N$ and $F(t_x) = \sum_{i=1}^N I(X_i \leq t_x)/N$, respectively.

Let \hat{Y} , \hat{X} , and \hat{R}_x be the sample means corresponding to population means \bar{Y} , \bar{X} , and \bar{R}_x , respectively. Let $\delta_1^2 = \sum_{i=1}^N (I(Y_i \leq t_y) - F(t_y))^2 / (N - 1)$, $\delta_2^2 = \sum_{i=1}^N (I(Y_i \leq t_x) - F(t_x))^2 / (N - 1)$, $\delta_3^2 = \sum_{i=1}^N (X_i - \bar{X})^2 / (N - 1)$, $\delta_4^2 = \sum_{i=1}^N (R_x - \bar{R}_x)^2 / (N - 1)$ be the population variances of $I(Y \leq t_y)$, $I(X \leq t_x)$, X and R_x , respectively. Let $C_1 = \delta_1 / F(t_y)$, $C_2 = \delta_2 / F(t_x)$, $C_3 = \delta_3 / \bar{X}$, $C_4 = \delta_4 / \bar{R}_x$ be the coefficients of variations of $I(Y \leq t_y)$, $I(X \leq t_x)$, X , and R_x , respectively. Let $\delta_{12} = \sum_{i=1}^N \{(I(Y_i \leq t_y) - F(t_y))(I(X_i \leq t_x) - F(t_x))\} / (N - 1)$, $\delta_{13} = \sum_{i=1}^N \{(I(Y_i \leq t_y) - F(t_y))(I(X_i - \bar{X}))\} / (N - 1)$,

$$\delta_{23} = \sum_{i=1}^N \frac{\{(I(Y_i \leq t_x) - F(t_x))(X_i - \bar{X})\}}{(N - 1)}, \quad (1)$$

$$\delta_{14} = \sum_{i=1}^N \frac{\{(I(Y_i \leq t_y) - F(t_y))(R_{x_i} - \bar{R}_x)\}}{(N - 1)}. \quad (2)$$

$\delta_{24} = \sum_{i=1}^N \{(I(Y_i \leq t_x) - F(t_x))(R_{x_i} - \bar{R}_x)\} / (N - 1)$ be the population covariances between $I(Y \leq t_y)$, $I(X \leq t_x)$, $I(Y \leq t_y)$, $I(X \leq t_x)$ and R_x , respectively. Let $\rho_{12} = \delta_{12} / (\delta_1 \delta_2)$, $\rho_{13} = \delta_{13} / (\delta_1 \delta_3)$, $\rho_{23} = \delta_{23} / (\delta_2 \delta_3)$, $\rho_{14} = \delta_{14} / (\delta_1 \delta_4)$,

$\rho_{24} = \delta_{24} / (\delta_2 \delta_4)$, be the correlation coefficients between $I(Y \leq t_y)$ and $I(X \leq t_x)$, $I(Y \leq t_y)$ and X , $I(X \leq t_x)$ and X , $I(Y \leq t_y)$ and R_x , $I(X \leq t_x)$ and R_x , respectively. Let $\rho_{1.23}^2 = (\rho_{12}^2 + \rho_{13}^2 - 2\rho_{12}\rho_{13}\rho_{23}) / (1 - \rho_{23}^2)$, $\rho_{1.24}^2 = (\rho_{12}^2 + \rho_{14}^2 - 2\rho_{12}\rho_{14}\rho_{24}) / (1 - \rho_{24}^2)$ be the population coefficients of multiple determination of $(Y \leq t_y)$ on $I(X \leq t_x)$ and X , $I(Y \leq t_y)$ on $I(X \leq t_x)$ and R_x , respectively.

To obtain the biases and mean squared errors (MSEs) of the adapted and proposed estimators of $F(t_y)$, we consider the following relative error terms. Let

$$\begin{aligned} \xi_0 &= \frac{\hat{F}(t_y) - F(t_y)}{F(t_y)}, \\ \xi_1 &= \frac{\hat{F}(t_x) - F(t_x)}{F(t_x)}, \\ \xi_2 &= \frac{\hat{X} - \bar{X}}{\bar{X}}, \\ \xi_3 &= \frac{\hat{R}_x - \bar{R}_x}{\bar{R}_x}. \end{aligned} \quad (3)$$

Such that $E(\xi_i) = 0$ for $i = 0, 1, 2, 3$. $V_{rstu} = E[\xi_0^r \xi_1^s \xi_2^t \xi_3^u]$

$$\begin{aligned} E(\xi_0^2) &= \lambda C_1^2, \\ E(\xi_1^2) &= \lambda C_2^2, \\ E(\xi_2^2) &= \lambda C_3^2, \\ E(\xi_3^2) &= \lambda C_4^2, \end{aligned} \quad (4)$$

$$E(\xi_0 \xi_1) = \lambda \rho_{12} C_1 C_2.$$

$$E(\xi_0 \xi_2) = \lambda \rho_{13} C_1 C_3,$$

$$E(\xi_0 \xi_3) = \lambda \rho_{14} C_1 C_4,$$

$$E(\xi_1 \xi_2) = \lambda \rho_{23} C_2 C_3, \quad (5)$$

$$E(\xi_1 \xi_3) = \lambda \rho_{24} C_2 C_4,$$

$$\text{where } \lambda = \frac{(N - n)}{(nN)}.$$

3. Existing Estimators

In this section, we briefly review some existing estimators of $F(t_y)$.

- (1) The conventional unbiased mean per unit estimator of $F(t_y)$ is

$$\hat{F}(t_y) = \frac{1}{n} \sum_{i=1}^n (Y_i \leq t_y). \quad (6)$$

The variance of $\hat{F}(t_y)$ is given by the following equation:

$$\text{Var}(\hat{F}(t_y)) = \lambda F^2(t_y) C_1^2. \quad (7)$$

(2) The traditional ratio estimator of $F(t_y)$ is

$$\widehat{F}(t_y)_R = \widehat{F}(t_y) \left(\frac{F(t_x)}{\widehat{F}(t_x)} \right). \quad (8)$$

The bias and MSE of $\widehat{F}(t_y)_R$, to the first order of approximation, respectively, are given by the following equation:

$$\text{Bias}(\widehat{F}(t_y)_R) \cong \lambda F(t_y)(C_2^2 - \rho_{12}C_1C_2), \quad (9)$$

$$\text{MSE}(\widehat{F}(t_y)_R) \cong \lambda F^2(t_y)(C_1^2 + C_2^2 - 2\rho_{12}C_1C_2). \quad (10)$$

The ratio estimator $\widehat{F}(t_y)_R$ performs better than $\widehat{F}(t_y)$, in terms of MSE, if $\rho_{12} > C_2/(2C_1)$.

(3) Reference [29] suggested the usual product estimator of $F(t_y)$:

$$\widehat{F}(t_y)_P = \widehat{F}(t_y) \left(\frac{\widehat{F}(t_x)}{F(t_x)} \right). \quad (11)$$

The bias and MSE of $\widehat{F}(t_y)_P$, to the first order of approximation are given by the following equation:

$$\text{Bias}(\widehat{F}(t_y)_P) = \lambda F(t_y)\rho_{12}C_1C_2, \quad (12)$$

$$\text{MSE}(\widehat{F}(t_y)_P) \cong \lambda F^2(t_y)(C_1^2 + C_2^2 + 2\rho_{12}C_1C_2). \quad (13)$$

The product estimator $\widehat{F}(t_y)_P$ is better than $\widehat{F}(t_y)$, in terms of MSE, if $\rho_{12} < -C_2/(2C_1)$.

(4) The conventional difference estimator of $F(t_y)$ is

$$\widehat{F}(t_y)_{Reg} = \widehat{F}(t_y) + m\{F(t_x) - \widehat{F}(t_x)\}, \quad (14)$$

where m is an unknown constant. The minimum variance of $\widehat{F}(t_y)_{Reg}$ at the optimum value of $m_{(opt)} = \rho_{12}(\delta_1/\delta_2)$, is given by the following equation:

$$\text{Var}_{\min}(\widehat{F}(t_y)_{Reg}) = \lambda F^2(t_y)C_1^2(1 - \rho_{12}^2). \quad (15)$$

(5) Reference [4] suggested an improved difference-type estimator of $F(t_y)$, given by the following equation:

$$\widehat{F}(t_y)_{R,D} = m_1\widehat{F}(t_y) + m_2\{F(t_x) - \widehat{F}(t_x)\}, \quad (16)$$

where m_1 and m_2 are unknown constants.

The optimum values of m_1 and m_2 are

$$m_{1(opt)} = \frac{1}{1 + \lambda C_1^2(1 - \rho_{12}^2)}, \quad (17)$$

$$m_{2(opt)} = \frac{F(t_y)\rho_{12}C_1}{F(t_x)C_2\{1 + \lambda C_1^2(1 - \rho_{12}^2)\}}.$$

The bias and minimum MSE of $\widehat{F}(t_y)_{R,D}$, to the first order of approximation, respectively, are given by the following equation:

$$\text{Bias}(\widehat{F}(t_y)_{R,D}) = F(t_y)(m_1 - 1), \quad (18)$$

$$\text{MSE}_{\min}(\widehat{F}(t_y)_{R,D}) \cong \frac{\lambda F^2(t_y)C_1^2(1 - \rho_{12}^2)}{1 + \lambda C_1^2(1 - \rho_{12}^2)}. \quad (19)$$

(6) Reference [30] suggested the exponential ratio-type and product-type estimators are given by the following equation:

$$\widehat{F}(t_y)_{BT,R} = \widehat{F}(t_y) \exp\left(\frac{F(t_x) - \widehat{F}(t_x)}{\widehat{F}(t_x) + F(t_x)}\right), \quad (20)$$

$$\widehat{F}(t_y)_{BT,P} = \widehat{F}(t_y) \exp\left(\frac{\widehat{F}(t_x) - F(t_x)}{\widehat{F}(t_x) + F(t_x)}\right). \quad (21)$$

The biases and MSEs of $\widehat{F}(t_y)_{BT,R}$ and $\widehat{F}(t_y)_{BT,P}$, to the first order of approximation, respectively, are given by the following equation:

$$\text{Bias}(\widehat{F}(t_y)_{BT,R}) \cong \lambda F(t_y) \left(\frac{3C_2^2}{8} - \frac{\rho_{12}C_1C_2}{2} \right), \quad (22)$$

$$\text{MSE}(\widehat{F}(t_y)_{BT,R}) \cong \frac{\lambda F^2(t_y)}{4} (4C_1^2 + C_2^2 - 4\rho_{12}C_1C_2), \quad (23)$$

$$\text{Bias}(\widehat{F}(t_y)_{BT,P}) \cong \lambda F(t_y) \left(\frac{\rho_{12}C_1C_2}{2} - \frac{C_2^2}{8} \right), \quad (24)$$

$$\text{MSE}(\widehat{F}(t_y)_{BT,P}) \cong \frac{\lambda F^2(t_y)}{4} (4C_1^2 + C_2^2 + 4\rho_{12}C_1C_2). \quad (25)$$

(7) Reference [14] suggested a generalized class of ratio-type exponential estimators as follows:

$$\begin{aligned} (\widehat{F}(t_y)_{GK}) = \{ & m_3 \widehat{F}(t_y) + m_4 (F(t_x) \\ & - \widehat{F}(t_x)) \} \exp\left(\frac{F(t_x) - \widehat{F}(t_x)}{F(t_x) + \widehat{F}(t_x)}\right), \end{aligned} \quad (26)$$

where m_3 and m_4 are unknown constants.

The optimum values of m_3 and m_4 , determined by minimizing (26) are given by the following equation:

$$m_{3(\text{opt})} = \frac{8 - \lambda^2 C_2^2}{8\{1 + \lambda C_1^2(1 - \rho_{12}^2)\}}, \quad (27)$$

$$m_{4(\text{opt})} = F(t_y) \frac{\lambda^3 C_2^3 + 8\rho_{12} C_1 - \lambda^2 \rho_{12} C_1 C_2^2 - 4C_2\{1 - \lambda C_1^2(1 - \rho_{12}^2)\}}{8F(t_x) C_2\{1 + \lambda C_1^2(1 - \rho_{12}^2)\}}. \quad (28)$$

The bias of $(\widehat{F}(y)_{GK})$, are given by the following equation:

$$\text{Bias}(\widehat{F}(t_y)_{GK}) \cong F(t_y) - m_3 F(t_y) + \frac{3}{8} m_3^2 F(t_y) \lambda C_1^2 + \frac{1}{2} m_4 F(t_x) \lambda C_2^2, \quad (29)$$

$$-\frac{1}{2} m_3 F(t_y) \lambda \rho_{12} C_1 C_2. \quad (30)$$

The minimum MSE of $\widehat{F}(y)_{GK}$ at the optimum values of m_3 and m_4 is given by the following equation:

$$\begin{aligned} \text{MSE}(\widehat{F}(t_y)_{GK})_{\min} = \text{Var}_{\min}(\widehat{F}_{\text{Reg}}(t_y)) \\ - \frac{\lambda^2 F^2(t_y) \{C_2^2 + 8C_1^2(1 - \rho_{12}^2)\}^2}{64\{1 + \lambda C_1^2(1 - \rho_{12}^2)\}}, \end{aligned} \quad (31)$$

4. Proposed Estimator

Using the appropriate auxiliary information during the estimation stage or at the design stage improves an estimator's efficiency. The sample distribution function of the auxiliary variable has already been employed to increase the efficiency and quality of estimators. The study of [20] suggested using the rank of the auxiliary variable as an additional auxiliary variable to improve the precision of a population distribution function estimator. In this article, we used two auxiliary variables to estimate the finite distribution function; we need additional auxiliary information on the sample mean and sample distribution function of the auxiliary variable, as well as the sample distribution function of the study variable. In literature, auxiliary information using the distribution function has been rarely attempted, therefore we are motivated towards it. The principal advantage of our proposed generalized class of estimator is that it is more flexible, and efficient than the existing estimators.

4.1. First Proposed Estimator. On the lines of [31], we suggest a generalized class of exponential factor type estimators which contains many stable and efficient estimators. By combining the idea of [30, 31], the first estimator is given by the following equation:

$$\widehat{F}(t_y)_{\text{Prop1}}^{(k_1, k_2)} = \widehat{F}(t_y) \left[\exp\left(\frac{S_{11} - M_{11}}{S_{11} + M_{11}}\right) \exp\left(\frac{S_{22} - M_{22}}{S_{22} + M_{22}}\right) \right], \quad (32)$$

where

$$S_{11} = (A_1 + T_1)F(t_x) + fB_1\widehat{F}(t_x), \quad (33)$$

$$S_{22} = (A_2 + T_2)\overline{X} + fB_2\widehat{\overline{X}},$$

$$M_{11} = (A_1 + fB_1)F(t_x) + T_1\widehat{F}(t_x), \quad (34)$$

$$M_{22} = (A_2 + fB_2)\overline{X} + T_2\widehat{\overline{X}},$$

$$A_i = (K_i - 1)(K_i - 2), \quad (35)$$

$$B_i = (K_i - 1)(K_i - 4),$$

$$T_i = (K_i - 2)(K_i - 3)(K_i - 4), \quad (36)$$

$$f = \frac{n}{N},$$

Substituting different values of K_i ($i = 1, 2, 3, 4$) in Equation (32), we can generate many more types of estimators from our general proposed class of estimators, given in Table 1.

By solving $\widehat{F}(t_y)_{\text{Prop1}}^{(k_1, k_2)}$ given in (32) up-to first order of approximation, we have the following equation:

$$\widehat{F}(t_y)_{\text{Prop1}}^{(k_1, k_2)} = F(t_y) \left[\begin{array}{l} (1 + \xi_0) \left(1 + \frac{1}{2} \vartheta_1 \xi_1 - \frac{1}{4} \vartheta_1 \gamma_1 \xi_1^2 + \frac{1}{8} \vartheta_1^2 \xi_1^2 \right) \\ \left(1 + \frac{1}{2} \vartheta_2 \xi_2 - \frac{1}{4} \vartheta_2 \gamma_2 \xi_2^2 + \frac{1}{8} \vartheta_2^2 \xi_2^2 \right) \end{array} \right], \quad (37)$$

TABLE 1: Family members of the proposed class of estimators $\widehat{F}(t_y)_{Prop1}^{(k_1, k_2)}$.

S.No	K_1	K_2	Estimators
I	1	1	$\widehat{F}(t_y)_{Prop1}^{(1,1)} = \widehat{F}(t_y)\exp(F(t_x) - \widehat{F}(t_x)/F(t_x) + \widehat{F}(t_x))\exp(\overline{X} - \widehat{X}/\overline{X} + \widehat{X})$
II	1	2	$\widehat{F}(t_y)_{Prop1}^{(1,1)} = \widehat{F}(t_y)\exp(F(t_x) - \widehat{F}(t_x)/F(t_x) + \widehat{F}(t_x))\exp(\overline{X} - \widehat{X}/\overline{X} + \widehat{X})$
III	1	3	$\widehat{F}(t_y)_{Prop1}^{(1,3)} = \widehat{F}(t_y)\exp(F(t_x) - \widehat{F}(t_x)/F(t_x) + \widehat{F}(t_x))\exp(n(\overline{X} - \widehat{X})/2N\overline{X} - n(\widehat{X} + \overline{X}))$
IV	1	4	$\widehat{F}(t_y)_{Prop1}^{(1,4)} = \widehat{F}(t_y)\exp(F(t_x) - \widehat{F}(t_x)/F(t_x) + \widehat{F}(t_x))$
V	2	1	$\widehat{F}(t_y)_{Prop1}^{(2,1)} = \widehat{F}(t_y)\exp(\widehat{F}(t_x) - F(t_x)/\widehat{F}(t_x) + F(t_x))\exp(\overline{X} - \widehat{X}/\overline{X} + \widehat{X})$
VI	2	2	$\widehat{F}(t_y)_{Prop1}^{(2,2)} = \widehat{F}(t_y)\exp(\widehat{F}(t_x) - F(t_x)/\widehat{F}(t_x) + F(t_x))\exp(\overline{X} - \widehat{X}/\overline{X} + \widehat{X})$
VII	2	3	$\widehat{F}(t_y)_{Prop1}^{(2,3)} = \widehat{F}(t_y)\exp(\widehat{F}(t_x) - F(t_x)/\widehat{F}(t_x) + F(t_x))\exp(n(\overline{X} - \widehat{X})/2N\overline{X} - (n)(\widehat{X} + \overline{X}))$
VIII	2	4	$\widehat{F}(t_y)_{Prop1}^{(2,4)} = \widehat{F}(t_y)\exp(\widehat{F}(t_x) - F(t_x)/\widehat{F}(t_x) + F(t_x))$
IX	3	1	$\widehat{F}(t_y)_{Prop1}^{(3,1)} = \widehat{F}(t_y)\exp(n(F(t_x) - \widehat{F}(t_x)))/2NF(t_x) - (n(\widehat{F}(t_x) + F(t_x)))\exp(\frac{\overline{X} - \widehat{X}}{\overline{X} + \widehat{X}})$
X	3	2	$\widehat{F}(t_y)_{Prop1}^{(3,2)} = \widehat{F}(t_y)\exp(n(F(t_x) - \widehat{F}(t_x)))/2NF(t_x) - (n(\widehat{F}(t_x) + F(t_x)))\exp(\widehat{X} - \overline{X}/\widehat{X} + \overline{X})$
XI	3	3	$\widehat{F}(t_y)_{Prop1}^{(3,3)} = \widehat{F}(t_y)\exp(n(F(t_x) - \widehat{F}(t_x)))/2NF(t_x) - (n(\widehat{F}(t_x) + F(t_x)))\exp(n(\overline{X} - \widehat{X})/2N\overline{X} - n(\widehat{X} + \overline{X}))$
XII	3	4	$\widehat{F}(t_y)_{Prop1}^{(3,4)} = \widehat{F}(t_y)\exp(n(F(t_x) - \widehat{F}(t_x)))/2NF(t_x) - (n(\widehat{F}(t_x) + F(t_x)))$
XIII	4	1	$\widehat{F}(t_y)_{Prop1}^{(4,1)} = \widehat{F}(t_y)\exp(\overline{X} - \widehat{X}/\overline{X} + \widehat{X})$
XIV	4	2	$\widehat{F}(t_y)_{Prop1}^{(4,2)} = \widehat{F}(t_y)\exp(\widehat{X} - \overline{X}/\widehat{X} + \overline{X})$
XV	4	3	$\widehat{F}(t_y)_{Prop1}^{(4,3)} = \widehat{F}(t_y)\exp(n(\overline{X} - \widehat{X})/2N\overline{X} - n(\widehat{X} + \overline{X}))$
XVI	4	4	$\widehat{F}(t_y)_{Prop1}^{(4,4)} = \widehat{F}(t_y)$

where

Or

$$\begin{aligned}
 \vartheta_1 &= \frac{fB_1 - T_1}{A_1 + fB_1 + T_1}, \\
 \gamma_1 &= \frac{fB_1 - T_1}{A_1 + fB_1 + T_1}, \\
 \vartheta_2 &= \frac{fB_2 - T_2}{A_2 + fB_2 + T_2}, \\
 \gamma_2 &= \frac{fB_2 - T_2}{A_2 + fB_2 + T_2}.
 \end{aligned}
 \tag{38}$$

$$\left(\widehat{F}(t_y)_{Prop1}^{(k_1, k_2)} - F(t_y) \right) = F(t_y) \begin{bmatrix} \xi_0 + \frac{1}{2}\vartheta_1\xi_1 + \frac{1}{2}\vartheta_2\xi_2 + \frac{1}{2}\vartheta_1\xi_0\xi_1 + \frac{1}{2}\vartheta_2\xi_0\xi_2 \\ -\frac{1}{4}\vartheta_1\gamma_1\xi_1^2 - \frac{1}{4}\vartheta_2\gamma_2\xi_2^2 + \frac{1}{8}\vartheta_1^2\xi_1^2 + \frac{1}{8}\vartheta_2^2\xi_2^2 \end{bmatrix}.
 \tag{39}$$

Using (39), the bias and MSE of $\widehat{F}(t_y)_{Prop1}^{(k_1, k_2)}$ are given by the following equation:

$$\text{Bias}\left(\widehat{F}(t_y)_{Prop1}^{(k_1, k_2)}\right) \cong F(t_y)\lambda \left[\begin{array}{c} \frac{1}{2}\vartheta_1\rho_{12}C_1C_2 + \frac{1}{2}\vartheta_2\rho_{13}C_1C_3 + \frac{1}{8}C_2^2(\vartheta_1^2 - 2\vartheta_1\gamma_1) \\ + \frac{1}{8}C_3^2(\vartheta_2^2 - 2\vartheta_2\gamma_2) \end{array} \right], \quad (40)$$

$$\text{MSE}\left(\widehat{F}(t_y)_{Prop1}^{(k_1, k_2)}\right) \cong F^2(t_y)\lambda \left[\begin{array}{c} C_1^2 + \frac{1}{4}\vartheta_1^2C_2^2 + \frac{1}{4}\vartheta_2^2C_3^2 + \vartheta_1\rho_{12}C_1C_2 \\ + \vartheta_2\rho_{13}C_1C_3 + \frac{1}{2}\vartheta_1\vartheta_2\rho_{23}C_2C_3 \end{array} \right]. \quad (41)$$

Differentiate (40) with respect to ϑ_1 and ϑ_2 , we get the optimum values of ϑ_1 and ϑ_2 i.e.,

$$\begin{aligned} \vartheta_{1(opt)} &= \frac{2C_1(\rho_{12} - \rho_{23}\rho_{13})}{C_2(\rho_{23}^2 - 1)}, \\ \vartheta_{2(opt)} &= \frac{2C_1(\rho_{12} - \rho_{23}\rho_{12})}{C_2(\rho_{23}^2 - 1)}. \end{aligned} \quad (42)$$

Substituting the optimum values of $\vartheta_{1(opt)}$ and $\vartheta_{2(opt)}$ in Equation $\neq 20$, we get minimum MSE of $\widehat{F}(t_y)_{Prop1}^{(k_1, k_2)}$ and is given by the following equation:

$$\text{MSE}\left(\widehat{F}(t_y)_{Prop1}^{(k_1, k_2)}\right) \cong F^2(t_y)\lambda C_1^2 [1 - \rho_{1.23}^2], \quad (43)$$

where

$$\rho_{1.23}^2 = \frac{\rho_{12}^2 + \rho_{13}^2 - 2\rho_{12}\rho_{13}\rho_{23}}{1 - \rho_{23}^2}. \quad (44)$$

is the coefficient of multiple determination of $F(t_y)$ on $F(t_x)$ and X .

4.2. Second Proposed Estimator. To increase the efficiency of the estimators both at the design stage as well as at the estimation stage, we utilize the auxiliary information. When there exists a correlation between the study variable and the auxiliary variable, then the rank of the auxiliary variable is also correlated with the study variable. The rank of the auxiliary variable can be treated as a new auxiliary variable, and this information may also be used to increase the precision of the estimators. Based on the idea of rank, we propose a second new class of factor type estimators of the finite population distribution function. The estimator is given by the following equation:

$$\widehat{F}(t_y)_{Prop2}^{(k_1, k_2)} = \widehat{F}(t_y) \left[\exp\left(\frac{P_{11} - Q_{11}}{P_{11} + Q_{11}}\right) \exp\left(\frac{P_{22} - Q_{22}}{P_{22} + Q_{22}}\right) \right], \quad (45)$$

where

$$P_{11} = (A_1 + T_1)F(t_x) + fB_1\widehat{F}(t_x), \quad (46)$$

$$P_{22} = (A_2 + T_2)\widehat{R}_x + fB_2\widehat{R}_x,$$

$$Q_{11} = (A_1 + fB_1)F(t_x) + T_1\widehat{F}(t_x), \quad (47)$$

$$Q_{22} = (A_2 + fB_2)\widehat{R}_x + T_2\widehat{R}_x,$$

$$A_i = (K_i - 1)(K_i - 2), \quad (48)$$

$$B_i = (K_i - 1)(K_i - 4),$$

$$T_i = (K_i - 2)(K_i - 3)(K_i - 4), \quad (49)$$

$$f = \frac{n}{N}.$$

Substituting different values of K_i ($i=1, 2, 3, 4$) in Equation (45), we can generate many more different types of estimators from our general proposed class of estimators, given in Table 2.

Solving $\widehat{F}(t_y)_{Prop2}^{(k_1, k_2)}$ given in (45) in terms of errors, we have the following equation:

$$\widehat{F}(t_y)_{Prop2}^{(k_1, k_2)} = F(t_y) \left[\begin{array}{c} (1 + \xi_0) \left(1 + \frac{1}{2}\vartheta_1\xi_1 - \frac{1}{4}\vartheta_1\gamma_1\xi_1^2 + \frac{1}{8}\vartheta_1^2\xi_1^2 \right) \\ \left(1 + \frac{1}{2}\vartheta_2\xi_3 - \frac{1}{4}\vartheta_2\gamma_2\xi_3^2 + \frac{1}{8}\vartheta_2^2\xi_3^2 \right) \end{array} \right], \quad (50)$$

where

$$\vartheta_1 = \frac{fB_1 - T_1}{A_1 + fB_1 + T_1},$$

$$\gamma_1 = \frac{fB_1 - T_1}{A_1 + fB_1 + T_1}, \quad (51)$$

$$\vartheta_2 = \frac{fB_1 - T_1}{A_1 + fB_1 + T_1},$$

$$\gamma_2 = \frac{fB_1 - T_1}{A_1 + fB_1 + T_1}.$$

With first order approximation, we have the following equation:

TABLE 2: Family members of the proposed class of estimators $\widehat{F}(t_y)^{(k_1, k_2)}_{Prop2}$.

S.No	K_1	K_2	Estimators
I	1	1	$\widehat{F}(t_y)^{(1,1)}_{Prop2} = \widehat{F}(t_y)\exp(F(t_x) - \widehat{F}(t_x)/F(t_x) + \widehat{F}(t_x))\exp(\overline{R}_x - \widehat{R}_x/\overline{R}_x + \widehat{R}_x)$
II	1	2	$\widehat{F}(t_y)^{(1,2)}_{Prop2} = \widehat{F}(t_y)\exp(F(t_x) - \widehat{F}(t_x)/F(t_x) + \widehat{F}(t_x))\exp(\overline{R}_x - \widehat{R}_x/\overline{R}_x + \widehat{R}_x)$
III	1	3	$\widehat{F}(t_y)^{(1,1)}_{Prop2} = \widehat{F}(t_y)\exp(F(t_x) - \widehat{F}(t_x)/F(t_x) + \widehat{F}(t_x))\exp(n(\overline{R}_x - \widehat{R}_x)/2N\overline{R}_x - n(\widehat{R}_x + \overline{R}_x))$
IV	1	4	$\widehat{F}(t_y)^{(1,4)}_{Prop2} = \widehat{F}(t_y)\exp(F(t_x) - \widehat{F}(t_x)/F(t_x) + \widehat{F}(t_x))$
V	2	1	$\widehat{F}(t_y)^{(2,1)}_{Prop2} = \widehat{F}(t_y)\exp(\widehat{F}(t_x) - \widehat{F}(t_x)/\widehat{F}(t_x) + \widehat{F}(t_x))\exp(\overline{R}_x - \widehat{R}_x/\overline{R}_x + \widehat{R}_x)$
VI	2	2	$\widehat{F}(t_y)^{(2,2)}_{Prop2} = \widehat{F}(t_y)\exp(\widehat{F}(t_x) - \widehat{F}(t_x)/\widehat{F}(t_x) + \widehat{F}(t_x))\exp(\overline{R}_x - \widehat{R}_x/\overline{R}_x + \widehat{R}_x)$
VII	2	3	$\widehat{F}(t_y)^{(2,3)}_{Prop2} = \widehat{F}(t_y)\exp(\widehat{F}(t_x) - \widehat{F}(t_x)/\widehat{F}(t_x) + \widehat{F}(t_x))\exp(n(\overline{R}_x - \widehat{R}_x)/2N\overline{R}_x - n(\widehat{R}_x + \overline{R}_x))$
VIII	2	4	$\widehat{F}(t_y)^{(2,4)}_{Prop2} = \widehat{F}(t_y)\exp(\widehat{F}(t_x) - F(t_x)/\widehat{F}(t_x) + F(t_x))$
IX	3	1	$\widehat{F}(t_y)^{(3,1)}_{Prop2} = \widehat{F}(t_y)\exp(\widehat{F}(t_x) - F(t_x)/\widehat{F}(t_x) + F(t_x))\exp(\overline{R}_x - \widehat{R}_x/\overline{R}_x + \widehat{R}_x)$
X	3	2	$\widehat{F}(t_y)^{(3,2)}_{Prop2} = \widehat{F}(t_y)\exp(\widehat{F}(t_x) - F(t_x)/\widehat{F}(t_x) + F(t_x))\exp(\overline{R}_x - \widehat{R}_x/\overline{R}_x + \widehat{R}_x)$
XI	3	3	$\widehat{F}(t_y)^{(3,3)}_{Prop2} = \widehat{F}(t_y)\exp(\widehat{F}(t_x) - F(t_x)/\widehat{F}(t_x) + F(t_x))\exp(n(\overline{R}_x - \widehat{R}_x)/2N\overline{R}_x - n(\widehat{R}_x + \overline{R}_x))$
XII	3	4	$\widehat{F}(t_y)^{(3,4)}_{Prop2} = \widehat{F}(t_y)\exp(n\widehat{F}(t_x) - F(t_x)/(2N\widehat{F}(t_x) + nF(t_x)))$
XIII	4	1	$\widehat{F}(t_y)^{(4,1)}_{Prop2} = \widehat{F}(t_y)\exp(\overline{R}_x - \widehat{R}_x/\overline{R}_x + \widehat{R}_x)$
XIV	4	2	$\widehat{F}(t_y)^{(4,2)}_{Prop2} = \widehat{F}(t_y)\exp(\overline{R}_x - \widehat{R}_x/\overline{R}_x + \widehat{R}_x)$
XV	4	3	$\widehat{F}(t_y)^{(4,3)}_{Prop2} = \widehat{F}(t_y)\exp(\overline{R}_x - \widehat{R}_x/\overline{R}_x + \widehat{R}_x)$
XVI	4	4	$\widehat{F}(t_y)^{(4,4)}_{Prop2} = \widehat{F}(t_y)$

$$(\widehat{F}(t_y)^{(k_1, k_2)}_{Prop2} - F(t_y)) = F(t_y) \begin{bmatrix} \xi_0 + \frac{1}{2}\vartheta_1\xi_1 + \frac{1}{2}\vartheta_2\xi_3 + \frac{1}{2}\vartheta_1\xi_0\xi_1 + \frac{1}{2}\vartheta_2\xi_0\xi_3 \\ -\frac{1}{4}\vartheta_1\gamma_1\xi_1^2 - \frac{1}{4}\vartheta_2\gamma_2\xi_3^2 + \frac{1}{8}\vartheta_1^2\xi_1^2 + \frac{1}{8}\vartheta_2^2\xi_3^2 \end{bmatrix} \quad (52)$$

Using (52) the bias and MSE of $\widehat{F}(t_y)^{(k_1, k_2)}_{Prop2}$ are given by the following equation:

$$\text{Bias}(\widehat{F}(t_y)^{(k_1, k_2)}_{Prop2}) \cong F(t_y)\lambda \left[\frac{1}{2}\vartheta_1\rho_{12}C_1C_2 + \frac{1}{2}\vartheta_2\rho_{14}C_1C_4 + \frac{1}{8}C_2^2(\vartheta_1^2 - 2\vartheta_1\gamma_1) + \frac{1}{8}C_4^2(\vartheta_2^2 - 2\vartheta_2\gamma_2) \right] \quad (53)$$

$$\text{MSE}(\widehat{F}(t_y)^{(k_1, k_2)}_{Prop2}) \cong F^2(t_y)\lambda \left[C_1^2 + \frac{1}{4}\vartheta_1^2C_2^2 + \frac{1}{4}\vartheta_2^2C_4^2 + \vartheta_1\rho_{12}C_1C_2 + \vartheta_2\rho_{14}C_1C_4 + \frac{1}{2}\vartheta_1\vartheta_2\rho_{24}C_2C_4 \right] \quad (54)$$

Differentiate Equation (54) with respect to ϑ_1 and ϑ_2 for minimum MSE, we get the optimum values of ϑ_1 and ϑ_2 ,

$$\vartheta_{1(opt)} = \frac{2C_1(\rho_{14} - \rho_{24}\rho_{12})}{C_4(\rho_{24}^2 - 1)}, \quad (55)$$

$$\vartheta_{2(opt)} = \frac{2C_1(\rho_{14} - \rho_{24}\rho_{12})}{C_4(\rho_{24}^2 - 1)}.$$

Substituting the optimum values of $\vartheta_{1(opt)}$ and $\vartheta_{2(opt)}$ in Equation $\neq 25$, we get minimum MSE of $\widehat{F}(t_y)^{(k_1, k_2)}_{Prop2}$ is given by the following equation:

$$\text{MSE}(\widehat{F}(t_y)^{(k_1, k_2)}_{Prop2})_{\min} \cong F^2(t_y)\lambda C_1^2 [1 - \rho_{1.24}^2], \quad (56)$$

where

$$\rho_{1.24}^2 = \frac{\rho_{12}^2 + \rho_{14}^2 - 2\rho_{12}\rho_{14}\rho_{24}}{1 - \rho_{24}^2} \quad (57)$$

is the coefficient of multiple determination of $F(t_y)$ on $F(t_x)$ and R_x .

5. Theoretical Comparison

In this section, the adapted and proposed estimators of $F(t_y)$ are compared in terms of the minimum mean square error.

(1) From (7) and (43),

$$\text{MSE}(\widehat{F}(t_y)^{(k_1, k_2)}_{Prop1})_{\min} < \text{Var}(\widehat{F}(t_y)) \text{ if.}$$

$$\text{Var}(\widehat{F}(t_y)) - \text{MSE}(\widehat{F}(t_y)_{Prop1}^{(k_1, k_2)})_{\min} > 0 \text{ or if } \\ \lambda F^2(t_y)C_1^2 - F^2(t_y)\lambda C_1^2[1 - \rho_{1.23}^2] > 0, \text{ or if } \\ \rho_{1.23}^2 > 0$$

(2) From (10) and (43),

$$\text{MSE}(\widehat{F}(t_y)_{Prop1}^{(k_1, k_2)})_{\min} < \text{MSE}(\widehat{F}(t_y)_R) \text{ if } \\ \text{MSE}(\widehat{F}(t_y)_R) - \text{MSE}(\widehat{F}(t_y)_{Prop1}^{(k_1, k_2)})_{\min} > 0 \text{ or if } \\ F^2(t_y)\lambda[C_1^2 + C_2^2 - 2\rho_{12}C_1C_2] - F^2(t_y)\lambda C_1^2[1 - \\ \rho_{1.23}^2] > 0, \text{ or if } \\ [C_2^2 - 2\rho_{12}C_1C_2 + C_1^2\rho_{1.23}^2] > 0$$

(3) From (13) and (43),

$$\text{MSE}(\widehat{F}(t_y)_{Prop1}^{(k_1, k_2)})_{\min} < \text{MSE}(\widehat{F}(t_y)_P) \text{ if } \\ \text{MSE}(\widehat{F}(t_y)_P) - \text{MSE}(\widehat{F}(t_y)_{Prop1}^{(k_1, k_2)})_{\min} > 0 \text{ or if } \\ F^2(t_y)\lambda[C_1^2 + C_2^2 + 2\rho_{12}C_1C_2] - F^2(t_y)\lambda C_1^2[1 - \\ \rho_{1.23}^2] > 0, \text{ or if } \\ [C_2^2 + 2\rho_{12}C_1C_2 + C_1^2\rho_{1.23}^2] > 0$$

(4) From (15) and (43),

$$\text{MSE}(\widehat{F}(t_y)_{Prop1}^{(k_1, k_2)})_{\min} < \text{MSE}(\widehat{F}(t_y)_{Reg}) \text{ if } \\ \text{MSE}(\widehat{F}(t_y)_{Reg}) - \text{MSE}(\widehat{F}(t_y)_{Prop1}^{(k_1, k_2)})_{\min} > 0 \text{ or if } \\ F^2(t_y)\lambda[C_1^2 + (1/4)C_2^2 - \rho_{12}C_1C_2] - \\ F^2(t_y)\lambda C_1^2[1 - \rho_{1.23}^2] > 0, \text{ or if } \\ [1/4C_2^2 - \rho_{12}C_1C_2 + C_1^2\rho_{1.23}^2] > 0$$

(5) From (10) and (43),

$$\text{MSE}(\widehat{F}(t_y)_{Prop1}^{(k_1, k_2)})_{\min} < \text{MSE}(\widehat{F}(t_y)_{R,D}) \text{ if } \\ \text{MSE}(\widehat{F}(t_y)_{R,D}) - \text{MSE}(\widehat{F}(t_y)_{Prop1}^{(k_1, k_2)})_{\min} > 0 \text{ or if } \\ \lambda F^2(t_y)C_1^2(1 - \rho_{12}^2)/1 + \lambda C_1^2(1 - \rho_{12}^2) - \\ F^2(t_y)\lambda C_1^2[1 - \rho_{1.23}^2] > 0, \text{ or if } \\ \lambda C_1^2(\rho_{12}^2 + \rho_{1.23}^2) - \rho_{12}^2 + \rho_{1.23}^2 > 0$$

(6) From (23) and (43),

$$\text{MSE}(\widehat{F}(t_y)_{Prop1}^{(k_1, k_2)})_{\min} < \text{MSE}(\widehat{F}(t_y)_{BT,R}) \text{ if } \\ \text{MSE}(\widehat{F}(t_y)_{BT,R}) - \text{MSE}(\widehat{F}(t_y)_{Prop1}^{(k_1, k_2)})_{\min} > 0 \text{ or if } \\ \lambda F^2(t_y)(C_1^2 + (1/4)C_2^2 - \rho_{12}C_1^2C_2) - \\ F^2(t_y)\lambda C_1^2[1 - \rho_{1.23}^2] > 0, \text{ or if } \\ [1/4(C_2^2 + C_1^2(\rho_{1.23}^2 - \rho_{12}^2))] > 0$$

(7) From (25) and (43),

$$\text{MSE}(\widehat{F}(t_y)_{Prop1}^{(k_1, k_2)})_{\min} < \text{MSE}(\widehat{F}(t_y)_{BT,P}) \text{ if } \\ \text{MSE}(\widehat{F}(t_y)_{BT,P}) - \text{MSE}(\widehat{F}(t_y)_{Prop1}^{(k_1, k_2)})_{\min} > 0 \text{ or if } \\ F^2(t_y)\lambda[C_1^2 + (1/4)C_2^2 + \rho_{12}C_1C_2] - \\ F^2(t_y)\lambda C_1^2[1 - \rho_{1.23}^2] > 0, \text{ or if } \\ [1/4C_2^2 + \rho_{12}C_1C_2 + C_1^2\rho_{1.23}^2] > 0$$

(8) From (31) and (43),

$$\text{MSE}(\widehat{F}(t_y)_{Prop1}^{(k_1, k_2)})_{\min} < \text{MSE}(\widehat{F}(t_y)_{G,K}) \text{ if } \\ \text{MSE}(\widehat{F}(t_y)_{G,K}) - \text{MSE}(\widehat{F}(t_y)_{Prop1}^{(k_1, k_2)})_{\min} > 0 \text{ or if } \\ \lambda F^2(t_y)\{64C_1^2(1 - \rho_{12}^2) - \lambda^4 C_2^4 - 16\lambda^2 C_1^2 C_2^2(1 - \\ \rho_{12}^2)\}/64\{1 + \lambda C_1^2(1 - \rho_{12}^2)\} - F^2(t_y)\lambda C_1^2[1 - \\ \rho_{1.23}^2] > 0, \text{ or if } \\ \lambda F^2(t_y)\{64C_1^2(1 - \rho_{12}^2) - \lambda^4 C_2^4 - 16\lambda^2 C_1^2 C_2^2(1 - \\ (1 - \rho_{12}^2))\} - [F^2(t_y)\lambda C_1^2(1 - \rho_{1.23}^2)] > 0.$$

(9) From (7) and (56),

$$\text{MSE}(\widehat{F}(t_y)_{Prop2}^{(k_1, k_2)})_{\min} < \text{Var}(\widehat{F}(t_y)) \text{ if } \\ \text{Var}(\widehat{F}(t_y)) - \text{MSE}(\widehat{F}(t_y)_{Prop2}^{(k_1, k_2)})_{\min} > 0 \text{ or if } \\ \lambda F^2(t_y)C_1^2 - F^2(t_y)\lambda C_1^2[1 - \rho_{1.24}^2] > 0, \text{ or if}$$

$$\rho_{1.24}^2 > 0.$$

(10) From (10) and (56),

$$\text{MSE}(\widehat{F}(t_y)_{Prop2}^{(k_1, k_2)})_{\min} < \text{MSE}(\widehat{F}(t_y)_R) \text{ if } \\ \text{MSE}(\widehat{F}(t_y)_R) - \text{MSE}(\widehat{F}(t_y)_{Prop2}^{(k_1, k_2)})_{\min} > 0 \text{ or if } \\ F^2(t_y)\lambda[C_1^2 + C_2^2 - 2\rho_{12}C_1C_2] - F^2(t_y)\lambda C_1^2[1 - \\ \rho_{1.24}^2] > 0, \text{ or if } \\ [C_2^2 - 2\rho_{12}C_1C_2 + C_1^2\rho_{1.24}^2] > 0.$$

(11) From (13) and (56),

$$\text{MSE}(\widehat{F}(t_y)_{Prop2}^{(k_1, k_2)})_{\min} < \text{MSE}(\widehat{F}(t_y)_P) \text{ if } \\ \text{MSE}(\widehat{F}(t_y)_P) - \text{MSE}(\widehat{F}(t_y)_{Prop2}^{(k_1, k_2)})_{\min} > 0 \text{ or if } \\ F^2(t_y)\lambda[C_1^2 + C_2^2 + 2\rho_{12}C_1C_2] - F^2(t_y)\lambda C_1^2[1 - \\ \rho_{1.24}^2] > 0, \text{ or if } \\ [C_2^2 + 2\rho_{12}C_1C_2 + C_1^2\rho_{1.24}^2] > 0$$

(12) From (15) and (56),

$$\text{MSE}(\widehat{F}(t_y)_{Prop2}^{(k_1, k_2)})_{\min} < \text{MSE}(\widehat{F}(t_y)_{Reg}) \text{ if } \\ \text{MSE}(\widehat{F}(t_y)_{Reg}) - \text{MSE}(\widehat{F}(t_y)_{Prop2}^{(k_1, k_2)})_{\min} > 0 \text{ or if } \\ F^2(t_y)\lambda[C_1^2 + (1/4)C_2^2 - \rho_{12}C_1C_2] - \\ F^2(t_y)\lambda C_1^2[1 - \rho_{1.24}^2] > 0, \text{ or if } \\ [1/4C_2^2 - \rho_{12}C_1C_2 + C_1^2\rho_{1.24}^2] > 0$$

(13) From (19) and (56),

$$\text{MSE}(\widehat{F}(t_y)_{Prop1}^{(k_1, k_2)})_{\min} < \text{MSE}(\widehat{F}(t_y)_{R,D}) \text{ if } \\ \text{MSE}(\widehat{F}(t_y)_{R,D}) - \text{MSE}(\widehat{F}(t_y)_{Prop1}^{(k_1, k_2)})_{\min} > 0 \text{ or if } \\ \lambda F^2(t_y)C_1^2(1 - \rho_{12}^2)/1 + \lambda C_1^2(1 - \rho_{12}^2) - \\ F^2(t_y)\lambda C_1^2[1 - \rho_{1.24}^2] > 0, \text{ or if } \\ \lambda C_1^2(\rho_{12}^2 + \rho_{1.24}^2) - \rho_{12}^2 + \rho_{1.24}^2 > 0$$

(14) From (23) and (56),

$$\text{MSE}(\widehat{F}(t_y)_{Prop1}^{(k_1, k_2)})_{\min} < \text{MSE}(\widehat{F}(t_y)_{BT,R}) \text{ if } \\ \text{MSE}(\widehat{F}(t_y)_{BT,R}) - \text{MSE}(\widehat{F}(t_y)_{Prop1}^{(k_1, k_2)})_{\min} > 0 \text{ or if } \\ \lambda F^2(t_y)(C_1^2 + (1/4)C_2^2 - \rho_{12}C_1^2C_2) - \\ F^2(t_y)\lambda C_1^2[1 - \rho_{1.24}^2] > 0, \text{ or if } \\ [1/4(C_2^2 + C_1^2(\rho_{1.24}^2 - \rho_{12}^2))] > 0$$

(15) From (25) and (56),

$$\text{MSE}(\widehat{F}(t_y)_{Prop1}^{(k_1, k_2)})_{\min} < \text{MSE}(\widehat{F}(t_y)_{BT,P}) \text{ if } \\ \text{MSE}(\widehat{F}(t_y)_{BT,P}) - \text{MSE}(\widehat{F}(t_y)_{Prop1}^{(k_1, k_2)})_{\min} > 0 \text{ or if } \\ F^2(t_y)\lambda[C_1^2 + (1/4)C_2^2 + \rho_{12}C_1C_2] - F^2(t_y)\lambda C_1^2[1 - \\ \rho_{1.24}^2] > 0, \text{ or if } \\ [1/4C_2^2 + \rho_{12}C_1C_2 + C_1^2\rho_{1.24}^2] > 0$$

(16) From (31) and (56),

$$\text{MSE}(\widehat{F}(t_y)_{Prop2}^{(k_1, k_2)})_{\min} < \text{MSE}(\widehat{F}(t_y)_{G,K}) \text{ if } \\ \text{MSE}(\widehat{F}(t_y)_{G,K}) - \text{MSE}(\widehat{F}(t_y)_{Prop2}^{(k_1, k_2)})_{\min} > 0 \text{ or if } \\ \text{MSE}(\widehat{F}(t_y)_{Prop2}^{(k_1, k_2)})_{\min} \\ \lambda F^2(t_y)\{64C_1^2(1 - \rho_{12}^2) - \lambda^4 C_2^4 - 16\lambda^2 C_1^2 C_2^2(1 - \\ \rho_{12}^2)\}/64\{1 + \lambda C_1^2(1 - \rho_{12}^2)\} - F^2(t_y)\lambda C_1^2[1 - \rho_{1.24}^2] > \\ 0, \text{ or if } \\ \lambda F^2(t_y)\{64C_1^2(1 - \rho_{12}^2) - \lambda^4 C_2^4 - 16\lambda^2 C_1^2 C_2^2(1 - \\ \rho_{12}^2)\} - [F^2(t_y)\lambda C_1^2(1 - \rho_{1.24}^2)] > 0$$

6. Numerical Study

In this section, we conduct a numerical study to investigate the performances of the adapted and proposed DF estimators. For this purpose, three populations are

TABLE 3: Summary statistics.

Parameter	Value	Parameter	$\{t_x, t_y (\text{value})\}$			
			\bar{X}, \bar{Y}	$Q_1(x), Q_1(y)$	\bar{X}, \bar{Y}	$Q_3(x), Q_3(y)$
N	923	$F(t_y)$	0.7703	0.2556	0.5016	0.7497
n	180	C_1	0.5463	1.7070	0.9973	0.5780
λ	0.00447	$F(t_x)$	0.7693	0.2503	0.5005	0.7508
\bar{X}	11440.5	C_2	0.5480	1.7317	0.9995	0.5764
C_3	1.86453	ρ_{12}	0.8930	0.8711	0.8462	0.8930
\bar{R}_x	462.000	ρ_{13}	-0.6640	-0.2861	-0.4416	-0.6465
C_4	0.57703	ρ_{23}	-0.6753	-0.2838	-0.4480	-0.6563
		ρ_{14}	-0.7169	-0.7424	-0.8286	-0.7402
		ρ_{24}	-0.7298	-0.7503	-0.8660	-0.7492
		β_2	2.6333	2.3295	1.0000	2.3449

TABLE 4: Summary statistics.

Parameter	Value	Parameter	$\{t_x, t_y (\text{value})\}$			
			\bar{X}, \bar{Y}	$Q_1(x), Q_1(y)$	\bar{X}, \bar{Y}	$Q_3(x), Q_3(y)$
N	923	$F(t_y)$	0.7703	0.2556	0.5016	0.7497
n	180	C_1	0.5463	1.7070	0.9973	0.5780
λ	0.00447	$F(t_x)$	0.7291	0.2524	0.5016	0.7508
\bar{X}	333.165	C_2	0.6098	1.7218	0.9973	0.5764
C_3	1.32809	ρ_{12}	0.8727	0.1910	0.8917	0.9162
\bar{R}_x	462.000	ρ_{13}	-0.7385	-0.3656	-0.5366	-0.7221
C_4	0.57703	ρ_{23}	-0.7120	-0.1046	-0.5419	-0.7298
		ρ_{14}	-0.7223	-0.7424	-0.8486	-0.7430
		ρ_{24}	-0.7697	-0.7503	-0.8660	-0.7491
		β_2	2.0634	1.2990	1.0000	2.3449

TABLE 5: Summary statistics.

Parameter	Value	Parameter	$\{t_x, t_y (\text{value})\}$			
			\bar{X}, \bar{Y}	$Q_1(x), Q_1(y)$	\bar{X}, \bar{Y}	$Q_3(x), Q_3(y)$
N	69	$F(t_y)$	0.7246	0.2464	0.5072	0.7536
n	10	C_1	0.6209	1.7618	0.9928	0.5759
λ	0.08550	$F(t_x)$	0.7681	0.2464	0.5072	0.7536
\bar{X}	4954.44	C_2	0.5535	1.7618	0.9928	0.5759
C_3	1.42478	ρ_{12}	0.6607	0.7658	0.9420	0.7658
\bar{R}_x	35.0000	ρ_{13}	-0.7129	-0.3612	-0.5709	-0.7424
C_4	0.57321	ρ_{23}	-0.7745	-0.3650	-0.5427	-0.7584
		ρ_{14}	-0.7168	-0.7109	-0.8558	-0.7008
		ρ_{24}	-0.7310	-0.7464	-0.8660	-0.7464
		β_2	2.6144	2.3857	1.0000	2.3857

TABLE 6: PREs of distribution function estimators for population I, II, and III, when $F(t_y)$ and $F(t_x) = \bar{Y}, \bar{X}$.

Estimators	Population 1 (\bar{Y}, \bar{X})	Population 2 (\bar{Y}, \bar{X})	Population 3 (\bar{Y}, \bar{X})
$\hat{F}(t_y)$	100	100	100
$\hat{F}(t_y)_R$	465.57	336.04	162.16
$\hat{F}(t_y)_P$	25.47	23.31	33.34
$\hat{F}(t_y)_{BT,R}$	281.07	296.44	164.00
$\hat{F}(t_y)_{BT,P}$	46.57	43.75	55.94
$\hat{F}(t_y)_{Reg}$	493.79	419.59	177.46
$\hat{F}(t_y)_{Rao}$	493.93	419.73	180.76
$\hat{F}(t_y)_{G,K}$	493.99	419.78	181.17
$\hat{F}(t_y)_{Prop1}^{(k_1, k_2)}$	511.03	475.05	216.33
$\hat{F}(t_y)_{Prop2}^{(k_1, k_2)}$	517.03	430.94	224.16

TABLE 7: PREs of distribution function estimators for population I, II, and III, when $F(t_y)$ and $F(t_x) = Q_1(y), Q_1(x)$.

Estimators	Population 1 ($Q_1(y), Q_1(x)$)	Population 2 ($Q_1(y), Q_1(x)$)	Population 3 ($Q_1(y), Q_1(x)$)
$\widehat{F}(y)$	100	100	100
$\widehat{F}(y)_R$	382.08	411.14	213.53
$\widehat{F}(t_y)_P$	18.95	18.92	19.22
$\widehat{F}(t_y)_{BT,R}$	267.67	273.36	206.54
$\widehat{F}(t_y)_{BT,P}$	46.70	46.60	49.60
$\widehat{F}(y)_{Reg}$	414.63	442.7	241.84
$\widehat{F}(t_y)_{Rao}$	415.93	444.01	268.39
$\widehat{F}(t_y)_{G,K}$	415.95	444.03	268.65
$\widehat{F}(t_y)_{Prop1}^{(k_1,k_2)}$	417.48	447.26	246.42
$\widehat{F}(t_y)_{Prop2}^{(k_1,k_2)}$	448.24	475.38	270.52

TABLE 8: PREs of distribution function estimators for population I, II, and III, when $F(t_y)$ and $F(t_x) = \bar{Y}, \bar{X}$ or $Q_2(y), Q_2(x)$.

Estimators	Population 1 (\bar{Y}, \bar{X})	Population 2 (\bar{Y}, \bar{X})	Population 3 (\bar{Y}, \bar{X})
$\widehat{F}(t_y)$	100	100	100
$\widehat{F}(t_y)_R$	324.29	461.49	862.32
$\widehat{F}(t_y)_P$	27.06	26.45	25.79
$\widehat{F}(t_y)_{BT,R}$	248.08	279.06	324.69
$\widehat{F}(t_y)_{BT,P}$	47.64	46.69	45.62
$\widehat{F}(t_y)_{Reg}$	352.08	487.23	888.06
$\widehat{F}(t_y)_{Rao}$	352.53	488.37	896.49
$\widehat{F}(t_y)_{G,K}$	352.58	488.43	898.79
$\widehat{F}(t_y)_{Prop1}^{(k_1,k_2)}$	358.25	497.75	899.77
$\widehat{F}(t_y)_{Prop2}^{(k_1,k_2)}$	404.37	550.62	941.66

TABLE 9: PREs of distribution function estimators for population I, II, and III, when $F(t_y)$ and $F(t_x) = Q_3(y), Q_3(x)$.

Estimators	Population 1 ($Q_3(y), Q_3(x)$)	Population 2 ($Q_3(y), Q_3(x)$)	Population 3 ($Q_3(y), Q_3(x)$)
$\widehat{F}(t_y)$	100	100	100
$\widehat{F}(t_y)_R$	468.75	598.06	213.53
$\widehat{F}(t_y)_P$	26.04	25.73	27.78
$\widehat{F}(y)_{BT,R}$	279.25	298.47	206.54
$\widehat{F}(t_y)_{BT,P}$	46.75	46.25	49.60
$\widehat{F}(t_y)_{Reg}$	493.83	622.46	241.84
$\widehat{F}(y)_{Rao}$	493.98	622.6	244.68
$\widehat{F}(t_y)_{G,K}$	494.64	622.69	245.26
$\widehat{F}(t_y)_{Prop1}^{(k_1,k_2)}$	509.95	647.15	284.03
$\widehat{F}(t_y)_{Prop2}^{(k_1,k_2)}$	523.64	652.19	266.09

considered. The summary statistics of these populations are reported in Tables 3–5. The percentage relative efficiency (PRE) of the estimator $\widehat{F}_i(t_y)$ with respect to $\widehat{F}(t_y)$ is given by the following equation::

$$PRE(\widehat{F}(t_y)_i, \widehat{F}(t_y)) = \frac{\text{Var}(\widehat{F}(t_y))}{\text{MSE}(\widehat{F}_i(t_y))_{\min}} \times 100. \quad (58)$$

where $i = R, P, \text{Reg}, (R, D), (BT, R), (BT, P), (G, K), \text{Prop}_1, \text{Prop}_2$.

The PREs of the distribution function estimators, computed from three populations, are given in Tables 6–9

Population 1. (Source: [32]) Y: number of teachers and X: number of students.

Population 2. (Source: [32]) Y: Number of teachers and X: number of Schools.

Population 3. (Source: [6]) Y: the estimated number of fish caught by marine recreational fishermen in the year 1995 and X: the estimated number of fish caught by marine recreational fishermen in the year 1994.

TABLE 10: PREs of distribution function estimators using simulation for Populations I, II, and III, when $F(t_y)$ and $F(t_x) = \bar{Y}, \bar{X}$.

Estimators	Population 1 ($Q_1(y), Q_1(x)$)	Population 2 ($Q_1(y), Q_1(x)$)	Population 3 ($Q_1(y), Q_1(x)$)
$\hat{F}(y)$	100	100	100
$\hat{F}(y)_R$	22.69356	19.94125	32.53728
$\hat{F}(t_y)_P$	85.22727	176.8868	39.89362
$\hat{F}(t_y)_{BT,R}$	119.5219	187.7347	66.51885
$\hat{F}(t_y)_{BT,P}$	60.12024	50.83023	100.3345
$\hat{F}(y)_{Reg}$	120.6046	206.0017	106.8579
$\hat{F}(t_y)_{Rao}$	123.3073	208.7044	109.5606
$\hat{F}(t_y)_{G,K}$	123.3148	208.7171	109.5673
$\hat{F}(t_y)_{Prop1}^{(k_1,k_2)}$	121.8722	210.9814	111.9523
$\hat{F}(t_y)_{Prop2}^{(k_1,k_2)}$	129.5595	228.8673	120.3476

TABLE 11: PREs of distribution function estimators using simulation for populations I, II, and III, when $F(t_y)$ and $F(t_x) = Q_1(y), Q_1(x)$.

Estimators	Population 1 ($Q_1(y), Q_1(x)$)	Population 2 ($Q_1(y), Q_1(x)$)	Population 3 ($Q_1(y), Q_1(x)$)
$\hat{F}(y)$	100	100	100
$\hat{F}(y)_R$	34.68595	29.01847	91.81331
$\hat{F}(t_y)_P$	84.38889	183.5737	33.68077
$\hat{F}(t_y)_{BT,R}$	120.7592	190.3485	58.09967
$\hat{F}(t_y)_{BT,P}$	58.79905	50.79537	124.1445
$\hat{F}(y)_{Reg}$	121.9505	211.4561	124.8422
$\hat{F}(t_y)_{Rao}$	122.8161	211.863	125.2238
$\hat{F}(t_y)_{G,K}$	122.8186	211.8638	125.2243
$\hat{F}(t_y)_{Prop1}^{(k_1,k_2)}$	123.3559	220.4552	128.5401
$\hat{F}(t_y)_{Prop2}^{(k_1,k_2)}$	134.4569	249.8853	134.4446

In Tables 6–9 we use $\bar{Y}, \bar{X}, \tilde{Y}, \tilde{X}, Q_1(y), Q_1(x), Q_3(y)$ and $Q_3(x)$ for indicator functions of Y and X , respectively. Here,

$$F(t_y) = I(Y_i \leq t_y) = \begin{cases} 1 & \text{if } Y_i \leq t_y, \\ 0 & \text{if } Y_i > t_y. \end{cases} \quad (59)$$

$$F(t_x) = I(X_i \leq t_x) = \begin{cases} 1 & \text{if } X_i \leq t_x, \\ 0 & \text{if } X_i > t_x. \end{cases} \quad (60)$$

when we used \bar{Y} as indicator function of Y and \bar{X} as indicator function of X , we get PRE in Table 6, and when we used $Q_1(y)$ as indicator function of Y and $Q_1(x)$ as indicator function of X , we get PRE given in Table 7. And, similarly when we used \tilde{Y} as indicator function of Y and \tilde{X} as indicator function of X , we get PRE in Table 8, and when we used $Q_3(y)$ as indicator function of Y and $Q_3(x)$ as indicator function of X , we get PRE given in Table 9.

Here we take three data sets for numerical illustration, respectively.

In Tables 6–9, we observe that the proposed class of estimators are more precise than the existing estimators in terms of PREs.

TABLE 12: PREs of distribution function estimators using simulation for populations I, II, and III, when $F(t_y)$ and $F(t_x) = \tilde{Y}, \tilde{X}$ or $Q_2(y), Q_2(x)$.

Estimators	Population 1 (\tilde{Y}, \tilde{X})	Population 2 (\tilde{Y}, \tilde{X})	Population 3 (\tilde{Y}, \tilde{X})
$\hat{F}(t_y)$	100	100	100
$\hat{F}(t_y)_R$	35.20506	28.93099	90.53 863
$\hat{F}(t_y)_P$	86.2069	183.8235	34.53039
$\hat{F}(t_y)_{BT,R}$	120.4819	191.5709	58.89282
$\hat{F}(t_y)_{BT,P}$	59.88024	50.55612	124.6883
$\hat{F}(t_y)_{Reg}$	121.4182	212.7587	125.1101
$\hat{F}(t_y)_{Rao}$	122.3191	213.6596	125.6773
$\hat{F}(t_y)_{G,K}$	122.3215	213.6639	125.6783
$\hat{F}(t_y)_{Prop1}^{(k_1,k_2)}$	122.7554	220.0364	128.8903
$\hat{F}(t_y)_{Prop2}^{(k_1,k_2)}$	133.9751	251.1061	136.1882

7. Simulation Study

A simulation study is conducted to obtain the efficiency of the suggested estimators under simple random sampling when the auxiliary variables and rank of the auxiliary variable are used. We have generated three populations of size 1,000 from a multivariate normal distribution with different covariance matrices. All populations have different correlations, i.e., Population I is negatively correlated, Population II is positively correlated, and Population III has a strong positive correlation between X and Y variables. Population averages and covariance matrices are given as follows.

7.1. Population I

$$\mu_1 = \begin{bmatrix} 5 \\ 5 \end{bmatrix}, \quad (61)$$

$$\Sigma_1 = \begin{bmatrix} 4 & 9.6 \\ 9.6 & 64 \end{bmatrix}, \quad (62)$$

$$\rho_{XY} = 0.613800. \quad (63)$$

7.2. Population II

$$\mu_2 = \begin{bmatrix} 5 \\ 5 \end{bmatrix}, \quad (64)$$

$$\Sigma_2 = \begin{bmatrix} 2 & 4 \\ 10 & 4 \end{bmatrix}, \quad (65)$$

$$\rho_{XY} = 0.902645, \quad (66)$$

7.3. Population III

TABLE 13: PREs of distribution function estimators using simulation for populations I, II, and III, when $F(t_y)$ and $F(t_x) = Q_3(y)$, $Q_3(x)$.

Estimators	Population 1 ($Q_3(y), Q_3(x)$)	Population 2 ($Q_3(y), Q_3(x)$)	Population 3 ($Q_3(y), Q_3(x)$)
$\hat{F}(t_y)$	100	100	100
$\hat{F}(t_y)_R$	36.04662	29.47373	66.445174
$\hat{F}(t_y)_P$	77.47934	148.8095	39.0625
$\hat{F}(y)_{BT,R}$	111.6903	170.6485	65.35948
$\hat{F}(t_y)_{BT,P}$	62.31824	52.2466	103.0928
$\hat{F}(t_y)_{Reg}$	114.3888	178.8576	108.5069
$\hat{F}(y)_{Rao}$	114.6891	179.1579	108.8072
$\hat{F}(t_y)_{G,K}$	114.6899	177.6082	108.808
$\hat{F}(t_y)_{Prop1}^{(k_1,k_2)}$	115.6248	177.6082	109.7103
$\hat{F}(t_y)_{Prop2}^{(k_1,k_2)}$	121.7216	201.0239	128.5133

$$\mu_3 = \begin{bmatrix} 5 \\ 5 \end{bmatrix}, \quad (67)$$

$$\Sigma_3 = \begin{bmatrix} 4 & -9.7 \\ -7 & 65 \end{bmatrix}, \quad (68)$$

$$\rho_{XY} = -0628261. \quad (69)$$

The Percentage Relative Efficiency (PRE) is calculated as follows:

$$\text{PRE}(\hat{F}_i(t_y), \hat{F}(t_y)) = \frac{\text{Var}(\hat{F}(t_y))}{\text{MSE}(\hat{F}_i(t_y))_{\min}} \times 100. \quad (70)$$

In this study, we consider the generated population for summarizing the simulation procedures. The simulation results of PRE are given in Tables 10–13.

In Tables 10–13, it can be seen that the proposed estimators perform better than all existing estimators. The percent relative efficiency shows that the second proposed family of estimators with simple random sampling yields the best result when the variables Y and X have a positive correlation. Overall, we can conclude that the performance of the family of suggested estimators is better than all existing estimators.

8. Concluding Remarks

In this article, we proposed a generalized class of exponential factor type estimators, utilizing the supplementary information in the form of the mean and rank of the auxiliary variable for estimating the finite population distribution function. The expressions for biases and mean squared errors of the proposed generalized class of estimators are derived up to the first order of approximation. The proposed estimators $\hat{F}(t_y)_{Prop1}^{(k_1,k_2)}$ and $\hat{F}(t_y)_{Prop2}^{(k_1,k_2)}$ are compared to all existing estimators numerically and theoretically. Based on the simulation studies as well as on the real data sets, it is observed that the proposed class of estimators performed better than their existing counterparts and should be

preferable over the existing estimators available in the literature.

Data Availability

The data used to support the findings of this study are included within the text.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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