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On a Certain Subclass of Analytic Functions Defined by Touchard Polynomials

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Abstract: This paper focuses on the establishment of a new subfamily of analytic functions including Touchard polynomials. Then, we attempt to obtain geometric properties such as coefficient inequalities, distortion properties, extreme points, radii of starlikeness and convexity, partial sums, neighbourhood results and integral means' inequality for this class. The symmetry properties of the subfamily of functions established in the current paper may be examined as future research directions.

Keywords: analytic function; coefficient estimate; starlike; convexity; neighborhood; Touchard polynomial

MSC: 30C45; 30C50; 30C80

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1. Introduction

The application of special functions in Geometric function Theory is a current and interesting topic of research. It is often used in areas such as mathematics, physics, and engineering. As a result of De Branges' study [1], the classic Bieberbach problem is successfully solved by applying a generalized hypergeometric function. Several types of special functions, including generalized hypergeometric Gaussian functions (see [2–4]) and Gegenbauer polynomials, (see [5]) have been studied extensively.

In combinatorics, the Bell numbers B_k $(k \in \mathbb{N} \cup \{0\})$ count the number of ways a set with k elements can be partitioned into disjoint and nonempty subsets. These numbers go back to medieval Japan, but they are named after Eric Temple Bell, who wrote about them in the 1930s [6]. Since then, these numbers have been investigated by mathematicians. The numbers B_k can be generated by

$$e^{e^t-1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!} = 1 + t + t^2 + \frac{5}{6}t^3 + \frac{5}{8}t^4 + \frac{13}{30}t^5 + \frac{203}{720}t^6 + \cdots .$$

For various applications of the Bell polynomials in soliton theory, including links with bilinear and trilinear forms of nonlinear differential equations which possess soliton solutions, we can refer to [7–13] and closely related references therein. Hence, applications of the Bell polynomials to integrable nonlinear equations are considerably expected and any amendment on multilinear forms of soliton equations, even on exact solutions, would be favorable for interested researchers.

The Touchard polynomials are also named the exponential polynomials, and comprise a polynomial sequence of binomial type [14]. This is a new algorithm for solving linear and nonlinear integral equations. These polynomials were studied by Jacques Touchard and he generalized the Bell polynomials in order to examine various problems of enumeration of the permutations when the cycles possess certain properties. Moreover, he introduced and studied a class of related polynomials. An exponential generating function, recurrence relations and connections related to the other known polynomials were also examined. For some special cases, relations with the Stifling number of the first and second kind, as well as with other numbers recently examined, are derived. Finally, a combinatorial interpretation is discussed. Afterwards, various algebraic, combinatorial and probabilistic properties of these polynomials were examined by Chrysaphinou [14], Nazir et al. [15], Paris [16] and Touchard [17].

In general, the integral equations are difficult to be solved analytically, therefore in many equations we need to obtain the approximate solutions, and for this case, the "Touchard Polynomials method" for the solution of the linear "Volterra integro-differential equation" is implemented. The Touchard polynomials method has been applied to solve linear and nonlinear Volterra (Fredholm) integral equations. In a recent study, Abdullah et al. [18] presented two numerical methods based on Touchard and Laguerre polynomials to solve Abel integral equations. Touchard and Laguerre matrices are utilized to transform Abel integral equations into an algebraic system of linear equations. Further, Abdullah and Ali [19] provide some efficient numerical methods to solve linear Volterra integral equations and Volterra Integro differential equations of the first and second types, with exponential, singular, regular and convolution kernels. These methods are based on Touchard and Laguerre polynomials that convert these equations into a system of linear algebraic equations. On the other hand, there have been various papers on interesting applications of the Touchard polynomials in nonlinear Fredholm–Volterra integral equations [20] and soliton theory [8–10], comprising relations between bilinear and trilinear forms of nonlinear differential equations which hold soliton solutions.

Touchard polynomials, also known as an exponential generating polynomials created by Jacques Touchard [17] (see [21–23]) or polynomial sequences of Bell type (see [24,25]), are polynomial sequences of binomial type that represent a random variable X via a Poisson distribution with an expected value $\hbar$. Then, its nth moment is $E(X_\varrho) = \mathfrak{I}(\varrho, \hbar)$, resulting in the type:

$$\mathfrak{I}(\varrho, \hbar) = e^{\varrho} \sum_{\ell=0}^{\infty} \frac{\varrho^\ell \ell^\hbar}{\ell!} w^\ell, \quad w \in U. \tag{1}$$

The result of the second force is presented using the coefficients of Touchard polynomials

$$\phi_\varrho^\hbar(w) = w + \sum_{\ell=2}^{\infty} \frac{(\ell-1)^\hbar \varrho^{\ell-1}}{(\ell-1)!} e^{-\varrho} w^\ell, \quad w \in U, \tag{2}$$

where $\hbar \geq 0, \varrho > 0$ and by analyzing ratio tests, we find that the radius of convergence of the above series is infinity.

On the other hand, special functions, such as Hermite polynomial and Laguerre polynomial, have been studied in quantum optics. The special function operator can be found by converting the function variable into a light-field operator. This is relevant since the occurrence of nonconventional polynomials in describing the properties of light is in the kernel of quantum optics. In recent years, the operations of light-field operators have been generalized to that of a special function's operator. For instance, quite recently, they have been linked to the squeezed states of light. See references [26,27].

Consider H the family of analytic functions in the unit disk $U = \{w : |w| < 1\}$. We will define A as a class of functions $\vartheta \in H$ of the type

$$\vartheta(w) = w + \sum_{\ell=2}^{\infty} a_\ell w^\ell, \quad w \in U. \tag{3}$$

Let S be the subfamily of A that consists of functions that are normalised, $\vartheta(0) = 0 = \vartheta'(0) = 1$, and univalent in U. A's subclass, which consists of functions of the type

$$\vartheta(w) = w + \sum_{\ell=2}^{\infty} a_\ell w^\ell, \;\; a_\ell \geq 0. \tag{4}$$

T signifies the subfamily of S that consists of mapping of the type

$$\vartheta(w) = w - \sum_{\ell=2}^{\infty} a_\ell w^\ell, \;\; a_\ell \geq 0 \text{ and } \;\; w \in U \tag{5}$$

studied extensively by Silverman [28].

For $\vartheta \in A$ given by (4) and $g(w) = w + \sum_{\ell=2}^{\infty} b_\ell w^\ell$, their convolution indicated by $(\vartheta * g)$ is written by

$$(\vartheta * g)(w) = w + \sum_{\ell=2}^{\infty} a_\ell b_\ell w^\ell = (g * \vartheta)(w), \;\; w \in U.$$

The linear operator is now understood

$$K_c^\hbar : A \to A$$

and as a consequence of convolution

$$K_c^\hbar \vartheta(w) = \phi_\varrho^\hbar(w) * \vartheta(w) = w + \sum_{\ell=2}^{\infty} \Lambda_\ell^\hbar a_\ell w^\ell, \tag{6}$$

where $\phi_\varrho^\hbar(w)$ is the series given by (2) and

$$\Lambda_\ell^\hbar = \frac{(\ell-1)^\hbar c^{\ell-1}}{(\ell-1)!} e^{-c}.$$

Now, we establish the class $R_{\hbar,\jmath,\rho,c}^{\tau}(\aleph, \wp)$ of analytic functions by using the operator $K_c^\hbar$.

Definition 1. *A function $\vartheta(w)$ of the type (4) belongs to the class $R_{\hbar,\jmath,\rho,c}^{\tau}(\aleph, \wp)$ if it fulfills*

$$\left| \frac{\rho\left(K_c^\hbar \vartheta(w)\right)' + \jmath w\left(K_c^\hbar \vartheta(w)\right)'' - \rho}{\tau(\aleph - \wp) - \wp\left(\rho K_c^\hbar \vartheta(w)\right)' + \jmath w\left(K_c^\hbar \vartheta(w)\right)'' - \rho} \right| < 1,$$

where $0 \leq \jmath < 1, 0 \leq \rho < 1, \tau \in \mathbb{C} \setminus \{0\}, \hbar > 0, c > 0$ *and* $-1 \leq \wp < \aleph \leq 1$.

The goal of this research is to look at the geometric and symmetric characteristics of this class, such as coefficient inequalities, distortion properties, extreme points, radii of starlikeness and convexity, partial sums, neighbourhoods, and integral means' inequality.

2. Coefficient Inequality

Theorem 1. *The function $\vartheta(w)$ expressed by (4) belongs to the class $R^{\tau}_{\hbar,\jmath,\rho,c}(\aleph,\wp) \Leftrightarrow$*

$$\sum_{\ell=2}^{\infty}(1+\wp)\ell\{\rho+\jmath(\ell-1)\}\Lambda_{\ell}^{\hbar}a_{\ell} \leq |\tau(\aleph-\wp)|. \tag{7}$$

The outcome is sharp for the function

$$\vartheta(w) = w + \frac{|\tau(\aleph-\wp)|}{2(1+\wp)(\rho+\jmath)\Lambda_2^{\hbar}}w^2. \tag{8}$$

Proof. From the definition, for $|w|=1$, we obtain

$$\begin{aligned}
& |\rho\{K_c^{\hbar}\vartheta(w)\}' + \jmath w\{K_c^{\hbar}\vartheta(w)\}'' - \rho| \\
& - \left|\tau(\aleph-\wp) - \wp[\rho\{K_c^{\hbar}\vartheta(w)\}' + \jmath w\{K_c^{\hbar}\vartheta(w)\}'' - \rho]\right| \\
= & \left|\rho\left[1+\sum_{\ell=2}^{\infty}\ell\Lambda_{\ell}^{\hbar}a_{\ell}w^{\ell-1}\right] + \jmath w\left[\sum_{\ell=2}^{\infty}\Lambda_{\ell}^{\hbar}\ell(\ell-1)a_{\ell}w^{\ell-2}\right] - \rho\right| \\
& - \left|\tau(\aleph-\wp) - \wp\left[\rho\left\{1+\sum_{\ell=2}^{\infty}\ell\Lambda_{\ell}^{\hbar}a_{\ell}w^{\ell-1}\right\} + \jmath w\sum_{\ell=2}^{\infty}\ell(\ell-1)\Lambda_{\ell}^{\hbar}a_{\ell}w^{\ell-2} - \rho\right]\right| \\
\leq & \ \rho\sum_{\ell=2}^{\infty}\Lambda_{\ell}^{\hbar}\ell a_{\ell} + \jmath\sum_{\ell=2}^{\infty}\Lambda_{\ell}^{\hbar}\ell(\ell-1)a_{\ell} - |\tau(\aleph-\wp)| \\
& + \wp\left|\rho\sum_{\ell=2}^{\infty}\ell\Lambda_{\ell}^{\hbar}a_{\ell}w^{\ell-1} + \jmath w\sum_{\ell=2}^{\infty}\ell(\ell-1)\Lambda_{\ell}^{\hbar}a_{\ell}w^{\ell-2}\right| \\
\leq & \ \rho\sum_{\ell=2}^{\infty}\ell\Lambda_{\ell}^{\hbar}a_{\ell} + \jmath\sum_{\ell=2}^{\infty}\ell(\ell-1)\Lambda_{\ell}^{\hbar}a_{\ell} - |\tau(\aleph-\wp)| \\
& + \wp\rho\sum_{\ell=2}^{\infty}\ell\Lambda_{\ell}^{\hbar}a_{\ell} + \wp\jmath\sum_{\ell=2}^{\infty}\ell(\ell-1)\Lambda_{\ell}^{\hbar}a_{\ell} \\
\leq & \ (1+\wp)\rho\sum_{\ell=2}^{\infty}\ell\Lambda_{\ell}^{\hbar}a_{\ell} + (1+\wp)\jmath\sum_{\ell=2}^{\infty}\ell(\ell-1)\Lambda_{\ell}^{\hbar}a_{\ell} - |\tau(\aleph-\wp)| \\
\leq & \ 0.
\end{aligned}$$

Thus, we conclude from the maximum modulus theorem that $\vartheta \in R^{\tau}_{\hbar,\jmath,\rho,c}(\aleph,\wp)$. On the contrary, assume

$$\left|\frac{\rho\{K_c^{\hbar}\vartheta(w)\}' + \jmath w\{K_c^{\hbar}\vartheta(w)\}'' - \rho}{\tau(\aleph-\wp) - \wp\rho\{K_c^{\hbar}\vartheta(w)\}' + \jmath w\{K_c^{\hbar}\vartheta(w)\}'' - \rho}\right| < 1$$

that is,

$$\left|\frac{\rho\sum_{\ell=2}^{\infty}\ell\Lambda_{\ell}^{\hbar}a_{\ell}w^{\ell-1} + \jmath\sum_{\ell=2}^{\infty}\ell(\ell-1)\Lambda_{\ell}^{\hbar}a_{\ell}w^{\ell-1}}{\tau(\aleph-\wp) - \wp\left\{\rho\sum_{\ell=2}^{\infty}\ell\Lambda_{\ell}^{\hbar}a_{\ell}w^{\ell-1} + \jmath\sum_{\ell=2}^{\infty}\ell(\ell-1)\Lambda_{\ell}^{\hbar}a_{\ell}w^{\ell-1}\right\}}\right| < 1.$$

Since $|\Re(w)| < |w|$, we find

$$\Re\left[\frac{\sum_{\ell=2}^{\infty}\ell\{\rho+\jmath(\ell-1)\}\Lambda_{\ell}^{\hbar}a_{\ell}w^{\ell-1}}{|\tau(\aleph-\wp)| - \wp\sum_{\ell=2}^{\infty}\ell\{\rho+\jmath(\ell-1)\}\Lambda_{\ell}^{\hbar}a_{\ell}w^{\ell-1}}\right] < 1.$$

By the selection value of w on the real axis so that $K_c^{\hbar}\vartheta(w)$ is real.
If $w \to 1^-$,

$$\sum_{\ell=2}^{\infty} \ell\{\rho + \jmath(\ell-1)\}\Lambda_\ell^{\hbar} a_\ell \leq |\tau(\aleph - \wp)| - \wp \sum_{\ell=2}^{\infty} \ell\{\rho + \jmath(\ell-1)\}\Lambda_\ell^{\hbar} a_\ell$$
$$\leq |\tau(\aleph - \wp)|.$$

□

Corollary 1. *Let $\vartheta(w) \in R^{\tau}_{\hbar,\jmath,\rho,c}(\aleph, \wp)$. Then*

$$a_\ell \leq \frac{|\tau(\aleph - \wp)|}{(1+\wp)\ell\{\rho + \jmath(\ell-1)\}\Lambda_\ell^{\hbar}}, \quad \ell \geq 2.$$

3. Distortion Theorem

Saigo's fractional calculus operator $I_{0,w}^{\alpha,\varsigma,\eta}\vartheta$ of $\vartheta \in A$ is marked with Srivastava et al. [29] (also [30]):

Definition 2. *For real numbers $\alpha > 0, \varsigma$ and η, the fractional integral operator $I_{0,w}^{\alpha,\varsigma,\eta}$ is expressed by*

$$I_{0,w}^{\alpha,\varsigma,\eta}\vartheta(w) = \frac{w^{-\alpha-\varsigma}}{\Gamma(\alpha)} \int_0^w (w-\zeta)^{\alpha-1} {}_2F_1\left[\alpha+\varsigma, -\eta; \alpha; 1 - \frac{\zeta}{w}\right]\vartheta(\zeta)d\zeta,$$

where $\vartheta(w)$ is an analytic function in a simply connected region of the w-plane containing the origin with the order

$$\vartheta(w) = O(|w|^{\epsilon}), \ \ (w \to 0, \epsilon > max\{\{0, \varsigma - \eta\} - 1\})$$

and the multiplicity of $(w-\zeta)^{\alpha-1}$ is removed by requiring $\log(w-\zeta)$ to be real when $w - \zeta > 0$. The mentioned lemma from Srivastava et al. [29] is expected to establish the imbalances using Saigo's fractional operators.

Lemma 1. *Assume $\alpha > 0, \varsigma$ and η are real. If $\ell > max\{0, \varsigma - \eta\} - 1$, we obtain*

$$I_{0,w}^{\alpha,\varsigma,\eta} w^{\ell} = \frac{\Gamma(\ell+1)\Gamma(\ell-\varsigma+\eta+1)}{\Gamma(\ell-\varsigma+1)\Gamma(\ell+\alpha+\eta+1)} w^{\ell-\varsigma}. \tag{9}$$

Theorem 2. *Let $\vartheta \in R^{\tau}_{\hbar,\jmath,\rho,c}(\aleph, \wp)$. Then*

$$|I_{0,w}^{\alpha,\varsigma,\eta}\vartheta(w)| \leq \frac{\Gamma(2-\varsigma+\eta)|w|^{1-\varsigma}}{\Gamma(2-\varsigma)\Gamma(2+\alpha+\eta)}\left[1 + \frac{(2-\varsigma+\eta)|\tau(\aleph-\wp)||w|}{(2-\varsigma)(2+\alpha+\eta)(1+\wp)(\rho+\jmath)\Lambda_2^{\hbar}}\right] \tag{10}$$

and

$$|I_{0,w}^{\alpha,\varsigma,\eta}\vartheta(w)| \geq \frac{\Gamma(2-\varsigma+\eta)|w|^{1-\varsigma}}{\Gamma(2-\varsigma)\Gamma(2+\alpha+\eta)}\left[1 - \frac{(2-\varsigma+\eta)|\tau(\aleph-\wp)||w|}{(2-\varsigma)(2+\alpha+\eta)(1+\wp)(\rho+\jmath)\Lambda_2^{\hbar}}\right]. \tag{11}$$

These equalities are achieved for the function $\vartheta(w)$ given by (8).

Proof. Let $\vartheta \in R^{\tau}_{\hbar,\jmath,\rho,c}(\aleph, \wp)$.

The generalized Saigo [30] fractional integration of $\vartheta \in A$ for real numbers $\alpha > 0, \varsigma$ and η is expressed by

$$I_{0,w}^{\alpha,\varsigma,\eta}\vartheta(w) = \sum_{\ell=1}^{\infty} \frac{\Gamma(\ell+1)\Gamma(\ell-\varsigma+\eta+1)}{\Gamma(\ell-\varsigma+1)\Gamma(\ell+\alpha+\eta+1)} a_\ell w^{\ell-\varsigma}, \quad (a_1 = 1)$$

$$\Rightarrow \frac{\Gamma(2-\varsigma)\Gamma(2+\alpha+\eta)}{\Gamma(2-\varsigma+\eta)} w^{\varsigma} I_{0,w}^{\alpha,\varsigma,\eta}\vartheta(w) = w + \sum_{\ell=2}^{\infty} \wp^{\alpha,\varsigma,\eta}(\ell) a_\ell w^\ell,$$

where

$$\wp^{\alpha,\varsigma,\eta}(\ell) = \frac{\Gamma(\ell+1)\Gamma(\ell-\varsigma+\eta+1)\Gamma(2-\varsigma)\Gamma(2+\alpha+\eta)}{\Gamma(\ell-\varsigma+1)\Gamma(\ell+\alpha+\eta+1)\Gamma(2-\varsigma+\eta)}.$$

Therefore,

$$\frac{\wp^{\alpha,\varsigma,\eta}(\ell)}{\wp^{\alpha,\varsigma,\eta}(\ell+1)} = \frac{(\ell-\varsigma+1)(\ell+\alpha+\eta+1)}{(\ell+1)(\ell-\varsigma+\eta+1)} = \frac{1+\left(\frac{\alpha+\eta}{\ell+1}\right)}{1+\left(\frac{\eta}{\ell-\varsigma+1}\right)}.$$

Now, $(\alpha+\eta) > \eta$ and $\frac{1}{\ell+1} > \frac{1}{\ell-\varsigma+1}$ for $\varsigma < 0$. Theorefore,

$$\frac{\alpha+\eta}{\ell+1} > \frac{\eta}{\ell-\varsigma+1}$$

and

$$\wp^{\alpha,\varsigma,\eta}(\ell) > \wp^{\alpha,\varsigma,\eta}(\ell+1).$$

Thus, we find that $\wp^{\alpha,\varsigma,\eta}(\ell), \varsigma < 0$ is decreasing for ℓ. Then

$$\wp^{\alpha,\varsigma,\eta}(\ell) \le \wp^{\alpha,\varsigma,\eta}(2) = \frac{2(2-\varsigma+\eta)}{(2-\varsigma)(2+\alpha+\eta)}.$$

By using Theorem 1, we have

$$\sum_{\ell=2}^{\infty} a_\ell \le \frac{|\tau(\aleph-\wp)|}{2(1+\wp)(\rho+\jmath)\Lambda_2^{\hbar}}, \quad \ell \ge 2.$$

Thus

$$\left|\frac{\Gamma(2-\varsigma)\Gamma(2+\alpha+\eta)}{\Gamma(2-\varsigma+\eta)} w^{\varsigma} I_{0,w}^{\alpha,\varsigma,\eta}\vartheta(w)\right| \le |w| + \wp^{\alpha,\varsigma,\eta}(2)|w|^2 \sum_{\ell=2}^{\infty} a_\ell$$

$$\left|I_{0,w}^{\alpha,\varsigma,\eta}\vartheta(w)\right| \le \frac{\Gamma(2-\varsigma+\eta)|w|^{1-\varsigma}}{\Gamma(2-\varsigma)(2+\alpha+\eta)}\left[1+\frac{(2-\varsigma+\eta)|\tau(\aleph-\wp)||w|}{(2-\varsigma)(2+\alpha+\eta)(1+\wp)(\rho+\jmath)\Lambda_2^{\hbar}}\right].$$

We discover this by repeating the preceding procedures

$$\left|I_{0,w}^{\alpha,\varsigma,\eta}\vartheta(w)\right| \ge \frac{\Gamma(2-\varsigma+\eta)|w|^{1-\varsigma}}{\Gamma(2-\varsigma)\Gamma(2+\alpha+\eta)}\left[1-\frac{(2-\varsigma+\eta)|\tau(\aleph-\wp)||w|}{(2-\varsigma)(2+\alpha+\eta)(1+\wp)(\rho+\jmath)\Lambda_2^{\hbar}}\right].$$

□

4. Extreme Point

Theorem 3. *Let $\vartheta_1(w) = w$ and $\vartheta_\ell(w) = w + \frac{|\tau(\aleph-\wp)|}{\ell(1+\wp)\{\rho+\jmath(\ell-1)\}\Lambda_\ell^{\hbar}} w^\ell$. Then $\vartheta \in R_{\hbar,\jmath,\rho,c}^{\tau}(\aleph,\wp) \Leftrightarrow$ $\vartheta(w)$ can be described in the following way*

$$\vartheta(w) = \lambda_1\vartheta_1(w) + \sum_{\ell=2}^{\infty} \lambda_\ell \vartheta_\ell(w), \tag{12}$$

where

$$\lambda_1 + \sum_{\ell=2}^{\infty} \lambda_\ell = 1, \quad (\lambda_1 \ge 0, \lambda_\ell \ge 0).$$

Proof. Let $\vartheta(w)$ be given by (7). Then

$$\begin{aligned}\vartheta(w) &= \lambda_1 w + \sum_{\ell=2}^{\infty} \lambda_\ell w^\ell + \frac{|\tau(\aleph-\wp)|}{\ell(1+\wp)\{\rho+\jmath(\ell-1)\}\Lambda_\ell^\hbar}\lambda_\ell w^\ell \\ &= w + \sum_{\ell=2}^{\infty} t_\ell w^\ell,\end{aligned}$$

where $t_\ell = \frac{|\tau(\aleph-\wp)|\lambda_\ell}{\ell(1+\wp)\{\rho+\jmath(\ell-1)\}\Lambda_\ell^\hbar}$. Now,

$$\begin{aligned}\sum_{\ell=2}^{\infty} \frac{\ell(1+\wp)\{\rho+\jmath(\ell-1)\}\Lambda_\ell^\hbar}{|\tau(\aleph-\wp)|} t_\ell = \sum_{\ell=2}^{\infty} \lambda_\ell \\ =1-\lambda_1 < 1.\end{aligned}$$

Therefore, $\vartheta \in R^{\tau}_{\hbar,\jmath,\rho,c}(\aleph,\wp)$.

Conversely, assume $\vartheta \in R^{\tau}_{\hbar,\jmath,\rho,c}(\aleph,\wp)$. Then by (7)

$$a_\ell < \frac{|\tau(\aleph-\wp)|}{\ell(1+\wp)\{\rho+\jmath(\ell-1)\}\Lambda_\ell^\hbar}, \quad \ell \geq 2.$$

So, if we set

$$\lambda_\ell = \frac{\ell(1+\wp)\{\rho+\jmath(\ell-1)\}\Lambda_\ell^\hbar a_\ell}{|\tau(\aleph-\wp)|} < 1, \ \ \ell \geq 2$$

and $\lambda_1 = 1 - \sum_{\ell=2}^{\infty} \lambda_\ell$, then,

$$\vartheta(w) = w + \sum_{\ell=2}^{\infty} a_\ell w^\ell = w + \sum_{\ell=2}^{\infty} \frac{|\tau(\aleph-\wp)|}{\ell(1+\wp)\{\rho+\jmath(\ell-1)\}\Lambda_\ell^\hbar} w^\ell,$$

$$\vartheta(w) = \lambda_1 \vartheta_1(w) + \sum_{\ell=2}^{\infty} \lambda_\ell \vartheta_\ell(w),$$

which leads to (12). □

5. Radii of Starlikeness and Convexity

Theorem 4. *Assume $\vartheta \in R^{\tau}_{\hbar,\jmath,\rho,c}(\aleph,\wp)$. Then $\vartheta(w)$ is starlike of order α $(0 \leq \alpha < 1)$ in $|w| < r_1$, where*

$$r_1 = \inf_\ell \left[\frac{(1-\alpha)\ell(1+\wp)\{\rho+\jmath(\ell-1)\}\Lambda_\ell^\hbar}{(\ell-\alpha)|\tau(\aleph-\wp)|}\right]^{\frac{1}{\ell-1}}.$$

Proof. For $0 \leq \alpha < 1$, we have to show that

$$\left|\frac{w\vartheta'(w)}{\vartheta(w)} - 1\right| < 1-\alpha.$$

That is, for $\vartheta(w) = w + \sum_{\ell=2}^{\infty} a_\ell w^\ell$,

$$\frac{\sum_{\ell=2}^{\infty} a_\ell(\ell-1)|w|^{\ell-1}}{1-\sum_{\ell=2}^{\infty} a_\ell|w|^{\ell-1}} < 1-\alpha$$

or, alternatively $\sum_{\ell=2}^{\infty} a_\ell\left(\frac{\ell-\alpha}{1-\alpha}\right)|w|^{\ell-1} < 1$, which holds if

$$|w|^{\ell-1} < \left[\frac{(1-\alpha)\ell(1+\wp)\rho+\jmath(\ell-1)\Lambda_\ell^\hbar}{(\ell-\alpha)|\tau(\aleph-\wp)|}\right],$$

$$r_1 = \inf_\ell\left[\frac{(1-\alpha)\ell(1+\wp)\rho+\jmath(\ell-1)\Lambda_\ell^\hbar}{(\ell-\alpha)|\tau(\aleph-\wp)|}\right]^{\frac{1}{\ell-1}}.$$

□

Noting the fact that $\vartheta(w)$ is convex $\Leftrightarrow$ $w\vartheta'(w)$ is starlike, we arrive at Theorem 5.

Theorem 5. *Let $\vartheta \in R^\tau_{\hbar,\jmath,\rho,c}(\aleph,\wp)$. Then, ϑ is convex of order α $(0 \le \alpha < 1)$ in $|w| < r_2$, where*

$$r_2 = \inf_\ell\left[\frac{(1-\alpha)(1+\wp)\{\rho+\jmath(\ell-1)\}\Lambda_\ell^\hbar}{(\ell-\alpha)|\tau(\aleph-\wp)|}\right]^{\frac{1}{\ell-1}}.$$

6. Partial Sums

Inspired by the work of Silverman [31] and Silvia [32], we explore partial sums of functions in $R\hbar,\jmath,\rho,c^\tau(\aleph,\wp)$ and derive sharp lower limits on the ratios of real component of $\vartheta(w)$ to $\vartheta q(w)$ and $\vartheta'(w)$ to $\vartheta' q(w)$.

Theorem 6. *Assume $\vartheta(w) \in R^\tau_{\hbar,\jmath,\rho,c}(\aleph,\wp)$ is given by (4). Consider the partial sums $\vartheta_1(w)$ and $\vartheta_q(w)$ by*

$$\vartheta_1(w) = w \text{ and } \vartheta_q(w) = w + \sum_{\ell=2}^{q} a_\ell w^\ell,\ (q \in \mathbb{N}\setminus\{1\}).$$

Assume $\sum_{\ell=2}^{\infty} d_\ell|a_\ell| \le 1$, where

$$d_\ell = \frac{(1+\wp)\ell\{\rho+\jmath(\ell-1)\}\Lambda_\ell^\hbar}{|\tau(\aleph-\wp)|}. \tag{13}$$

Then, $\vartheta \in R^\tau_{\hbar,\jmath,\rho,c}(\aleph,\wp)$. Furthermore,

$$\Re\left[\frac{\vartheta(w)}{\vartheta_q(w)}\right] > 1-\frac{1}{d_{q+1}},\ w \in U,\ q \in \mathbb{N} \tag{14}$$

and

$$\Re\left[\frac{\vartheta_q(w)}{\vartheta(w)}\right] > \frac{d_{q+1}}{1+d_{q+1}}. \tag{15}$$

Proof. For the above coefficients d_ℓ, ensure that $d_{\ell+1} > d_\ell > 1$. As a consequence, by employing the assertion (13), we find

$$\sum_{\ell=2}^{q} |a_\ell| + d_{q+1} \sum_{\ell=q+1}^{\infty} |a_\ell| \leq \sum_{\ell=2}^{\infty} d_\ell |a_\ell| \leq 1. \tag{16}$$

By setting

$$\begin{aligned} g_1(w) &= d_{q+1}\left[\frac{\vartheta(w)}{\vartheta_q(w)} - \left(1 - \frac{1}{d_{q+1}}\right)\right] \\ &= 1 + \frac{d_{q+1} \sum_{\ell=q+1}^{\infty} a_\ell w^{\ell-1}}{1 + \sum_{\ell=2}^{q} a_\ell w^{\ell-1}} \end{aligned}$$

and applying (16), we find that

$$\left|\frac{g_1(w)-1}{g_1(w)+1}\right| \leq \frac{d_{q+1} \sum_{\ell=q+1}^{\infty} |a_\ell|}{2 - 2\sum_{\ell=2}^{q} |a_\ell| - d_{q+1} \sum_{\ell=q+1}^{\infty} |a_\ell|} \leq 1,$$

which yields the assertion (14).

In order to see that

$$\vartheta(w) = w + \frac{w^{q+1}}{d_{q+1}} \tag{17}$$

gives sharp outcome, we recognise for $w = re^{\frac{i\pi}{q}}$ that

$$\frac{\vartheta(w)}{\vartheta_q(w)} = 1 + \frac{w^q}{d_{q+1}} \to 1 - \frac{1}{d_{q+1}} \text{ as } w \to 1^-.$$

Similarly, if we take

$$\begin{aligned} g_2(w) &= (1 + d_{q+1})\left(\frac{\vartheta_q(w)}{\vartheta(w)} - \frac{d_{q+1}}{1 + d_{q+1}}\right) \\ &= 1 - \frac{(1 + d_{\ell+1}) \sum_{\ell=q+1}^{\infty} a_\ell w^{\ell-1}}{1 + \sum_{\ell=2}^{\infty} a_\ell w^{\ell-1}} \end{aligned}$$

and making use of (16), we conclude that

$$\left|\frac{g_2(w)-1}{g_2(w)+1}\right| \leq \frac{(1 + d_{q+1}) \sum_{\ell=q+1}^{\infty} |a_\ell|}{2 - 2\sum_{\ell=2}^{q} |a_\ell| - (1 - d_{q+1}) \sum_{\ell=q+1}^{\infty} |a_\ell|} \leq 1,$$

which leads to the assertion (15).

The bound in (15) is sharp for each $q \in \mathbb{N}$ with the external function $\vartheta(w)$ given by (17). □

Theorem 7. *If $\vartheta(w)$ of the form (4) fulfills the condition (7), then*

$$\Re\left[\frac{\vartheta'(w)}{\vartheta'_q(w)}\right] \geq 1 - \frac{q+1}{d_{q+1}}.$$

Proof. By setting

$$\begin{aligned} g_3(w) &= d_{q+1}\left[\frac{\vartheta'(w)}{\vartheta'_q(w)}\right] - \left(1 - \frac{q+1}{d_{q+1}}\right) \\ &= \frac{1 + \frac{d_{q+1}}{q+1}\sum\limits_{\ell=q+1}^{\infty} \ell a_\ell w^{\ell-1} + \sum\limits_{\ell=2}^{\infty} \ell a_\ell w^{\ell-1}}{1 + \sum\limits_{\ell=2}^{\infty} \ell a_\ell w^{\ell-1}} \\ &= 1 + \frac{\frac{d_{q+1}}{q+1}\sum\limits_{\ell=q+1}^{\infty} \ell a_\ell w^{\ell-1}}{1 + \sum\limits_{\ell=2}^{\infty} \ell a_\ell w^{\ell-1}} \end{aligned}$$

$$\Longrightarrow \quad \left|\frac{g_3(w)-1}{g_3(w)+1}\right| \leq \frac{\frac{d_{q+1}}{q+1}\sum\limits_{\ell=q+1}^{\infty} \ell|a_\ell|}{2 - 2\sum\limits_{\ell=2}^{q} \ell|a_\ell| - \frac{d_{q+1}}{q+1}\sum\limits_{\ell=q+1}^{\infty} \ell|a_\ell|}.$$

Now, $\left|\frac{g_3(w)-1}{g_3(w)+1}\right| \leq 1$, if

$$\sum_{\ell=2}^{q} \ell|a_\ell| + \frac{d_{q+1}}{q+1}\sum_{\ell=q+1}^{\infty} \ell|a_\ell| \leq 1. \tag{18}$$

Since the L.H.S of (18) is bounded above by $\sum\limits_{\ell=2}^{q} d_\ell|a_\ell|$ if

$$\sum_{\ell=2}^{q} (d_\ell - \ell)|a_\ell| - \frac{d_{q+1}}{q+1}\sum_{\ell=q+1}^{\infty} \ell|a_\ell| \geq 0 \tag{19}$$

and the proof is complete. □

The outcome is sharp for the extremal function $\vartheta(w) = w + \frac{w^{q+1}}{d_{q+1}}$.

Theorem 8. *If $\vartheta(w)$ of the type (4) fulfills the condition (7), then*

$$\Re\left[\frac{\vartheta'_q(w)}{\vartheta'(w)}\right] \geq \frac{d_{q+1}}{q+1+d_{q+1}}.$$

Proof. By setting

$$\begin{aligned} g_4(w) &= [q+1+d_{q+1}]\left[\frac{\vartheta'_q(w)}{\vartheta'(w)} - \frac{d_{q+1}}{q+1+d_{q+1}}\right] \\ &= 1 - \frac{\left(1 + \frac{d_{q+1}}{q+1}\right)\sum\limits_{\ell=q+1}^{\infty} \ell a_\ell w^{\ell-1}}{1 + \sum\limits_{\ell=2}^{q} \ell a_\ell w^{\ell-1}} \end{aligned}$$

and making use of (19), we conclude that

$$\left|\frac{g_4(w)-1}{g_4(w)+1}\right| \leq \frac{\left(1+\frac{d_{q+1}}{q+1}\right)\sum_{\ell=q+1}^{\infty}\ell|a_\ell|}{2-2\sum_{\ell=2}^{q}\ell|a_\ell|-\left(1+\frac{d_{q+1}}{q+1}\right)\sum_{\ell=q+1}^{\infty}\ell|a_\ell|} \leq 1,$$

which proves Theorem 8. □

7. Neighborhood Result

Rucscheweyh [33] developed and investigated the concept of analytic function neighborhood, which is stated clearly.

Definition 3. *For $\vartheta \in A$ of the type (4) and $\mu \geq 0$, we establish a (n,μ)-neighborhood of a mapping ϑ by*

$$N_{n,\mu}(\vartheta) = \left\{g : g \in A, g(w) = w + \sum_{\ell=n+1}^{\infty} b_\ell w^\ell \text{ and } \sum_{\ell=n+1}^{\infty} \ell|a_\ell - b_\ell| \leq \mu\right\}. \tag{20}$$

Particularly, for the identity function $e(w) = w$, we arrive

$$N_{n,\mu}(e) = \left\{g : g \in A, g(w) = w + \sum_{\ell=n+1}^{\infty} b_\ell w^\ell \text{ and } \sum_{\ell=n+1}^{\infty} \ell|b_\ell| \leq \mu\right\}, \tag{21}$$

where $n \in \mathbb{N} \setminus \{1\}$.

Theorem 9. *Let $\vartheta \in R^{\tau}_{\hbar,\jmath,\rho,c}(\aleph,\wp)$. If*

$$\mu = \frac{|\tau(\aleph-\wp)|}{(1+\wp)(\rho+n\jmath)\Lambda^{\hbar}_{n+1}},$$

then

$$R^{\tau}_{\hbar,\jmath,\rho,c}(\aleph,\wp) \subset N_{n,\mu}(e).$$

Proof. For a function $\vartheta \in R^{\tau}_{\hbar,\jmath,\rho,c}(\aleph,\wp)$ of the type (4), Theorem 1 immediately yields

$$\sum_{\ell=n+1}^{\infty}(1+\wp)\ell\{\rho+\jmath(\ell-1)\}\Lambda^{\hbar}_{\ell}a_\ell \leq |\tau(\aleph-\wp)|, \text{ where}, n \in \mathbb{N}\setminus\{1\}$$

$$(1+\wp)(\rho+n\jmath)\Lambda^{\hbar}_{n+1}\sum_{\ell=n+1}^{\infty}\ell a_\ell \leq |\tau(\aleph-\wp)|$$

$$\sum_{\ell=n+1}^{\infty}\ell a_\ell \leq \frac{|\tau(\aleph-\wp)|}{(1+\wp)(\rho+n\jmath)\Lambda^{\hbar}_{n+1}} = \mu.$$

□

A mapping $\vartheta \in A$ belongs to the class $R^{\tau,\alpha}_{\hbar,\jmath,\rho,c}(\aleph,\wp)$, if there exists a mapping $g \in R^{\tau}_{\hbar,\jmath,\rho,c}(\aleph,\wp)$ such that

$$\left|\frac{\vartheta(w)}{g(w)} - 1\right| < 1-\alpha, \quad (w \in U, \ 0 < \alpha < 1). \tag{22}$$

Now, we determine the neighborhood for the class $R^{\tau,\alpha}_{\hbar,\jmath,\rho,c}(\aleph,\wp)$.

Theorem 10. *If* $g \in R^{\tau}_{\hbar,\jmath,\rho,c}(\aleph,\wp)$ *and*

$$\alpha = 1 - \frac{\mu(1+\wp)(\rho+n\jmath)\Lambda^{\hbar}_{n+1}}{n(1+\wp)(\rho+n\jmath)\Lambda^{\hbar}_{n+1} - |\tau(\aleph-\wp)|}, \tag{23}$$

then

$$N_{n,\mu}(g) \subset R^{\tau,\alpha}_{\hbar,\jmath,\rho,c}(\aleph,\wp).$$

Proof. Assume $\vartheta \in N_{n,\mu}(g)$. From the Definition 3, we arrive at

$$\sum_{\ell=n+1}^{\infty} \ell|a_\ell - b_\ell| \leq \mu,$$

which implies

$$\sum_{\ell=n+1}^{\infty} |a_\ell - b_\ell| \leq \frac{\mu}{n+1}, \quad n \in \mathbb{N}.$$

Next, since $g \in R^{\tau}_{\hbar,\jmath,\rho,c}(\aleph,\wp)$, we have

$$\sum_{\ell=n+1}^{\infty} b_\ell \leq \frac{|\tau(\aleph-\wp)|}{(n+1)(1+\wp)(\rho+n\jmath)\Lambda^{\hbar}_{n+1}}.$$

Now,

$$\begin{aligned}
\left|\frac{\vartheta(w)}{g(w)} - 1\right| &\leq \frac{\sum\limits_{\ell=n+1}^{\infty} |a_\ell - b_\ell|}{1 - \sum\limits_{\ell=n+1}^{\infty} b_\ell} \\
&\leq \frac{\mu}{(n+1)\left[1 - \frac{|\tau(\aleph-\wp)|}{(n+1)(1+\wp)(\rho+n\jmath)\Lambda^{\hbar}_{n+1}}\right]} \\
&\leq \frac{\mu(1+\wp)(\rho+n\jmath)\Lambda^{\hbar}_{n+1}}{(n+1)(1+\wp)(\rho+n\jmath)\Lambda^{\hbar}_{n+1} - |\tau(\aleph-\wp)|} \\
&\leq 1-\alpha,
\end{aligned}$$

provided that α is given precisely by (23).

Thus, $\vartheta \in R^{\tau,\alpha}_{\hbar,\jmath,\rho,c}(\aleph,\wp)$ for α given by (23). □

8. Integral Means' Inequality

Silverman [28] (see, e.g., [34]) obtained that the mapping $\vartheta_2(w) = w - \frac{w^2}{2}$ is often external over the family T and used this mapping to resolve the integral means' inequality, estimated in [35],

$$\int_0^{2\pi} |\vartheta(re^{i\theta})|^{\eta} d\theta \leq \int_0^{2\pi} |\vartheta_2(re^{i\theta})|^{\eta} d\theta, \tag{24}$$

for all $\vartheta \in T, \eta > 0$ and $0 < r < 1$. Afterwards, he displayed the proposition for the subfamilies $S^*(\alpha)$ and $K(\alpha)$ of T.

Lemma 2 ([36]). *If* ϑ, g *are analytic in* U *with* $\vartheta(w) \prec g(w)$, *then*

$$\int_0^{2\pi} |\vartheta(re^{i\theta})|^{\eta} d\theta \leq \int_0^{2\pi} |g(re^{i\theta})|^{\eta} d\theta, \tag{25}$$

where $\eta \geq 0$, $w = re^{i\theta}$ *and* $0 < r < 1$.

Application of Lemma 2 to the mapping of ϑ in $R^{\tau}_{\hbar,\jmath,\rho,c}(\aleph,\wp)$, yields the next outcome.

Theorem 11. *Assume $\eta > o$. If $\vartheta \in R^{\tau}_{\hbar,\jmath,\rho,c}(\aleph,\wp)$ is given by (4) and $\vartheta_2(w)$ is defined by*

$$\begin{aligned}\vartheta_2(w) &= w + \frac{|\tau(\aleph-\wp)|}{2(1+\wp)(\rho+\jmath)\Lambda_2^{\hbar}}w^2 \\ &= w + \frac{1}{\phi_{\wp}^{\aleph}(2,\hbar,\jmath,\rho,c,\tau)}w^2,\end{aligned} \tag{26}$$

where $\phi_{\wp}^{\aleph}(2,\hbar,\jmath,\rho,c,\tau) = \frac{2(1+\wp)(\rho+\jmath)\Lambda_2^{\hbar}}{|\tau(\aleph-\wp)|}$, then

$$\int_0^{2\pi}|\vartheta(w)|^{\eta}d\theta \le \int_0^{2\pi}|\vartheta_2(w)|^{\eta}d\theta,\ \ for\ \ w = re^{i\theta}, 0<r<1. \tag{27}$$

Proof. Mapping ϑ of the type (4) is equivalent to proving that

$$\int_0^{2\pi}\left|1+\sum_{\ell=2}^{\infty}a_{\ell}w^{\ell-1}\right|^{\eta}d\theta \le \int_0^{2\pi}\left|1+\frac{1}{\phi_{\wp}^{\aleph}(2,\hbar,\jmath,\rho,c,\tau)}w\right|^{\eta}d\theta.$$

By Lemma 2, it is enough to show that

$$1+\sum_{\ell=2}^{\infty}a_{\ell}w^{\ell-1} \prec 1+\frac{1}{\phi_{\wp}^{\aleph}(2,\hbar,\jmath,\rho,c,\tau)}w.$$

Setting

$$1+\sum_{\ell=2}^{\infty}a_{\ell}w^{\ell-1} = 1+\frac{1}{\phi_{\wp}^{\aleph}(2,\hbar,\jmath,\rho,c,\tau)}w(w)$$

and from Theorem 7, we obtain

$$|w(w)| \le \left|\sum_{\ell=2}^{\infty}\phi_{\wp}^{\aleph}(2,\hbar,\jmath,\rho,c,\tau)a_{\ell}w^{\ell-1}\right| \le |w|\sum_{\ell=2}^{\infty}\phi_{\wp}^{\aleph}(2,\hbar,\jmath,\rho,c,\tau)a_{\ell} \le |w|$$

which completes the proof. □

9. Conclusions

The Touchard polynomials draw the attention of many researchers. They have many applications in the theory of geometric function. In particular, the Touchard polynomial $T_n(\hbar)$ is the nth moment of a random variable X which has Poisson distribution and expected value $\hbar$. Hence, we analyze and give some essential information on the properties of the Touchard polynomials in Geometric Function Theory. Firstly, we establish of a new subfamily of analytic functions including Touchard polynomials. Afterwards, we obtain coefficient inequalities, distortion properties, extreme points, radii of starlikeness and convexity, partial sums and neighbourhood outcomes. Finally, integral means' inequality related to Touchard polynomials are obtained.

The theory of Touchard polynomials is analyzed in the framework of operational techniques. The interest in such polynomials is pointed out, taking into account their explicit relations, integral representations, and summation formulae. This research can be continued by using the other special families of polynomials and extended to find new relations for generalized polynomials.

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