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Extended dissipativity synchronization for Markovian jump recurrent neural networks via memory sampled-data control and its application to circuit theory

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Abstract

The problem of synchronization with extended dissipativity for Markovian Jump Recurrent Neural Networks (MJRNNs) is investigated. For MJRNNs, a new memory sampled - data extended dissipative control approach is suggested here. Some sufficient conditions in terms of Linear Matrix Inequalities (LMIs) are acquired by suitably establishing a relevant Lyapunov - Krasovskii functional (LKF), wherein the master and the slave system of MJRNNs are quadratically stable. At last, a numerical section is provided, along with one of the applications in circuit theory that clearly illustrates the efficacy of the proposed method's performance.

Keywords: Extended Dissipativity, Markovian Jump Recurrent Neural Networks, Memory sampled - data control, Synchronization.
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1. Introduction

In the last few decades, Neural Networks (NNs), primarily Recurrent NNs [25], Cellular NNs [20] and Hopfield NNs [1, 4, 5] have been skillfully utilized in signal processing, image analysis, cognition,

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fault detection, inferencing, and optimization techniques [15, 7, 32]. Interpretation of the stabilisation of the equilibrium point of NNs is required in all of these research works. NNs may exert network mode switching (jumping); i.e, a NN may have discrete modes that toggle between them at distinct intervals. It is demonstrated that a Markovian chain can anticipate such type of jumping. Numerous notable results on the processing and analysis of Markovian jump systems [13, 23, 39, 40] and also the results of dynamical analyses on Markovian Jump Recurrent Neural Networks (MJRNNs) can indeed be found in the literature [2]. It really is important to note that the references cited earlier in this section assume that all data about transition probabilities in the jumping technique is correct.

Because time delays are a common origin of the system's instability and poor conduct in NNs [8], a popular area of study is stability analysis for NNs with time delays. Furthermore, the time delay in NNs is caused by the low efficiency of processing information as well as the time it would take neurons to interact [10, 12]. It was recently demonstrated that when the time delay as well as network conditions are properly chosen, NNs could also exert advanced and unforeseen dynamic behavior [21]. The synchronization problem in many coupled systems and networks has been studied, with notable results. Synchronization in coupled delayed NNs had been established to be a vital step across both core science and technical implementation, such as neurosciences, encrypted communications, and harmonic oscillation production [22, 9]. There has been a surge in interest in investigating the dynamical features of NNs due to its universal implementations in many areas in recent times [17].

In recent years, sampled - data control, which has the important and practical advantages of high reliability, high accuracy, and more stable operation, has become a hot - spot for high - speed development of computer technology. By building a fuzzy sampled-data controller with various system has been investigated in [3, 6, 19]. In [28, 34], the researchers investigated the synchronization problem for NNs in non - fragile sampled - data sense with time delays on continuous and discrete time under sampling variable dependent of the input delay approach. Indeed, the investigation into the synchronization problem for Recurrent NNs with multiple time - delays using sampling control for finding a few less conservative conditions to attain the master system in sync with the slave system, which really stands as the principal motivation behind this paper, is valid and fascinating.

In complex systems, the concept of dissipativity [11] is more general and it generalises the concept of a Lyapunov stability function. Furthermore, it has widespread applications in a variety of fields, including stability theory and robust control theory. For this purpose, researchers have recently focused on the extended dissipativity concept [6, 26], which unifies passivity, $(\bar{\mathcal{Q}}-\mathcal{S}-\mathfrak{R})$ dissipativity, H_∞ performance and perfomance of $L_2 - L_\infty$ by fixing the weighting matrices.

The following is an outline of the structure of the paper: Section II contains a brief model description of MJRNNs as well as preliminary results. Section III provides some key results and appropriate conditions for determining controller gains. Section IV includes numerical findings to showcase the applicability of the model designed.

1.1. Notations:

The Euclidean space is denoted as $\mathbb{R}^n$, which is n-dimensional. $\mathbb{N}$ stands for natural numbers and $\mathbb{Z}$ denotes the integers, $\mathbb{R}^{m\times n}$ is the $m \times n$ real matrix and superscript (T) stands for transpose of any matrix. The Euclidean norm in $\mathbb{R}^n$ is given as $\|.\|$. Moreover, $sym(\cdot)$ denotes the symmetric matrix.

2. Problem statement and preliminaries

Let $\Im = \{1, 2, \cdots, s\}$ be a finite set with $\{\mathbf{e}_t, t \geq 0\}$, which is a right continuous Markov process on probability space defined on $\Im$. The transition probability matrix $\Lambda = \Lambda_{ij}$ is given by:

$$P(\mathbf{e}_{t+\Delta} = j|\mathbf{e}_t = i) = \begin{cases} \Lambda_{ij}\Delta + o(\Delta), & i \neq j; \\ 1 + \Lambda_{ij}\Delta + o(\Delta), & i = j; \end{cases}$$

where $\Delta > 0$, $\lim\limits_{\Delta\to 0} \frac{o(\Delta)}{\Delta} = 0$. $\Lambda_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ and

$$\Lambda_{ij} = -\sum_{j=1, j\neq i}^{m} \Lambda_{ij}.$$

Consider the uncertain MJRNNs:

$$\dot{r}(t) = -\tilde{R}(\mathbf{e}_t)r(t) + \tilde{S}(\mathbf{e}_t)g(r(t)) + \sum_{\mathfrak{m}=1}^{n} \tilde{T}_{\mathfrak{m}}(\mathbf{e}_t)g(r(t-\tau_{\mathfrak{m}}(t))) + \mathcal{V}(t), \tag{2.1}$$

where $r(t) = [r_1(t), \cdots, r_n(t)]^T \in \mathbb{R}^n$ represents the neuron state vector and $g(.) = [g_1(.), \cdots, g_n(.)]^T \in \mathbb{R}^n$ indicates activation functions, n represents number of neurons, $\tau_{\mathfrak{m}}(t) = [\tau_{\mathfrak{m}1}(t), \cdots, \tau_{\mathfrak{m}}(t)]^T, \tau_{jk}(t) \geq 0, \mathfrak{m} = 1, 2, \cdots, n, k = 1, 2, \cdots N$ signifies time-varying delays. The external input is $\mathcal{V}(t) = [\mathcal{V}_1(t), \cdots, \mathcal{V}_n(t)]$; $\tilde{R}(\mathbf{e}_t) = R(\mathbf{e}_t) + \Delta R(\mathbf{e}_t), \tilde{S}(\mathbf{e}_t) = S(\mathbf{e}_t) + \Delta S(\mathbf{e}_t), \tilde{T}_{\mathfrak{m}}(\mathbf{e}_t) = T_{\mathfrak{m}}(\mathbf{e}_t) + \Delta T_{\mathfrak{m}}(\mathbf{e}_t)$ for $\mathfrak{m} = 1, 2, \cdots, n$; $R(\mathbf{e}_t), S(\mathbf{e}_t)$ and $T_{\mathfrak{m}}(\mathbf{e}_t)(\mathfrak{m} = 1, \cdots, n)$ are constant matrices and $\Delta R(\mathbf{e}_t), \Delta S(\mathbf{e}_t)$ and $\Delta T_{\mathfrak{m}}(\mathbf{e}_t)$ are time varying parameter uncertainties with suitable dimensions and they are assumed to satisfy

$$[\Delta R(\mathbf{e}_t), \Delta S(\mathbf{e}_t), \Delta T_1(\mathbf{e}_t), \cdots, \Delta T_n(\mathbf{e}_t)] = M(\mathbf{e}_t)C(\mathbf{e}_t)(t)[N_a(\mathbf{e}_t), N_b(\mathbf{e}_t), N_{c1}(\mathbf{e}_t), \cdots, N_{cn}(\mathbf{e}_t)],$$

where $M(\mathbf{e}_t), N_a(\mathbf{e}_t), N_b(\mathbf{e}_t), N_{c1}(\mathbf{e}_t), \cdots, N_{cn}(\mathbf{e}_t)$ are known constant matrices for all $\mathbf{e}_t \in \Im$; unknown matrix function $C(\mathbf{e}_t)(t)$ satisfies $C^T(\mathbf{e}_t)(t)C(\mathbf{e}_t)(t) \leq I$ for all $\mathbf{e}_t \in \Im$. Time - varying delays are bounded as given here:

$$0 \leq \tau_{\mathfrak{m}}(t) \leq \tau_{\mathfrak{m}}, \quad \text{for } \mathfrak{m} = 1, 2, \cdots, n \tag{2.2}$$

and $\dot{\tau}_{\mathfrak{m}}(t) \leq \sigma_{\mathfrak{m}}$, where $\tau_{\mathfrak{m}}$ and $\sigma_{\mathfrak{m}}$ refer to constants.
The neuron activation functions $g_i(\cdot)$ satisfies the following condition:

$$a_i^- \leq \frac{g_i(s_1) - g_i(s_2)}{s_1 - s_2} \leq a_i^+. \tag{2.3}$$

Here, a_i^- and a_i^+ denote constants, $s_1, s_2 \in \mathbb{R}$, $s_1 \neq s_2$ and $g_i(0) = 0$.

The master system is now referred to as (2.1). Then, the corresponding slave system is given as follows:

$$\dot{v}(t) = -\tilde{R}(\mathbf{e}_t)v(t) + \tilde{S}(\mathbf{e}_t)g(v(t)) + \sum_{\mathfrak{m}=1}^{n} \tilde{T}_{\mathfrak{m}}(\mathbf{e}_t)g(v(t-\tau_{\mathfrak{m}}(t))) + \mathcal{V}(t) + \mathfrak{u}(t) + B(\mathbf{e}_t)\bar{\omega}(t), \tag{2.4}$$

where $\mathfrak{u}(t)$ denotes control input and external disturbance is given as $\bar{\omega}(t)$. Consider the model $\mathfrak{c}(t) = v(t) - r(t)$ is taken to be the error system. Then, the synchronization error system is given by:

$$\dot{\mathfrak{c}}(t) = -\tilde{R}(\mathbf{e}_t)\mathfrak{c}(t) + \tilde{S}(\mathbf{e}_t)\hbar(\mathfrak{c}(t)) + \sum_{\mathfrak{m}=1}^{n} \tilde{T}_m(\mathbf{e}_t)\hbar(\mathfrak{c}(t-\tau_m(t))) + \mathfrak{u}(t) + B(\mathbf{e}_t)\bar{\omega}(t), \tag{2.5}$$

where $\hbar(\mathfrak{c}(t)) = g(v(t)) - g(r(t))$. Now, the neuron activation functions $\hbar_i(.)$ satisfies the following condition:

$$a_i^- \leq \frac{\hbar_i(r_i)}{r_i} \leq a_i^+, \tag{2.6}$$

for all $r_i \neq 0$ and $g_i(0) = 0$.

The memory sampled-data control is used to synchronize uncertain MJRNNs and it is constructed as shown below:

$$\mathfrak{u}(t) \equiv K(\mathbf{e}_t)\mathfrak{c}(t_k - \theta), \quad t \in [t_k, t_{k+1}), \tag{2.7}$$

where θ denotes the signal transmission delay and the sampling instants $t_{k+1} - t_k > 0$. If we denote $t_{k+1} - t_k$ by h_k, then we obtain $h_k \leq h$ holds, given $h > 0$ signifies the largest sampling interval. Substituting (2.7) in (2.5), it follows that

$$\dot{\mathfrak{c}}(t) = -\tilde{R}(\mathbf{e}_t)\mathfrak{c}(t) + \tilde{S}(\mathbf{e}_t)\hbar(\mathfrak{c}(t)) + \sum_{\mathfrak{m}=1}^{n} \tilde{T}_{\mathfrak{m}}(\mathbf{e}_t)\hbar(\mathfrak{c}(t - \tau_{\mathfrak{m}}(t))) + K(\mathbf{e}_t)\mathfrak{c}(t_k - \theta) + B(\mathbf{e}_t)\bar{\omega}(t), \tag{2.8}$$

$$f(t) = \mathfrak{c}(t) + \mathfrak{c}(t - \tau_n(t)), \tag{2.9}$$

where $f(t)$ indicates the output error system.

Assumption 2.1. *The following are the conditions satisfied by matrices $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and ε_4:*
1. $\varepsilon_1 = \varepsilon_1^T \leq 0, \varepsilon_3 = \varepsilon_3^T > 0, \varepsilon_4 = \varepsilon_4^T \geq 0$,
2. $(||\varepsilon_1|| + ||\varepsilon_2||) . ||\varepsilon_4|| = 0$.

The following required definitions and lemmas will use in our main results.

Definition 2.1. *[33] Let $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and ε_4 be the matrices which satisfy Assumption (2.1). Then, the NN (2.8) with (2.9) is extended dissipative, if there is $\sigma > 0$ so that, for every $t_l \geq 0$, the subsequent condition holds*

$$\int_0^{t_l} \mathfrak{H}(t)dt \geq \sup \; f^T(t)\varepsilon_4 f(t) + \sigma, 0 \leq t \leq t_l, \tag{2.10}$$

where $\mathfrak{H}(t) = f^T(t)\varepsilon_1 f(t) + 2f^T(t)\varepsilon_2\bar{\omega}(t) + \bar{\omega}^T(t)\varepsilon_3\bar{\omega}(t)$.

Definition 2.2. *[33] The system (2.8) with $\bar{\omega}(t) = 0$ is known to be quadratically stable, if $\varpi > 0$ exists such that $\mathcal{L}V(\mathfrak{c}(t)) \leq -\varpi|\mathfrak{c}(t)|^2$ holds.*

Lemma 2.3. *[24] Given scalars $\gamma \in (0, 1)$, matrix $Z \in \mathbb{R}^{n\times n} > 0$, two matrices P_1 and $P_2 \in \mathbb{R}^{n\times m}$. Define, for $\eta \in \mathbb{R}^m$, then, the condition is true:*

$$\varpi(\gamma, Z) = \frac{1}{\gamma}\xi^T P_1^T Z P_1\xi + \frac{1}{1-\gamma}\xi^T P_2^T Z P_2\xi. \tag{2.11}$$

Then, if there is a matrix $E \in \mathbb{R}^{n\times n}$ such that $\begin{bmatrix} Z & E \\ \star & Z \end{bmatrix} > 0$, *then*

$$min_{\gamma\in(0,1)}\varpi(\gamma, Z) \geq \begin{bmatrix} P_1\xi \\ P_2\xi \end{bmatrix}^T \begin{bmatrix} Z & E \\ \star & Z \end{bmatrix} \begin{bmatrix} P_1\xi \\ P_2\xi \end{bmatrix}.$$

Lemma 2.4. *[18] Given* $\mathfrak{c} : [a_1, a_2] \to \mathbb{R}^n$ *a differentiable function. For* $M \in \mathbb{R}^n > 0, G_1, G_2 \in \mathbb{R}^{3n\times n}$*, the inequality holds:*

$$-\int_{a_1}^{a_2} \dot{\mathfrak{c}}^T(r) M \dot{\mathfrak{c}}(r) dr \le \varepsilon^T(a_1, a_2)\psi(a_1, a_2)\varepsilon(a_1, a_2), \tag{2.12}$$

where $\varepsilon(a_1, a_2) = [\mathfrak{c}^T(a_2), \mathfrak{c}^T(a_1), \int_{a_1}^{a_2} \mathfrak{c}^T(r)dr]^T$, $\psi(a_1, a_2) = (a_2-a_1)(G_1M^{-1}G_1^T + \frac{(a_2-a_1)^2}{3}G_2M^{-1}G_2^T)$ $-$ *sym*$[G_2, G_2, 0]) +$ *sym*$([G_1, -G_1, 2G_2])$.

Lemma 2.5. *[19] Let* $A = A^T, \mathfrak{U}, V$ *and* $R = R^T > 0$ *be the given matrix with suitable dimensions. Then,*

$$A + \mathfrak{U}\mathcal{W}(t)V + V^T\mathcal{W}^T(t)\mathfrak{U}^T < 0,$$

for all $\mathcal{W}(t)$ *satisfying* $\mathcal{W}^T(t)\mathcal{W}(t) \le I$ *iff there exists a scalar* $\beta > 0$ *such that*

$$A + \beta\mathfrak{U}\mathfrak{U}^T + \beta^{-1}V^TV < 0.$$

Lemma 2.6. *[27] Let the differentiable function be* $\mathfrak{c} : [c, d] \to \mathbb{R}^n$*, integer* $l \in \mathbb{N}$*,* $k \in \mathbb{Z}_{\ge 0}$*, a matrix* $W > 0 \in \mathbb{R}^{n\times n}$*,* $\xi \in \mathbb{R}^{kn}$*, and matrices* $M_i \in \mathbb{R}^{kn\times n}$*, the succeeding condition holds* $(i = 1, \cdots, m+1)$*:*

$$-\int_c^d \dot{\mathfrak{c}}^T(r) W \dot{\mathfrak{c}}(r) \le \sum_{i=1}^{m+1} \frac{d-c}{2i-1}\xi^T M_i W^{-1} M_i + sym(M_i\psi_{i-1}(c, d))\xi, \tag{2.13}$$

$$\psi_i(c, d) = \begin{cases} \mathfrak{c}(d) - \mathfrak{c}(c), i = 0; \\ \mathfrak{c}(d) - (-1)^i\mathfrak{c}(c) - \sum_{j=1}^{i} l^i \frac{j!}{(d-c)^j}\sigma_{(j-1)}(c, d), i \in \mathbb{N}. \end{cases}$$

$$\sigma_k(c, d) = \int_{r_o}^d \int_{r_1}^d \cdots \int_{r_k}^d \mathfrak{c}(r_{m+1})dr_{m+1}\cdots dr_1, \quad r_o = c;\ \mathfrak{l}_{\mathfrak{j}}^{\mathfrak{i}} = (-1)^{\mathfrak{j}+\mathfrak{i}}\begin{pmatrix} \mathfrak{i} \\ \mathfrak{j} \end{pmatrix}\begin{pmatrix} \mathfrak{i}+\mathfrak{j} \\ \mathfrak{j} \end{pmatrix}.$$

The following remark is derived from above Lemma (2.6) on substituting $k = 1$.

Remark 2.7. *Let* $\mathfrak{c} : [c, d] \to \mathbb{R}^n$ *be a differentiable function. For integers* $k \in \mathbb{N}$*,* $0 < W \in \mathbb{R}^{n\times n}$*, any vector* $\xi \in \mathbb{R}^{kn}$*, and for any matrices* $M_r, M_s \in \mathbb{R}^{kn\times n}$*, the succeeding condition holds:*

$$-\int_c^d \dot{\mathfrak{c}}^T(s) W \dot{\mathfrak{c}}(s) \le \xi^T[(d-c)(M_rW^{-1}M_r^T + \frac{1}{3}M_sW^{-1}M_s^T) + sym(M_r\mathfrak{D}_1 + M_s\mathfrak{D}_2)]\xi, \tag{2.14}$$

where $\mathfrak{D}_1\xi = \mathfrak{c}(d) - \mathfrak{c}(c), \mathfrak{D}_2\xi = \mathfrak{c}(d) + \mathfrak{c}(c) - \frac{2}{d-c}\int_c^d \mathfrak{c}(r)dr$.

Remark 2.8. *To achieve the extended dissipativity condition, the following conditions are the values assigned to the weighting matrices, which helps in getting a general solution.*
(1) When $\varepsilon_1 = \varepsilon_2 = \sigma = 0, \varepsilon_3 = \tilde{\gamma}^2 I$ *and* $\varepsilon_4 = I$*, (2.10) reduces to* $\mathcal{L}_2 - \mathcal{L}_\infty$ *performance.*
(2) When $\varepsilon_1 = -I$*,* $\varepsilon_2 = \varepsilon_4 = \sigma = 0$ *and* $\varepsilon_3 = \tilde{\gamma}^2 I$ *(2.10) becomes* H_∞ *performance.*
(3) When $\varepsilon_1 = \varepsilon_4 = \sigma = 0, \varepsilon_2 = I$ *and* $\varepsilon_3 = \tilde{\gamma}$*, (2.10) executes passivity performance.*
(4) When $\varepsilon_1 = \varepsilon_4 = \sigma = 0$*,* $\varepsilon_2 = I$ *and* $\varepsilon_3 = \tilde{\gamma}$ *(2.10) becomes Mixed* H_∞ *and Passivity performance.*
(5)When $\varepsilon_1 = \bar{\mathcal{Q}}$*,* $\varepsilon_2 = \mathcal{S}, \varepsilon_3 = \mathfrak{R} - \tilde{\alpha}I$ *and* $\varepsilon_4 = 0$*, (2.10) degenerates to* $(\bar{\mathcal{Q}} - \mathcal{S} - \mathfrak{R})$ *Dissipativity performance.*

3. Main results

In this section, the memory sampled-data control is constructed in the following theorems, and few stability conditions are shown to verify whether the error system (2.5) of MJRNNs is synchronized and extended dissipative. The following are some notations used:
$\mathfrak{e}_i = [0_{r\times(i-1)r} \quad I_r \quad 0_{r\times(8+3J-i)r}]^T, i = 1, 2, \cdots, 7+3J,$
$\theta_1(t) = [\mathfrak{c}^T(t), \mathfrak{c}^T(t_k), \int_{t_k}^{t} \mathfrak{c}^T(r)dr]^T, \quad \theta_2(t) = [\mathfrak{c}^T(t_k), \dot{\mathfrak{c}}^T(t), \mathfrak{c}^T(t_k-\theta)]^T, \theta_3(t) = \mathfrak{c}(t) - \mathfrak{c}(t_k-\theta),$
$\theta_4(t) = [\dot{\mathfrak{c}}^T(t), 0, \mathfrak{c}^T(t)]^T, \quad \theta_5(t) = [\mathfrak{c}^T(t_k), 0, \mathfrak{c}^T(t_k-\theta)]^T, \quad \theta_6(t) = \mathfrak{c}(t) - \mathfrak{c}(t-\theta)$

We assume that $\Delta R = \Delta S = \Delta T_1 = \Delta T_2 = ... = \Delta T_n = \Delta K = 0$.

Theorem 3.1. *For given scalars $h > 0$, $\theta > 0$, $\tau_{\mathfrak{m}} > 0$, $\sigma_{\mathfrak{m}} < 1$ and given control gain matrix $K(\mathbf{e}_t)$, the error system attains quadratically stability and extended dissipativity synchronization, if there exist symmetric matrices $Z > 0, \bar{Z}_{22} > 0, Y_i (i = 1, \cdots, 6), Q_i(\mathbf{e}_t)(i = 1, 2, 3), G_1, G_2 \in \mathbb{R}^{(8+3J)n\times n}, M_{rm} > 0, N_{rm} > 0, M_{sm} > 0, N_{sm} > 0$, $\bar{Z} = \begin{pmatrix} \bar{Z}_{11} & \bar{Z}_{12} & \bar{Z}_{13} \\ \star & \bar{Z}_{22} & \bar{Z}_{23} \\ \star & \star & \bar{Z}_{33} \end{pmatrix} > 0$, and diagonal matrices $A > 0, A_1 > 0, \cdots, A_n > 0$ so that the subsequent LMIs hold:*

$$\begin{bmatrix} Y_3 & E \\ \star & Y_3 \end{bmatrix} > 0, \tag{3.1}$$

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \star & \Gamma_{22} \end{bmatrix} > 0, \tag{3.2}$$

$$\Xi = \begin{bmatrix} \Upsilon & \mathfrak{F}_{12} & \mathfrak{F}_{13} & \mathfrak{F}_{14} & \mathfrak{F}_{15} & hG_1 & h^2G_2 \\ \star & \mathfrak{F}_{22} & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \mathfrak{F}_{33} & 0 & 0 & 0 & 0 \\ \star & \star & \star & \mathfrak{F}_{44} & 0 & 0 & 0 \\ \star & \star & \star & \star & \mathfrak{F}_{55} & 0 & 0 \\ \star & \star & \star & \star & \star & -h\bar{Z}_{22} & 0 \\ \star & \star & \star & \star & \star & \star & -3h\bar{Z}_{22} \end{bmatrix} < 0, \tag{3.3}$$

where

$$\Upsilon = \Upsilon_1 + \Upsilon_{2a} + \Upsilon_{2b} + \Upsilon_{3a} + \Upsilon_{3b} + \Upsilon_4 + \Upsilon_5 + \Upsilon_c + \Upsilon_e,$$

$$\begin{aligned}
\Upsilon_1 =& -2\mathfrak{e}_1^T X(\mathbf{e}_t)\mathfrak{e}_4 + 2\mathfrak{e}_{5+J}^T L\mathfrak{e}_4 + \mathfrak{e}_{5+J}^T \sum_{\mathfrak{m}=1}^{n} L_{\mathfrak{m}}\mathfrak{e}_{5+J} - (1-\sigma_1)\mathfrak{e}_{6+J}^T L_1\mathfrak{e}_{6+J} - \cdots - (1-\sigma_n)\mathfrak{e}_{5+2J}^T L_n\mathfrak{e}_{5+2J} \\
&+ \mathfrak{e}_1^T \sum_{\mathfrak{m}=1}^{n} H_{\mathfrak{m}}\mathfrak{e}_1 - (1-\sigma_1)\mathfrak{e}_5^T H_1\mathfrak{e}_5 - \cdots - (1-\sigma_n)\mathfrak{e}_{4+J}^T H_n\mathfrak{e}_{4+J} + \sum_{\mathfrak{m}=1}^{n} \mathfrak{e}_4^T \tau_{\mathfrak{m}} B_{\mathfrak{m}}\mathfrak{e}_4 + \sum_{j\in\Im} \Lambda_{ij}\mathfrak{e}_1^T X(\mathbf{e}_t)\mathfrak{e}_1 \\
&- \aleph_1 + \aleph_2 + sym(\chi_1^T\chi_2),\\
\Upsilon_{2a} =& 2h \begin{bmatrix} \mathfrak{e}_1 \\ \mathfrak{e}_2 \\ \mathfrak{e}_3 \end{bmatrix}^T Z \begin{bmatrix} \mathfrak{e}_4 \\ 0 \\ \mathfrak{e}_1 \end{bmatrix} + h \begin{bmatrix} \mathfrak{e}_2 \\ \mathfrak{e}_4 \\ \mathfrak{e}_{7+2J} \end{bmatrix}^T \bar{Z} \begin{bmatrix} \mathfrak{e}_2 \\ \mathfrak{e}_4 \\ \mathfrak{e}_{7+2J} \end{bmatrix} - \begin{bmatrix} \mathfrak{e}_1 \\ \mathfrak{e}_2 \\ \mathfrak{e}_3 \end{bmatrix}^T Z \begin{bmatrix} \mathfrak{e}_1 \\ \mathfrak{e}_2 \\ \mathfrak{e}_3 \end{bmatrix} - \begin{bmatrix} \mathfrak{e}_2 \\ 0 \\ \mathfrak{e}_{7+2J} \end{bmatrix}^T \\
&\times \bar{Z} \begin{bmatrix} \mathfrak{e}_2 \\ \mathfrak{e}_0 \\ \mathfrak{e}_{7+2J} \end{bmatrix},\\
\Upsilon_{2b} =& (\mathfrak{e}_1 - \mathfrak{e}_2)^T \bar{Z}_{12}^T \mathfrak{e}_2 - (\mathfrak{e}_1 - \mathfrak{e}_2)^T \bar{Z}_{23}\mathfrak{e}_{6+2J},\\
\Upsilon_{3a} =& \mathfrak{e}_1^T (Y_1 + Y_2)\mathfrak{e}_1 - (1-\sigma_1)\mathfrak{e}_5^T Y_1\mathfrak{e}_5 - \cdots - (1-\sigma_{\mathfrak{m}})\mathfrak{e}_{4+J}^T Y_1\mathfrak{e}_{4+J} - \mathfrak{e}_{6+2J}^T Y_2\mathfrak{e}_{6+2J} - \cdots \\
&- \mathfrak{e}_{5+3J}^T Y_2\mathfrak{e}_{5+3J} - \mathfrak{e}_{6+2J}^T Y_2\mathfrak{e}_{6+2J} + \mathfrak{e}_4^T (\sum_{\mathfrak{m}=1}^{n} \tau_{\mathfrak{m}}^2 Y_3)\mathfrak{e}_4,\\
\Upsilon_{3b} =& -\sum_{s=5}^{4+J}\sum_{d=6+2J}^{5+3J} \begin{bmatrix} \mathfrak{e}_1 - \mathfrak{e}_s \\ \mathfrak{e}_s - \mathfrak{e}_d \end{bmatrix}^T \begin{bmatrix} Y_3 & E \\ \star & Y_3 \end{bmatrix} \begin{bmatrix} \mathfrak{e}_1 - \mathfrak{e}_s \\ \mathfrak{e}_s - \mathfrak{e}_d \end{bmatrix} - (\mathfrak{e}_1 - \mathfrak{e}_2)^T Z_{23}\mathfrak{e}_{7+2J},\\
\Upsilon_4 =& \mathfrak{e}_1^T Y_4\mathfrak{e}_1 - \mathfrak{e}_{7+3J}^T Y_4\mathfrak{e}_{7+3J} + \mathfrak{e}_4^T \theta^2\mathfrak{e}_4 - [\mathfrak{e}_1 - \mathfrak{e}_{7+3J}]^T Y_5[\mathfrak{e}_1 - \mathfrak{e}_{7+3J}],\\
\Upsilon_5 =& h^2\mathfrak{e}_4^T Y_6\mathfrak{e}_4 - \frac{\pi^2}{4}(\mathfrak{e}_1 - \mathfrak{e}_{6+3J})^T Y_6(\mathfrak{e}_1 - \mathfrak{e}_{6+3J}),\\
\Upsilon_c =& -2[\varepsilon_1\mathfrak{e}_1 - \mathfrak{e}_{5+J}]^T A[\mathfrak{e}_{5+J} - \varepsilon_2\mathfrak{e}_1] - 2[\varepsilon_1\mathfrak{e}_5 - \mathfrak{e}_{6+J}]^T A_1[\mathfrak{e}_{6+J} - \varepsilon_2\mathfrak{e}_5] - \cdots - 2[\varepsilon_1\mathfrak{e}_{4+J} - \mathfrak{e}_{5+2J}]^T \\
&A_n[\mathfrak{e}_{5+2J} - \varepsilon_2\mathfrak{e}_{4+J}],\\
\Upsilon_e =& [\mathfrak{e}_1 \ \ \mathfrak{e}_{4+J}]^T \varepsilon_1 [\mathfrak{e}_1 \ \ \mathfrak{e}_{4+J}] + 2[\mathfrak{e}_1 \ \ \mathfrak{e}_{4+J}]\varepsilon_2\bar{\omega}^T + \bar{\omega}^T\varepsilon_3\bar{\omega},\\
\Gamma_{11} =& \alpha\mathfrak{P} - \varepsilon_4, \quad \Gamma_{12} = [\Gamma_{12}^1, \Gamma_{12}^2, \cdots, \Gamma_{12}^n], \quad \Gamma_{12}^l = -\varepsilon_4, \quad \Gamma_{22} = [\Gamma_{22}^1, \cdots, \Gamma_{22}^n], \quad \Gamma_{22}^l = (1-\alpha)\mathfrak{P} - \varepsilon_4,\\
\mathfrak{F}_{12} =& \sum_{\mathfrak{m}=1}^{n} \tilde{\tau}_c M_{r\mathfrak{m}}, \quad \mathfrak{F}_{13} = \frac{1}{3}\sum_{\mathfrak{m}=1}^{n} \tilde{\tau}_c M_{s\mathfrak{m}}, \quad \mathfrak{F}_{14} = \sum_{\mathfrak{m}=1}^{n} \tau_{\mathfrak{m}} N_{r\mathfrak{m}}; \quad \mathfrak{F}_{15} = \frac{1}{3}\sum_{\mathfrak{m}=1}^{n} \tau_{\mathfrak{m}} N_{s\mathfrak{m}},\\
\mathfrak{F}_{22} =& -\sum_{\mathfrak{m}=1}^{n} \tilde{\tau}_c B_{\mathfrak{m}}; \quad \mathfrak{F}_{33} = -\frac{1}{3}\sum_{\mathfrak{m}=1}^{n} \tilde{\tau}_c B_{\mathfrak{m}}, \quad \mathfrak{F}_{44} = -\sum_{\mathfrak{m}=1}^{n} \tau_{\mathfrak{m}} B_{\mathfrak{m}}; \quad \mathfrak{F}_{55} = -\frac{1}{3}\sum_{\mathfrak{m}=1}^{n} \tau_{\mathfrak{m}} B_{\mathfrak{m}},\\
\aleph_1 =& sym[G_2, G_2, 0, 0, \cdots, 0], \quad \aleph_2 = sym[G_1, -G_1, 2G_2, 0, \cdots, 0],\\
\chi_1 =& [Q_1(\mathbf{e}_t), Q_2(\mathbf{e}_t), 0, Q_3(\mathbf{e}_t), 0, \cdots, 0],\\
\chi_2 =& [R(\mathbf{e}_t), 0, 0, -I, T_1(\mathbf{e}_t), \cdots, T_n(\mathbf{e}_t), S(\mathbf{e}_t), 0, \cdots, 0, K(\mathbf{e}_t), B(\mathbf{e}_t), 0], \tilde{\tau}_c = \tau_{\mathfrak{m}} - \tau_{\mathfrak{m}}(t).
\end{aligned}$$

Proof . Consider the following LKF:

$$V((\mathbf{e}_t, \mathfrak{c}(t)) = \sum_{k=1}^{5} V_k(\mathbf{e}_t, \mathfrak{c}(t)), \qquad t \in [t_k, t_{k+1}), \tag{3.4}$$

$$\begin{aligned}V_1((\mathbf{e}_t,\mathfrak{c}(t)) =&\mathfrak{c}^T(t)X(\mathbf{e}_t)\mathfrak{c}(t)+2\sum_{k=1}^{N}l_k\int_0^{\mathfrak{c}_k(t)}h_{\mathfrak{m}}(r)dr+\sum_{\mathfrak{m}=1}^{n}\int_{t-\tau_{\mathfrak{m}}(t)}^{t}\Big((\hbar^T(\mathfrak{c}(r))L_{\mathfrak{m}}\hbar(\mathfrak{c}(r))+(\mathfrak{c}^T(r))\\&\times H_{\mathfrak{m}}(\mathfrak{c}(r)))\Big)dr+\sum_{\mathfrak{m}=1}^{n}\int_{-\tau_{\mathfrak{m}}}^{0}\int_{t+s}^{t}\dot{\mathfrak{c}}^T(r)B_{\mathfrak{m}}\dot{\mathfrak{c}}(r)drds,\\
V_2((\mathbf{e}_t,\mathfrak{c}(t)) =&(t_{k+1}-t)\theta_1^TZ\theta_1(t)+(t_{k+1}-t)\int_{t_k}^{t}\theta_2^T(r)\bar{Z}\theta_2(r)dr,\\
V_3((\mathbf{e}_t,\mathfrak{c}(t)) =&\sum_{\mathfrak{m}=1}^{n}\int_{t-\tau_{\mathfrak{m}}(t)}^{t}\mathfrak{c}^T(r)Y_1\mathfrak{c}(r)dr+\sum_{\mathfrak{m}=1}^{n}\int_{t-\tau_{\mathfrak{m}}}^{t}\mathfrak{c}^T(r)Y_2\mathfrak{c}(r)dr+\sum_{\mathfrak{m}=1}^{n}\tau_{\mathfrak{m}}\int_{-\tau_{\mathfrak{m}}}^{0}\int_{t+s}^{t}\dot{\mathfrak{c}}^T(r)Y_3\dot{\mathfrak{c}}(r)drds,\\
V_4((\mathbf{e}_t,\mathfrak{c}(t)) =&\int_{t-\theta}^{t}\mathfrak{c}^T(r)Y_4\mathfrak{c}(r)dr+\theta\int_{-\theta}^{0}\int_{t+s}^{t}\dot{\mathfrak{c}}^T(r)Y_5\dot{\mathfrak{c}}(r)drds,\\
V_5((\mathbf{e}_t,\mathfrak{c}(t)) =&h^2\int_{t_k-\theta}^{t}\dot{\mathfrak{c}}^T(r)Y_6\dot{\mathfrak{c}}(r)dr-\frac{\pi^2}{4}\int_{t_k-\theta}^{t-\theta}\theta_3^T(r)Y_6\theta_3(r)dr,\end{aligned}$$

where $Z=\begin{bmatrix}\dfrac{Z_1^T+Z_1-Z_2^T-Z_2}{2} & Z_2 & Z_3\\ * & \dfrac{-Z_1^T-Z_1-Z_2^T-Z_2}{2} & Z_4\\ * & * & Z_5^T+Z_5\end{bmatrix}$.

The time-derivative of $V_k(\mathbf{e}_t,\mathfrak{c}(t))$ $(k=1,\cdots,5)$ for the system:

$$\begin{aligned}\mathcal{L}V_1(\mathbf{e}_t,\mathfrak{c}(t))\le&-2\mathfrak{c}^T(t)X(\mathbf{e}_t)\dot{\mathfrak{c}}(t)+\sum_{j\in\Im}\Lambda_{ij}\mathfrak{c}^T(t)X(\mathbf{e}_t)\mathfrak{c}(t)+2\hbar^T(\mathfrak{c}(t))L\dot{\mathfrak{c}}(t)+\sum_{\mathfrak{m}=1}^{n}\Big(\hbar^T(\mathfrak{c}(t))\times L_{\mathfrak{m}}\hbar(\mathfrak{c}(t))\\&-(1-\sigma_{\mathfrak{m}})\hbar^T(\mathfrak{c}(t-\tau_{\mathfrak{m}}(t)))L_{\mathfrak{m}}\hbar(\mathfrak{c}(t-\tau_{\mathfrak{m}}(t)))\Big)+\sum_{\mathfrak{m}=1}^{n}\Big(\mathfrak{c}^T(t)H_{\mathfrak{m}}\mathfrak{c}(t)-(1-\sigma_{\mathfrak{m}})\mathfrak{c}^T(t\\&-\tau_{\mathfrak{m}}(t))H_{\mathfrak{m}}\mathfrak{c}(t-\tau_{\mathfrak{m}}(t))\Big)+\sum_{\mathfrak{m}=1}^{n}\tau_{\mathfrak{m}}\dot{\mathfrak{c}}^T(t)B_{\mathfrak{m}}\dot{\mathfrak{c}}(t)-\sum_{\mathfrak{m}=1}^{n}\int_{t-\tau_{\mathfrak{m}}}^{t}\dot{\mathfrak{c}}^T(r)B_{\mathfrak{m}}\dot{\mathfrak{c}}(r)dr.\end{aligned}$$

The last integral term in the previous inequality, by using Remark (2.7) will result as follows:

$$\begin{aligned}-\sum_{\mathfrak{m}=1}^{n}\int_{t-\tau_{\mathfrak{m}}}^{t}\dot{\mathfrak{c}}^T(r)B_{\mathfrak{m}}\dot{\mathfrak{c}}(r)dr=&\sum_{\mathfrak{m}=1}^{n}\left[-\int_{t-\tau_{\mathfrak{m}}}^{t-\tau_{\mathfrak{m}}(t)}\dot{\mathfrak{c}}^T(r)B_{\mathfrak{m}}\dot{\mathfrak{c}}(r)dr-\int_{t-\tau_{\mathfrak{m}}(t)}^{t}\dot{\mathfrak{c}}^T(r)B_{\mathfrak{m}}\dot{\mathfrak{c}}(r)dr\right]\\
\le&\xi^T(t)\Bigg[\sum_{\mathfrak{m}=1}^{n}(\tau_{\mathfrak{m}}-\tau_{\mathfrak{m}}(t))\Big(M_{r\mathfrak{m}}B_{\mathfrak{m}}^{-1}M_{r\mathfrak{m}}^T+\frac{1}{3}M_{s\mathfrak{m}}B_{\mathfrak{m}}^{-1}M_{s\mathfrak{m}}^T\Big)\\
&+\mathrm{sym}(M_{r\mathfrak{m}}\mathfrak{O}_1^{\mathfrak{m}}+M_{s\mathfrak{m}}\mathfrak{O}_2^{\mathfrak{m}})+\sum_{\mathfrak{m}=1}^{n}\tau_{\mathfrak{m}}(t)\Big(N_{r\mathfrak{m}}B_{\mathfrak{m}}^{-1}N_{r\mathfrak{m}}^T\\
&+\frac{1}{3}N_{s\mathfrak{m}}B_{\mathfrak{m}}^{-1}N_{s\mathfrak{m}}^T\Big)+\mathrm{sym}(N_{r\mathfrak{m}}\mathfrak{O}_3^{\mathfrak{m}}+N_{s\mathfrak{m}}\mathfrak{O}_4^{\mathfrak{m}})\Bigg]\xi(t),\end{aligned}$$

where

$$\begin{aligned}
\mathfrak{O}_1^{\mathfrak{m}}\xi =&\mathfrak{c}(t-\tau_{\mathfrak{m}}(t))-\mathfrak{c}(t-\tau_{\mathfrak{m}})),\\
\mathfrak{O}_2^{\mathfrak{m}}\xi =&\mathfrak{c}(t-\tau_{\mathfrak{m}}(t))+\mathfrak{c}(t-\tau_{\mathfrak{m}})-\frac{2}{\tau_{\mathfrak{m}}-\tau_{\mathfrak{m}}(t)}\int_{t-\tau_{\mathfrak{m}}}^{t-\tau_{\mathfrak{m}}(t)}\mathfrak{c}(r)dr,\\
\mathfrak{O}_3^{\mathfrak{m}}\xi =&\mathfrak{c}(t)-\mathfrak{c}(t-\tau_{\mathfrak{m}}(t))\text{and}\\
\mathfrak{O}_4^{\mathfrak{m}}\xi =&\mathfrak{c}(t)+\mathfrak{c}(t-\tau_{\mathfrak{m}}(t))-\frac{2}{\tau_{\mathfrak{m}}(t)}\int_{t-\tau_{\mathfrak{m}}(t)}^{t}\mathfrak{c}(r)dr,\ \text{ for }\ \mathfrak{m}=1,2,\cdots,n.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\mathcal{L}V_1(\mathbf{e}_t,\mathfrak{c}(t)) \leq&\xi^T(t)\Bigg(\Upsilon_1+\sum_{\mathfrak{m}=1}^{n}(\tau_{\mathfrak{m}}-\tau_{\mathfrak{m}}(t))\Bigg(M_{r\mathfrak{m}}B_{\mathfrak{m}}^{-1}M_{r\mathfrak{m}}^T+\frac{1}{3}M_{s\mathfrak{m}}B_{\mathfrak{m}}^{-1}M_{s\mathfrak{m}}^T\Bigg) \qquad (3.5)\\
&+\sum_{\mathfrak{m}=1}^{n}\tau_{\mathfrak{m}}(t)\Bigg(N_{r\mathfrak{m}}B_{\mathfrak{m}}^{-1}N_{r\mathfrak{m}}^T+\frac{1}{3}N_{s\mathfrak{m}}B_{\mathfrak{m}}^{-1}N_{s\mathfrak{m}}^T\Bigg)\Bigg)\xi(t),\\
\mathcal{L}V_2(\mathbf{e}_t,\mathfrak{c}(t)) =&2(t_{k+1}-t)\theta_1^T(t)Z\theta_4(t)-\theta_1^T(t)Z\theta_1(t)+(t_{k+1}-t)\theta_2^T(t)\bar{Z}\theta_2(t)\\
&-(t-t_k)\theta_5^T(t)\bar{Z}\theta_5(t)-(\mathfrak{c}(t)-\mathfrak{c}(t_k))^T\bar{Z}_{12}^T\mathfrak{c}(t_k)-(\mathfrak{c}(t)-\mathfrak{c}(t_k))^T\\
&\times\bar{Z}_{23}\mathfrak{c}(t_k-\theta)-\int_{t_k}^{t}\dot{\mathfrak{c}}^T(r)\bar{Z}_{22}\dot{\mathfrak{c}}(r)dr. \qquad (3.6)
\end{aligned}$$

Using Lemma (2.4) and also due to the reason that $\bar{Z}_{22}>0$, the succeeding inequality shown below has been obtained.

$$-\int_{t_k}^{t}\dot{\mathfrak{c}}(r)\bar{Z}_{22}\dot{\mathfrak{c}}(r)dr \leq\xi^T(t)\aleph\xi(t), \qquad (3.7)$$

where $\aleph=(t-t_k)\bigg(G_1\bar{Z}_{22}^{-1}G_1^T+\frac{h^2}{3}G_2\bar{Z}_{22}^{-1}-\aleph_1\bigg)+\aleph_2$.
Thus, we get

$$\begin{aligned}
&\mathcal{L}V_2(\mathbf{e}_t,\mathfrak{c}(t)) \leq\xi^T(r)(\Upsilon_{2a}+\Upsilon_{2b}+\aleph)\xi(t), \qquad (3.8)\\
&\mathcal{L}V_3(\mathbf{e}_t,\mathfrak{c}(t)) \leq\mathfrak{c}^T(t)(Y_1+Y_2)\mathfrak{c}(t)-\sum_{\mathfrak{m}=1}^{n}(1-\sigma_{\mathfrak{m}})\mathfrak{c}^T(t-\tau_{\mathfrak{m}}(t))Y_1\mathfrak{c}(t-\tau_{\mathfrak{m}}(t))\\
&\qquad-\sum_{\mathfrak{m}=1}^{n}\mathfrak{c}^T(t-\tau_{\mathfrak{m}})Y_2\mathfrak{c}(t-\tau_{\mathfrak{m}})+\dot{\mathfrak{c}}^T(t)\tau_{\mathfrak{m}}^2Y_3\dot{\mathfrak{c}}(t)-\sum_{\mathfrak{m}=1}^{n}\tau_{\mathfrak{m}}\int_{t-\tau_{\mathfrak{m}}}^{t}\dot{\mathfrak{c}}^T(r)Y_3\dot{\mathfrak{c}}(r)dr. \qquad (3.9)
\end{aligned}$$

Using Lemma (2.3), for $\mathfrak{m}=1,2,\cdots,n$, we get

$$\begin{aligned}
-\tau_{\mathfrak{m}}\int_{t-\tau_{\mathfrak{m}}}^{t}\dot{\mathfrak{c}}^T(r)Y_3\dot{\mathfrak{c}}(r)dr =&-\tau_{\mathfrak{m}}\int_{t-\tau_{\mathfrak{m}}(t)}^{t}\dot{\mathfrak{c}}^T(r)Y_3\dot{\mathfrak{c}}(r)dr-\tau_{\mathfrak{m}}\int_{t-\tau_{\mathfrak{m}}}^{t-\tau_{\mathfrak{m}}(t)}\dot{\mathfrak{c}}^T(r)Y_3\dot{\mathfrak{c}}(r)dr,\\
\leq&-\begin{bmatrix}\int_{t-\tau_{\mathfrak{m}}(t)}^{t}\dot{\mathfrak{c}}(r)dr\\ \int_{t-\tau_{\mathfrak{m}}}^{t-\tau_{\mathfrak{m}}(t)}\dot{\mathfrak{c}}(r)dr\end{bmatrix}^T\begin{bmatrix}Y_3 & E\\ \star & Y_3\end{bmatrix}\begin{bmatrix}\int_{t-\tau_{\mathfrak{m}}(t)}^{t}\dot{\mathfrak{c}}(r)dr\\ \int_{t-\tau_{\mathfrak{m}}}^{t-\tau_{\mathfrak{m}}(t)}\dot{\mathfrak{c}}(r)dr\end{bmatrix},
\end{aligned}$$

where E is any matrix with suitable dimension and satisfies $\begin{bmatrix} Y_3 & E \\ \star & Y_3 \end{bmatrix} > 0$.

$$\begin{aligned}
\mathcal{L}V_3(\mathbf{e}_t,\mathfrak{c}(t)) \leq & \xi^T(t)(\Upsilon_{3a}+\Upsilon_{3b})\xi(t), && (3.10)\\
\mathcal{L}V_4(\mathbf{e}_t,\mathfrak{c}(t)) \leq & \mathfrak{c}^T(t)Y_4\mathfrak{c}(t)-\mathfrak{c}^T(t-\theta)Y_4\mathfrak{c}(t-\theta)+\dot{\mathfrak{c}}^T(t)\theta^2Y_5\dot{\mathfrak{c}}(t)-\theta_6^T(t)Y_5\theta_6(t), && (3.11)\\
\leq & \xi^T(t)\Upsilon_4\xi(t),\\
\mathcal{L}V_5(\mathbf{e}_t,\mathfrak{c}(t)) = & h^2\mathfrak{c}^T(t)Y_6\dot{\mathfrak{c}}(t)-\frac{\pi^2}{4}\theta_3^T(t-\theta)Y_6\theta_3(t-\theta), && (3.12)\\
\leq & \xi^T(t)\Upsilon_5\xi(t).
\end{aligned}$$

For any matrices $Q_1(\mathbf{e}_t), Q_2(\mathbf{e}_t)$ and $Q_3(\mathbf{e}_t)$, we get the equation as follows:

$$\begin{aligned}
0 = & 2[Q_1(\mathbf{e}_t)\mathfrak{c}^T(t)+Q_2(\mathbf{e}_t)\mathfrak{c}^T(t_k)+Q_3(\mathbf{e}_t)\dot{\mathfrak{c}}^T(t)][R(\mathbf{e}_t)\mathfrak{c}(t)+S(\mathbf{e}_t)\hbar(\mathfrak{c}(t))+\sum_{\mathfrak{m}=1}^{n}T_{\mathfrak{m}}(\mathbf{e}_t)\hbar(\mathfrak{c}(t-\tau_{\mathfrak{m}}(t)))\\
& +K(\mathbf{e}_t)\mathfrak{c}(t_k-\theta)+B(\mathbf{e}_t)\bar{\omega}(t)-\dot{\mathfrak{c}}(t)]. \qquad (3.13)
\end{aligned}$$

Now, the above equation can be rewritten as follows:

$$0 = 2[\xi^T(t)\chi_1^T\chi_2\xi(t)].$$

From (2.3), it is possible to deduce diagonal matrices $A \geq 0$ and $A_{\mathfrak{m}} \geq 0 (\mathfrak{m} = 1,\cdots,n)$ in a manner that the following holds:

$$2\left[\varepsilon_1\mathfrak{c}(t)-\hbar(\mathfrak{c}(t))\right]^T A\left[\hbar(\mathfrak{c}(t))-\varepsilon_2\mathfrak{c}(t)\right] \geq 0, \qquad (3.14)$$

$$2\sum_{\mathfrak{m}=1}^{n}\left[\varepsilon_1\mathfrak{c}(t-\tau_{\mathfrak{m}}(t))-\hbar(\mathfrak{c}(t-\tau_{\mathfrak{m}}(t)))\right]^T A_m\left[\hbar(\mathfrak{c}(t-\tau_{\mathfrak{m}}(t)))-\varepsilon_2\mathfrak{c}(t-\tau_{\mathfrak{m}}(t))\right] \geq 0. \qquad (3.15)$$

$$\begin{aligned}
\mathcal{L}V(\mathbf{e}_t,\mathfrak{c}(t))-\mathfrak{H}(\mathbf{e}_t)(t) < & \xi^T(t)\mho\xi(t)+\sum_{\mathfrak{m}=1}^{n}(\tau_{\mathfrak{m}}-\tau_{\mathfrak{m}}(t))\left(M_{r\mathfrak{m}}B_{\mathfrak{m}}^{-1}M_{r\mathfrak{m}}^T+\frac{1}{3}M_{s\mathfrak{m}}B_{\mathfrak{m}}^{-1}M_{s\mathfrak{m}}^T\right)\\
& +\sum_{\mathfrak{m}=1}^{n}\tau_{\mathfrak{m}}(t)\left(N_{r\mathfrak{m}}B_{\mathfrak{m}}^{-1}N_{r\mathfrak{m}}^T+\frac{1}{3}N_{s\mathfrak{m}}B_{\mathfrak{m}}^{-1}N_{s\mathfrak{m}}^T\right)+\mathrm{sym}(\chi_1^T\chi_2),\\
\mathcal{L}V(\mathbf{e}_t,\mathfrak{c}(t)) < & 0, \qquad (3.16)
\end{aligned}$$

where

$$\begin{aligned}
\mho = & \Upsilon+\bar{\Upsilon},\\
\Upsilon = & \Upsilon_1+\Upsilon_2+\Upsilon_{3a}+\Upsilon_{3b}+\Upsilon_4+\Upsilon_5+\Upsilon_c,\\
\bar{\Upsilon} = & G_1\bar{Z}_{22}^{-1}G_1^T+3hG_2\bar{Z}_{22}^{-1}G_2^T-h\aleph_1+h\aleph_2+\mathrm{sym}(\chi_1^T\chi_2) \text{ and}
\end{aligned}$$

$\xi(t) = [\mathfrak{c}^T(t), \mathfrak{c}^T(t_k), \int_{t_k}^{t}\mathfrak{c}^T(r)dr, \dot{\mathfrak{c}}^T(t), \mathfrak{c}^T(t-\tau_1(t)), \cdots, \mathfrak{c}^T(t-\tau_{\mathfrak{m}}(t)), \hbar^T(\mathfrak{c}(t)), \hbar^T(\mathfrak{c}(t-\tau_1(t))), \cdots,$ $\hbar^T(\mathfrak{c}(t-\tau_n(t))), \mathfrak{c}^T(t_k-\theta), \mathfrak{c}^T(t-\theta), \bar{\omega}(t)]$.

If $\mho < 0$, $\varpi > 0$ occurs, so that $\mho < -\varpi I$, then

$$\begin{aligned}
\mathcal{L}V(\mathbf{e}_t,\mathfrak{c}(t))-\mathfrak{H}(\mathbf{e}_t)(t) \leq -\varpi|\xi(t)|^2 \leq -\varpi|\mathfrak{c}(t)|^2,\\
\mathcal{L}V(\mathbf{e}_t,\mathfrak{c}(t)) \leq \mathfrak{H}(\mathbf{e}_t)(t)-\varpi|\mathfrak{c}(t)|^2.
\end{aligned}$$

Take $\bar{\omega}(t) = 0$. Thus, we attain

$$\mathfrak{H}(\mathbf{e}_t)(t) = f^T(t)\varepsilon_1 f(t).$$

Since $\varepsilon_1 \leq 0$ under Assumption (2.1), it yields $\mathcal{L}V(\mathbf{e}_t, \mathfrak{c}(t)) \leq -\varpi|\mathfrak{c}(t)|^2$. Hence, the system (2.5) is quadratically stable. Then, we have

$$\begin{aligned}&\mathcal{L}V(\mathbf{e}_t, \mathfrak{c}(t)) - \mathfrak{H}(t) \leq \xi^T(t)[\mho + \Upsilon_e]\xi(t),\\&\mathcal{L}V(\mathbf{e}_t, \mathfrak{c}(t)) - \mathfrak{H}(t) \leq 0.\end{aligned} \tag{3.17}$$

Integrate (3.17), we obtain

$$\int_0^t \mathfrak{H}(r)dr \geq V(\mathbf{e}_t, \mathfrak{c}(t)) - V(\mathbf{e}_t, \mathfrak{c}(0)) \geq \mathfrak{c}^T(t)\mathfrak{P}\mathfrak{c}(t) + \sigma, \tag{3.18}$$

where σ is taken as $\sigma = -V(\mathbf{e}_t, \mathfrak{c}(0)) - ||\mathfrak{P}|| \sup_{-\tau_\mathfrak{m} \leq r \leq 0} |\Gamma(r)|^2$. Two cases $\|\varepsilon_4\| = 0$ and $\|\varepsilon_4\| \neq 0$ are needed for proving that (2.10) is valid. If we consider $\|\varepsilon_4\| = 0$, then for any $t_l \geq 0$,

$$\int_0^{t_l} \mathfrak{H}(t)dt \geq \mathfrak{c}^T(t_l)\mathfrak{P}\mathfrak{c}(t_l) + \sigma \geq \sigma. \tag{3.19}$$

Hence, Theorem (3.1) holds. If $\|\varepsilon_4\| \neq 0$, as in Assumption (2.1), it can therefore be concluded that $\varepsilon_1 = 0,\ \varepsilon_2 = 0$ and $\varepsilon_3 > 0$. If $t_l \geq t \geq 0$, we get

$$\int_0^{t_l} \mathfrak{H}(t)dt \geq \int_0^t \mathfrak{H}(s)ds \geq \mathfrak{c}^T(t)\mathfrak{P}\mathfrak{c}(t) + \sigma. \tag{3.20}$$

Also, we have $0 < t - \tau_n(t) \leq t_l$.
Thus,

$$\int_0^{t_l} \mathfrak{H}(t)dt \geq \mathfrak{c}^T(t - \tau_n(t))\mathfrak{P}\mathfrak{c}(t - \tau_n(t)) + \sigma. \tag{3.21}$$

If $t \leq \tau_n(t)$, we get $-\tau_n \leq t - \tau_n(t) \leq 0$, then

$$\begin{aligned}\sigma + \mathfrak{c}^T(t - \tau_n(t))\mathfrak{P}\mathfrak{c}(t - \tau_n(t)) \leq&\sigma + \|\mathfrak{P}\||\mathfrak{c}(t - \tau_n(t))|^2,\\ \leq&\sigma + \|\mathfrak{P}\| \sup_{-\tau_\mathfrak{m} \leq s \leq 0} |\Gamma(s)|^2,\\ =&- V(\mathbf{e}_t, \mathfrak{c}(0)) \leq \int_0^{t_l} \mathfrak{H}(\mathbf{e}_t)(t)dt.\end{aligned}$$

Thus, (3.21) holds for any $t_l \geq t \geq 0$. From (3.20) and (3.21), there exists $0 < \alpha < 1$ that satisfies

$$\int_0^{t_l} \mathfrak{H}(t)dt \geq \sigma + \alpha\mathfrak{c}^T(t)\mathfrak{P}\mathfrak{c}(t) + (1 - \alpha)\mathfrak{c}^T(t - \tau_n(t))\mathfrak{P}\mathfrak{c}(t - \tau_n(t)). \tag{3.22}$$

$$f^T(t)\varepsilon_4 f(t) = -\begin{bmatrix}\mathfrak{c}(t)\\ \mathfrak{c}(t - \tau_n(t))\end{bmatrix}^T \Gamma \begin{bmatrix}\mathfrak{c}(t)\\ \mathfrak{c}(t - \tau_n(t))\end{bmatrix} + \alpha\mathfrak{c}^T(t)\mathfrak{P}\mathfrak{c}(t) + (1 - \alpha)\mathfrak{c}^T(t - \tau_n(t))\mathfrak{P}\mathfrak{r}(t - \tau_n(t)),$$

for $\Gamma > 0$, then

$$f^T(t)\varepsilon_4 f(t) \leq \alpha \mathfrak{c}^T(t)\mathfrak{P}\mathfrak{c}(t) + (1-\alpha)\mathfrak{c}^T(t-\tau_n(t))\mathfrak{P}\mathfrak{c}(t-\tau_n(t)).$$

So, for $t \geq 0, t_l \geq 0$, which satisfies $t_l \geq t$,

$$\int_0^{t_l} \mathfrak{H}(\mathbf{e}_t)(t)dt \geq f^T(t)\varepsilon_4 f(t) + \sigma.$$

Therefore, (2.10) holds for any $t_l \geq 0$. For $\|\varepsilon_4\| = 0$ and $\|\varepsilon_4\| \neq 0$, the system (2.5) under concern is extended dissipative. Hence the proof. □

Theorem 3.2. *For given $h > 0$, $\beta > 0$, $\theta > 0$, $\tau_{\mathfrak{m}} > 0$ and $\sigma_m < 1$, the error system attains quadratically stability and extended dissipativity synchronization, if subsequent symmetric matrices $\hat{Z} > 0, \hat{\hat{Z}}_{22} > 0, \hat{Y}_i(i = 1, \cdots, 6), \hat{Q}_i(\mathbf{e}_t)(i = 1, 2, 3), \hat{G}_1, \hat{G}_2 \in \mathbb{R}^{(8+3J)n\times n}, M_{r\mathfrak{m}} > 0, M_{s\mathfrak{m}} > 0, N_{r\mathfrak{m}} > 0, N_{s\mathfrak{m}} > 0$, symmetric matrix $\hat{\hat{Z}} = \begin{bmatrix} \hat{\hat{Z}}_{11} & \hat{\hat{Z}}_{12} & \hat{\hat{Z}}_{13} \\ \star & \hat{\hat{Z}}_{22} & \hat{\hat{Z}}_{23} \\ \star & \star & \hat{\hat{Z}}_{33} \end{bmatrix}$ and diagonal matrices $A > 0, A_1 > 0, \cdots, A_n > 0$ exist such that the following LMIs hold:*

$$\begin{bmatrix} \hat{Y}_3 & \hat{E} \\ \star & \hat{Y}_3 \end{bmatrix} > 0 \tag{3.23}$$

$$\hat{\Gamma} = \begin{bmatrix} \hat{\Gamma}_{11} & \hat{\Gamma}_{12} \\ \star & \hat{\Gamma}_{22} \end{bmatrix} > 0, \tag{3.24}$$

$$\begin{bmatrix} \hat{\Upsilon} & \tilde{\mathfrak{F}}_{12} & \tilde{\mathfrak{F}}_{13} & \tilde{\mathfrak{F}}_{14} & \tilde{\mathfrak{F}}_{15} & h\hat{G}_1 & h^2\hat{G}_2 & \beta\Omega_1^T(\mathbf{e}_t) & \Omega_2^T(\mathbf{e}_t) \\ \star & \tilde{\mathfrak{F}}_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \tilde{\mathfrak{F}}_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \tilde{\mathfrak{F}}_{44} & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \tilde{\mathfrak{F}}_{55} & 0 & & 0 & 0 \\ \star & \star & \star & \star & \star & -h\hat{\hat{Z}}_{22} & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -3h\hat{\hat{Z}}_{22} & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & -\beta I & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & -\beta I \end{bmatrix} < 0, \tag{3.25}$$

where

$$\begin{aligned} \hat{\Upsilon} =& \hat{\Upsilon}_1 + \hat{\Upsilon}_{2a} + \hat{\Upsilon}_{2b} + \hat{\Upsilon}_{3a} + \hat{\Upsilon}_{3b} + \hat{\Upsilon}_4 + \hat{\Upsilon}_5 + \hat{\Upsilon}_c + \hat{\Upsilon}_e, \\ \hat{\Upsilon}_1 =& -2\mathfrak{e}_1^T\hat{X}(\mathbf{e}_t)\mathfrak{e}_4 + 2\mathfrak{e}_{5+J}^T\hat{L}\mathfrak{e}_4 + \mathfrak{e}_{5+J}^T\sum_{\mathfrak{m}=1}^{n}\hat{L}_{\mathfrak{m}}\mathfrak{e}_{5+J} - (1-\sigma_1)\mathfrak{e}_{6+J}^T\hat{L}_1\mathfrak{e}_{6+J} - \cdots - (1-\sigma_n)\mathfrak{e}_{5+2J}^T\hat{L}_n\mathfrak{e}_{5+2J} \\ &+ \mathfrak{e}_1^T\sum_{\mathfrak{m}=1}^{n}\hat{H}_{\mathfrak{m}}\mathfrak{e}_1 - (1-\sigma_1)\mathfrak{e}_5^T\hat{H}_1\mathfrak{e}_5 - \cdots - (1-\sigma_n)\mathfrak{e}_{4+J}^T\hat{H}_n\mathfrak{e}_{4+J} + \sum_{\mathfrak{m}=1}^{n}\mathfrak{e}_4^T\tau_{\mathfrak{m}}\hat{B}_m\mathfrak{e}_4 + \sum_{j\in\Im}\Lambda_{ij}\mathfrak{e}_1^T\hat{X}(\mathbf{e}_t)\mathfrak{e}_1 \\ &- \hat{\aleph}_1 + \hat{\aleph}_2 + sym(\hat{\chi}_1^T\hat{\chi}_2), \\ \hat{\Upsilon}_{2a} =& 2h\begin{bmatrix}\mathfrak{e}_1\\ \mathfrak{e}_2\\ \mathfrak{e}_3\end{bmatrix}^T\hat{Z}\begin{bmatrix}\mathfrak{e}_4\\ 0\\ \mathfrak{e}_1\end{bmatrix} + h\begin{bmatrix}\mathfrak{e}_2\\ \mathfrak{e}_4\\ \mathfrak{e}_{7+2J}\end{bmatrix}^T\hat{\hat{Z}}\begin{bmatrix}\mathfrak{e}_2\\ \mathfrak{e}_4\\ \mathfrak{e}_{7+2J}\end{bmatrix} - \begin{bmatrix}\mathfrak{e}_1\\ \mathfrak{e}_2\\ \mathfrak{e}_3\end{bmatrix}^T\hat{Z}\begin{bmatrix}\mathfrak{e}_1\\ \mathfrak{e}_2\\ \mathfrak{e}_3\end{bmatrix} - \begin{bmatrix}\mathfrak{e}_2\\ 0\\ \mathfrak{e}_{7+2J}\end{bmatrix}^T\hat{\hat{Z}}\begin{bmatrix}\mathfrak{e}_2\\ \mathfrak{e}_0\\ \mathfrak{e}_{7+2J}\end{bmatrix}, \end{aligned}$$

$$
\begin{aligned}
\hat{\Upsilon}_{2b} =&(\mathfrak{e}_1-\mathfrak{e}_2)^T\hat{\hat{Z}}_{12}^T\mathfrak{e}_2-(\mathfrak{e}_1-\mathfrak{e}_2)^T\hat{\hat{Z}}_{23}\mathfrak{e}_{6+2J},\\
\hat{\Upsilon}_{3a} =&\mathfrak{e}_1^T(\hat{Y}_1+\hat{Y}_2)\mathfrak{e}_1-(1-\sigma_1)\mathfrak{e}_5^T\hat{Y}_1\mathfrak{e}_5-\cdots-(1-\sigma_m)\mathfrak{e}_{4+J}^T\hat{Y}_1\mathfrak{e}_{4+J}-\mathfrak{e}_{6+2J}^T\hat{Y}_2\mathfrak{e}_{6+2J}-\cdots-\mathfrak{e}_{5+3J}^T\hat{Y}_2\mathfrak{e}_{5+3J}\\
&-\mathfrak{e}_{6+2J}^T\hat{Y}_2\mathfrak{e}_{6+2J}+\mathfrak{e}_4^T(\sum_{\mathfrak{m}=1}^{n}\tau_{\mathfrak{m}}^2\hat{Y}_3)\mathfrak{e}_4,\\
\hat{\Upsilon}_{3b} =&-\sum_{s=5}^{4+J}\sum_{d=6+2J}^{5+3J}\begin{bmatrix}\mathfrak{e}_1-\mathfrak{e}_s\\ \mathfrak{e}_s-\mathfrak{e}_d\end{bmatrix}^T\begin{bmatrix}\hat{Y}_3 & \hat{E}\\ \star & \hat{Y}_3\end{bmatrix}\begin{bmatrix}\mathfrak{e}_1-\mathfrak{e}_s\\ \mathfrak{e}_s-\mathfrak{e}_d\end{bmatrix}-(\mathfrak{e}_1-\mathfrak{e}_2)^T\hat{Z}_{23}\mathfrak{e}_{7+2J},\\
\hat{\Upsilon}_4 =&\mathfrak{e}_1^T\hat{Y}_4\mathfrak{e}_1-\mathfrak{e}_{7+3J}^T\hat{Y}_4\mathfrak{e}_{7+3J}+\mathfrak{e}_4^T\theta^2\mathfrak{e}_4-[\mathfrak{e}_1-\mathfrak{e}_{7+3J}]^T\hat{Y}_5[\mathfrak{e}_1-\mathfrak{e}_{7+3J}],\\
\hat{\Upsilon}_5 =&h^2\mathfrak{e}_4^T\hat{Y}_6\mathfrak{e}_4-\frac{\pi^2}{4}(\mathfrak{e}_1-\mathfrak{e}_{6+3J})^T\hat{Y}_6(\mathfrak{e}_1-\mathfrak{e}_{6+3J}),\\
\hat{\Upsilon}_c =&-2[\varepsilon_1\mathfrak{e}_1-\mathfrak{e}_{5+J}]^TA[\mathfrak{e}_{5+J}-\varepsilon_2\mathfrak{e}_1]-2[\varepsilon_1\mathfrak{e}_5-\mathfrak{e}_{6+J}]^TA_1[\mathfrak{e}_{6+J}-\varepsilon_2\mathfrak{e}_5]-\cdots-2[\varepsilon_1\mathfrak{e}_{4+J}-\mathfrak{e}_{5+2J}]^T\\
&A_n[\mathfrak{e}_{5+2J}-\varepsilon_2\mathfrak{e}_{4+J}],\\
\hat{\Upsilon}_e =&[\mathfrak{e}_1\ \ \mathfrak{e}_{4+J}]^T\varepsilon_1[\mathfrak{e}_1\ \ \mathfrak{e}_{4+J}]+2[\mathfrak{e}_1\ \ \mathfrak{e}_{4+J}]\varepsilon_2\bar{\omega}^T+\bar{\omega}^T\varepsilon_3\bar{\omega},\\
\hat{\Gamma}_{11} =&\alpha\mathfrak{P}-\varepsilon_4,\ \ \hat{\Gamma}_{12}=[\hat{\Gamma}_{12}^1,\hat{\Gamma}_{12}^2,\cdots,\hat{\Gamma}_{12}^n],\ \ \hat{\Gamma}_{12}^l=-\varepsilon_4,\ \ \hat{\Gamma}_{22}=[\hat{\Gamma}_{22}^1,\cdots,\hat{\Gamma}_{22}^n],\ \ \hat{\Gamma}_{22}^l=(1-\alpha)\mathfrak{P}-\varepsilon_4,\\
\tilde{\mathfrak{F}}_{12} =&\sum_{\mathfrak{m}=1}^{n}\tilde{\tau}_cM_{r\mathfrak{m}},\quad \tilde{\mathfrak{F}}_{13}=\frac{1}{3}\sum_{\mathfrak{m}=1}^{n}\tilde{\tau}_cM_{s\mathfrak{m}},\quad \tilde{\mathfrak{F}}_{14}=\sum_{\mathfrak{m}=1}^{n}\tau_{\mathfrak{m}}N_{r\mathfrak{m}},;\quad \tilde{\mathfrak{F}}_{15}=\frac{1}{3}\sum_{\mathfrak{m}=1}^{n}\tau_{\mathfrak{m}}N_{s\mathfrak{m}},\\
\tilde{\mathfrak{F}}_{22} =&-\sum_{\mathfrak{m}=1}^{n}\tilde{\tau}_c\hat{B}_m;\quad \tilde{\mathfrak{F}}_{33}=-\frac{1}{3}\sum_{\mathfrak{m}=1}^{n}\tilde{\tau}_c\hat{B}_m,\quad \tilde{\mathfrak{F}}_{44}=-\sum_{\mathfrak{m}=1}^{n}\tau_{\mathfrak{m}}\hat{B}_m;\quad \tilde{\mathfrak{F}}_{55}=-\frac{1}{3}\sum_{\mathfrak{m}=1}^{n}\tau_{\mathfrak{m}}\hat{B}_m,\\
\hat{\aleph}_1 =&sym[\hat{G}_2,\hat{G}_2,0,0,\cdots,0],\quad \hat{\aleph}_2=sym[\hat{G}_1,-\hat{G}_1,2\hat{G}_2,0,\cdots,0]\\
\hat{\chi}_1(\mathbf{e}_t) =&[M(\mathbf{e}_t),\epsilon_1M(\mathbf{e}_t),0,\epsilon_2M(\mathbf{e}_t),0,\cdots,0],\ ,\ \tilde{\tau}_c=\tau_{\mathfrak{m}}-\tau_{\mathfrak{m}}(t)\\
\hat{\chi}_2(\mathbf{e}_t) =&[R(\mathbf{e}_t)\bar{\mathcal{Q}}^{-1}(\mathbf{e}_t),0,0,-\bar{\mathcal{Q}}^{-1}(\mathbf{e}_t),T_1(\mathbf{e}_t)\bar{\mathcal{Q}}^{-1}(\mathbf{e}_t),\cdots,T_n(\mathbf{e}_t)\bar{\mathcal{Q}}^{-1}(\mathbf{e}_t),S(\mathbf{e}_t)\bar{\mathcal{Q}}^{-1}(\mathbf{e}_t),0,\cdots,0,\\
&\bar{K}(\mathbf{e}_t),B(\mathbf{e}_t)\bar{\mathcal{Q}}^{-1}(\mathbf{e}_t),0],\\
V(\mathbf{e}_t) =&[N_a(\mathbf{e}_t),0,0,0,N_{c1}(\mathbf{e}_t),\cdots,,N_{cn}(\mathbf{e}_t),N_b(\mathbf{e}_t),0,\cdots,0,N(\mathbf{e}_t),0,0].
\end{aligned}
$$

Furthermore, $K(\mathbf{e}_t)=\bar{\mathcal{Q}}^{-1}(e_t)\bar{K}(\mathbf{e}_t)$ *are the control gain matrices.*

Proof . The controller to be built takes the form as follows:

$$\mathfrak{u}(t)=(K(\mathbf{e}_t)+\Delta K(\mathbf{e}_t))\mathfrak{c}(t_k-\theta),\tag{3.26}$$

where $K(\mathbf{e}_t)$ is the control gain matrix which has to be found and $\Delta K(t)$ takes the form $\Delta K(t)=M(\mathbf{e}_t)C(\mathbf{e}_t)(t)N(\mathbf{e}_t)$. A real - valued matrix to represent the controller gain fluctuation is $\Delta K(\mathbf{e}_t)$. We set $Q_2(\mathbf{e}_t)=\epsilon_1Q_1(\mathbf{e}_t),Q_3(\mathbf{e}_t)=\epsilon_2Q_1(\mathbf{e}_t),Q_1(\mathbf{e}_t)=\bar{\mathcal{Q}}(\mathbf{e}_t)$. Replacing $\tilde{R}(\mathbf{e}_t),\tilde{S}(\mathbf{e}_t),\tilde{T}_1(\mathbf{e}_t),...,\tilde{T}_n(\mathbf{e}_t)$ by $R(\mathbf{e}_t)+M(\mathbf{e}_t)C(\mathbf{e}_t)(t))N_a(\mathbf{e}_t),S(\mathbf{e}_t)+M(\mathbf{e}_t)C(\mathbf{e}_t)(t))N_b(\mathbf{e}_t),T_1(\mathbf{e}_t)+M(\mathbf{e}_t)C(\mathbf{e}_t)(t))N_{c1}(\mathbf{e}_t),\cdots,T_n(\mathbf{e}_t)+M(\mathbf{e}_t)C(\mathbf{e}_t)(t))N_{cn}(\mathbf{e}_t)$ respectively and proceeding as in Theorem (3.1), we get

$$\Xi+\hat{\chi}_1(\mathbf{e}_t)C(\mathbf{e}_t)(t)V(\mathbf{e}_t)+V^T(\mathbf{e}_t)C^T(\mathbf{e}_t)(t)\hat{\chi}_1^T(\mathbf{e}_t)<0.\tag{3.27}$$

where Ξ is defined in (3.3). By Lemma (2.5), there exist $\beta>0$ such that

$$\Xi+\beta\Omega_1(\mathbf{e}_t)\Omega_1^T(\mathbf{e}_t)+\beta^{-1}\Omega_2^T(\mathbf{e}_t)\Omega_2(\mathbf{e}_t)<0.\tag{3.28}$$

$$\begin{aligned}\Omega_1(\mathbf{e}_t) =&[M^T(\mathbf{e}_t)\bar{\mathcal{Q}}^{-T}(\mathbf{e}_t), \epsilon_1 M^T(\mathbf{e}_t)\bar{\mathcal{Q}}^{-T}(\mathbf{e}_t), 0, \epsilon_2 M^T(\mathbf{e}_t)\bar{\mathcal{Q}}^{-T}(\mathbf{e}_t), 0, \cdots, 0]^T,\\ \Omega_2(\mathbf{e}_t) =&[N_a(\mathbf{e}_t)\bar{\mathcal{Q}}^{-1}(\mathbf{e}_t), 0, 0, 0, N_{c1}(\mathbf{e}_t)\bar{\mathcal{Q}}^{-1}(\mathbf{e}_t), \cdots, , N_{cn}(\mathbf{e}_t)\bar{\mathcal{Q}}^{-1}(\mathbf{e}_t), N_b(\mathbf{e}_t)\bar{\mathcal{Q}}^{-1}(\mathbf{e}_t), 0, \cdots, 0,\\ &N(\mathbf{e}_t)\bar{\mathcal{Q}}^{-1}(\mathbf{e}_t), 0, 0].\end{aligned}$$

Pre and post multiplying (3.1, 3.2) by diag$\{\bar{\mathcal{Q}}^{-1}(\mathbf{e}_t), \bar{\mathcal{Q}}^{-1}(\mathbf{e}_t)\}$, they get converted into (3.23, 3.24) respectively. We define $\hat{\bar{Z}} = \text{diag}\{\bar{\mathcal{Q}}^{-1}(\mathbf{e}_t), \bar{\mathcal{Q}}^{-1}(\mathbf{e}_t), \bar{\mathcal{Q}}^{-1}(\mathbf{e}_t)\}\bar{Z}\text{diag}\{\bar{\mathcal{Q}}^{-1}(\mathbf{e}_t), \bar{\mathcal{Q}}^{-1}(\mathbf{e}_t), \bar{\mathcal{Q}}^{-1}(\mathbf{e}_t)\}, \hat{\bar{Z}}_{22} = \bar{\mathcal{Q}}^{-1}(\mathbf{e}_t)\bar{Z}_{22}\bar{\mathcal{Q}}^{-1}(\mathbf{e}_t), \hat{Y}_i = \bar{\mathcal{Q}}^{-1}(\mathbf{e}_t)Y_i\bar{\mathcal{Q}}^{-1}(\mathbf{e}_t), (i = 1, \cdots, 6); \hat{L} = \mathcal{Q}^{-1}(\mathbf{e}_t)L\mathcal{Q}^{-1}(\mathbf{e}_t), \hat{X}(\mathbf{e}_t) = \mathcal{Q}^{-1}(\mathbf{e}_t)X(\mathbf{e}_t) \times \mathcal{Q}^{-1}(\mathbf{e}_t); \hat{L}_m = \bar{\mathcal{Q}}^{-1}(\mathbf{e}_t)L_m\bar{\mathcal{Q}}^{-1}(\mathbf{e}_t), (m = 1\cdots n)$.

Set $\Pi_1 = \text{diag}\{\overbrace{\mathcal{Q}^{-1}(\mathbf{e}_t)\cdots\mathcal{Q}^{-1}(\mathbf{e}_t)}^{8+3J}\}, \Pi_2 = \text{diag}\{\Pi_1, I, I, I, I, \mathcal{Q}^{-1}(\mathbf{e}_t), \mathcal{Q}^{-1}(\mathbf{e}_t), I, I\}$. Then, $\hat{G}_i = \Pi_1 G_i\bar{\mathcal{Q}}^{-1}(\mathbf{e}_t), (i = 1, 2)$. Pre and post multiply (3.28) by Π_2 and also applying Schur complement to the obtained result, we get (3.25).

Hence the proof. □

Corollary 3.3. *Consider the error system (2.8) with $\mathfrak{m} = 1$. When there are no uncertainties, external disturbance and $\mathfrak{u}(t) = 0$, the system (2.8) gets transformed into following:*

$$\dot{\mathfrak{c}}(t) = -R\mathfrak{c}(t) + S\hbar(\mathfrak{c}(t)) + T_1\hbar(\mathfrak{c}(t - \tau_1(t))). \tag{3.29}$$

Proof . *The proof follows from Theorem (3.1) and hence it is omitted.* □

4. Numerical examples

We provide numerical simulations in this to illustrate the appropriateness of suggested approach and benefits of our research methods in view of the conditions obtained in the prior section.

Example 4.1. The MJRNNs with multiple time-varying delays into consideration:

$$\dot{\mathfrak{c}}(t) = -\tilde{R}(\mathbf{e}_t)\mathfrak{c}(t) + \tilde{S}(\mathbf{e}_t)\hbar(\mathfrak{c}(t)) + \sum_{\mathfrak{m}=1}^{2}\tilde{T}_{\mathfrak{m}}(\mathbf{e}_t)\hbar(\mathfrak{c}(t - \tau_{\mathfrak{m}}(t))) + \mathfrak{u}(t) + B(\mathbf{e}_t)\bar{\omega}(t), \tag{4.1}$$

where

$$\begin{aligned}
R_1 &= \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \ S_1 = \begin{bmatrix} 1 & 0.3 \\ -1 & 0.2 \end{bmatrix}, \ T_{11} = \begin{bmatrix} 0.4 & 0.6 \\ 0.5 & 0.3 \end{bmatrix},\\
T_{21} &= \begin{bmatrix} 0.4 & 0.3 \\ 0.2 & 0.8 \end{bmatrix}, \ B_1 = \begin{bmatrix} 0.3 & -0.3 \\ -0.2 & 0.5 \end{bmatrix},\\
R_1 &= \begin{bmatrix} 3.2 & 0 \\ 0 & 2.3 \end{bmatrix}, \ S_1 = \begin{bmatrix} 1.2 & 0.3 \\ -1 & 0.24 \end{bmatrix}, \ T_{12} = \begin{bmatrix} 0.5 & 0.3 \\ 0.6 & 0.3 \end{bmatrix},\\
T_{22} &= \begin{bmatrix} 0.5 & 0.2 \\ 0.5 & 0.5 \end{bmatrix}, \ B_2 = \begin{bmatrix} 0.4 & -0.7 \\ -0.4 & 0.2 \end{bmatrix},\\
N_a&(\mathbf{e}_t) = N_b(\mathbf{e}_t) = N_{c1}(\mathbf{e}_t) = \cdots, N_{cn}(\mathbf{e}_t) = diag\{0.1, 0.1\},
\end{aligned}$$

Let the activation function satisfy $a_1^- = -0.1,\ a_1^+ = 0.1,\ a_2^- = -0.2,\ a_2^+ = 0.2$. We take $h = 0.2$ and further time - delays are of the form $\tau_1(t) = 0.3sin(t)+0.2,\ \tau_2(t) = 0.1sin(t)+0.5,\ \tau_1 = 0.5,\ \tau_2 = 0.6$. MATLAB LMI Toolbox has been used in solving the obtained results in Theorem (3.2). We assign values using weighting matrices $\varepsilon_1, \varepsilon_2, \varepsilon_3$, and ε_4 and the extended dissipativity analysis of the system (4.1) is done.

$\mathcal{L}_2 - \mathcal{L}_\infty$ performance: $\varepsilon_1 = 0, \varepsilon_2 = 0, \varepsilon_3 = \tilde{\gamma}^2 I, \varepsilon_4 = I$, and $\sigma = 0$. We calculate the following control gain matrix by solving the feasibility problem for LMI in Theorem (3.2).

$$K_1 = \begin{bmatrix} 0.3455 & 0.0034 \\ 0.0321 & 0.0342 \end{bmatrix}, K_2 = \begin{bmatrix} 0.1232 & 0.0032 \\ 0.0121 & 0.0452 \end{bmatrix}.$$

Thus, Figure 4.1 shows that the system converges to zero with our parameter values and shows

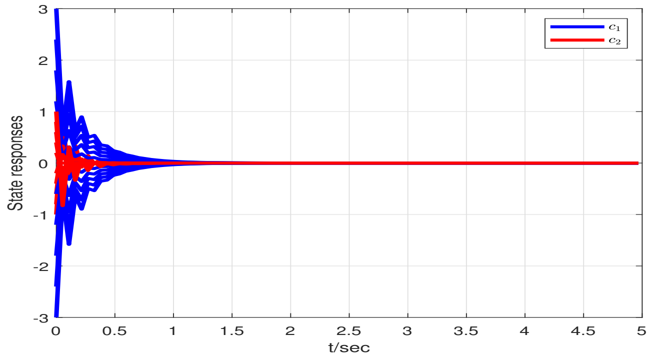

Figure 4.1: State trajectories showing $\mathcal{L}_2 - \mathcal{L}_\infty$ performance.

$\mathcal{L}_2 - \mathcal{L}_\infty$ performance, indicating that the designed controller is efficient. The results of the simulation clearly explain that the proposed methodology is effective.

H_∞ performance: On assigning the values $\varepsilon_1 = -I,\ \varepsilon_2 = 0,\ \varepsilon_3 = \tilde{\gamma}^2 I,\ \varepsilon_4 = 0$, and $\sigma = 0$, we get

$$K_1 = \begin{bmatrix} 0.1245 & 0.0044 \\ 0.0128 & 0.1254 \end{bmatrix},\ K_2 = \begin{bmatrix} 0.1452 & 0.0032 \\ 0.0453 & 0.4213 \end{bmatrix},$$

which is the desired control gain matrix. Hence from Figure 4.2, it is evident that the system operates

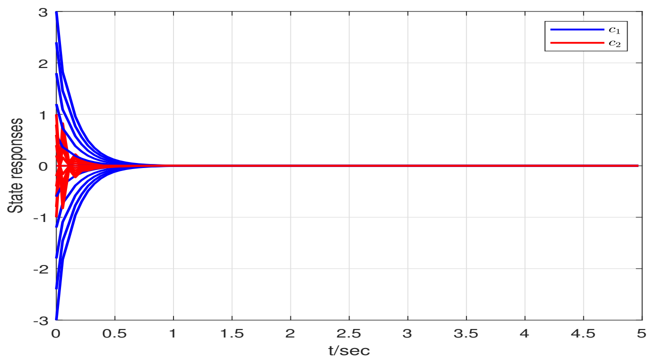

Figure 4.2: State trajectories depicting H_∞ performance.

well.

Passivity performance: $\varepsilon_1 = 0,\ \varepsilon_2 = I,\ \varepsilon_3 = \tilde{\gamma},\ \varepsilon_4 = 0$, and $\sigma = 0$. With these parameter inputs, we get

$$K_1 = \begin{bmatrix} 0.6577 & 0.0254 \\ 0.0254 & 0.0547 \end{bmatrix}, K_2 = \begin{bmatrix} 0.2345 & 0.3242 \\ 0.0312 & 0.5634 \end{bmatrix}.$$

The generated state responses in the presence of disturbance $\bar{\omega}(t)$ under randomly initialized condi-

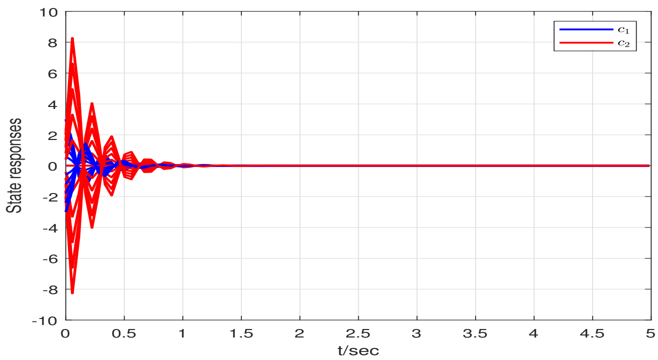

Figure 4.3: Passivity performance : State trajectories.

tions are depicted in Figure 4.3. All of these researches indicate that the controller is well-designed.

Mixed H_∞ and Passivity performance: The weighting matrices take the values $\varepsilon_1 = 0,\ \varepsilon_2 = I,\ \varepsilon_3 = \tilde{\gamma},\ \varepsilon_4 = 0$, and $\sigma = 0$.

$$K_1 = \begin{bmatrix} 0.1248 & 0.1567 \\ 0.6745 & 0.4578 \end{bmatrix}, K_2 = \begin{bmatrix} 0.1032 & 0.0451 \\ 0.0021 & 0.1096 \end{bmatrix}$$

Thus, it is evident from Figure 4.4 that system is simplified to Mixed H_∞ and Passivity performance.

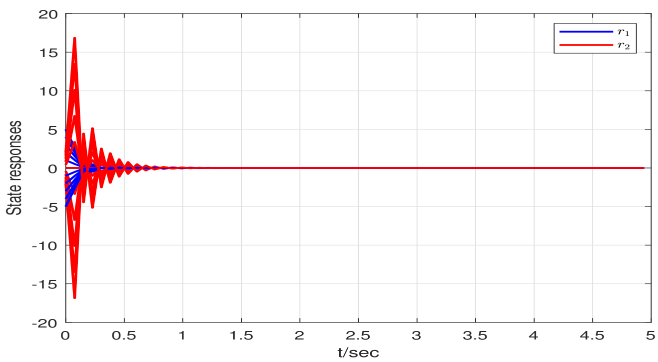

Figure 4.4: Mixed H_∞ and passivity performance : State trajectories.

$(\bar{\mathcal{Q}} - \mathcal{S} - \mathfrak{R})$ Dissipativity: $\varepsilon_1 = \bar{\mathcal{Q}},\ \varepsilon_2 = \mathcal{S},\ \varepsilon_3 = \mathfrak{R} - \tilde{\alpha} I$, and $\varepsilon_4 = 0$ with

$$\bar{\mathcal{Q}} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \ \mathcal{S} = \begin{bmatrix} 0.3 & 0 \\ 0.4 & 0.25 \end{bmatrix}, \ \mathfrak{R} = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}.$$

The following is the resultant matrix obtained:

$$K_1 = \begin{bmatrix} 5.5477 & 0.1521 \\ 0.0412 & 3.5461 \end{bmatrix}, K_2 = \begin{bmatrix} 1.0212 & 0.4021 \\ 0.0544 & 2.2321 \end{bmatrix}$$

and dissipativity performance is $\alpha = 0.0072$. The initial condition $[-5, 5]^T$ is to probe the numerical

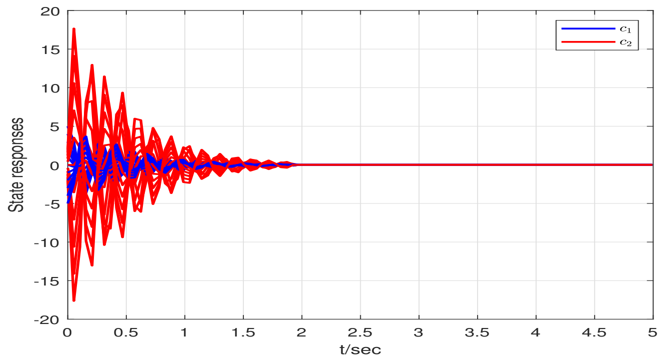

Figure 4.5: State responses showing the performance of $(\bar{\mathcal{Q}} - \mathcal{S} - \mathfrak{R})$ dissipativity.

results of $\mathfrak{u}(t)$. The state response of the dynamical system (4.1) which reduces to $(\bar{\mathcal{Q}} - \mathcal{S} - \mathfrak{R})$ dissipativity performance is seen in Figure 4.5. The simulation results allow us to reach a conclusion that the state trajectories very well converge to zero. Eventually, Figures. 4.1 - 4.5 not just justify the system's (4.1) control performance, but it also show the huge benefits of our newly constructed control.

Example 4.2. *The differential equations shown below describe continuous time MJRNNs with N units [14]:*

$$\begin{aligned} \dot{\mathfrak{c}}_i(t) =& \frac{\mathfrak{c}_i(t)}{\mathcal{R}_i \mathcal{C}_i} + \sum_{j=1}^{N} \tilde{S}_j(\mathfrak{e}_\mathfrak{t})(t) + \mathfrak{u}_i(t), \\ x_i(t) =& \hbar_i(\mathfrak{c}_i(t)). \end{aligned} \tag{4.2}$$

As it is shown in Figure 4.6, the system (4.2) could be enabled with the aid of an analog resistance-capacitance network circuit with reference to the results obtained in [14]. Here, the input voltage of the i^{th} amplifier is $\mathfrak{c}_i$ and the output voltage of the i^{th} amplifier is $\mathcal{V}_i = \hbar_i(\mathfrak{c}_i(t))$. Every operational amplifier possess two output terminals $\mathcal{V}_i$ and $-\mathcal{V}_i$. $\mathcal{R}_i$ and $\mathcal{W}_{ij}$ are described as given below:

$$\begin{aligned} \frac{1}{\mathcal{R}_i} =& \frac{1}{\rho_i} + \sum_{j=1}^{N} \frac{1}{\mathcal{R}_{ij}}, \\ \mathcal{W}_{ij} =& \begin{cases} +\dfrac{1}{\mathcal{R}_{ij}}, \textit{when } \mathcal{R}_{ij} \textit{ connected to } \mathcal{V}_j; \\ -\dfrac{1}{\mathcal{R}_{ij}}, \textit{when } \mathcal{R}_{ij} \textit{ connected to } -\mathcal{V}_j. \end{cases} \end{aligned}$$

Thus, the system (4.2) takes the subsequent form:

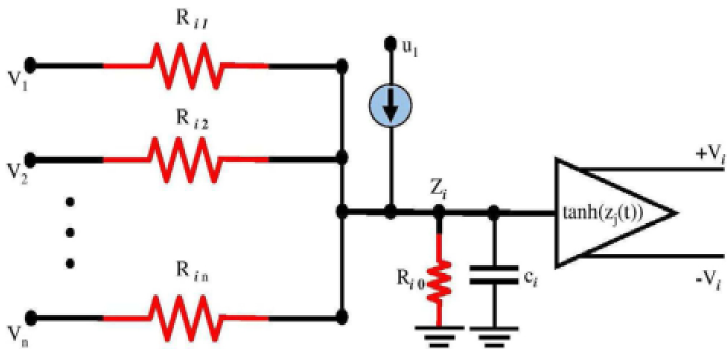

Figure 4.6: RNN Analog circuit representation for i^{th} neuron.

$$\dot{\mathfrak{c}}(t) = -R\mathfrak{c}(t) + S\hbar(\mathfrak{c}(t)) + T\mathfrak{u}, \tag{4.3}$$

where $R = \begin{bmatrix} \dfrac{1}{\mathcal{R}_1\mathcal{C}_1} & 0 \\ 0 & \dfrac{1}{\mathcal{R}_2\mathcal{C}_2} \end{bmatrix}, S = \begin{bmatrix} \dfrac{\tilde{S}_{11}}{\mathcal{C}_1} & \dfrac{\tilde{S}_{12}}{\mathcal{C}_1} \\ \dfrac{\tilde{S}_{21}}{\mathcal{C}_2} & \dfrac{1}{\tilde{S}_{21}/\mathcal{C}_2} \end{bmatrix}, T = \begin{bmatrix} \dfrac{1}{\mathcal{C}_1} & 0 \\ 0 & \dfrac{1}{\mathcal{C}_2} \end{bmatrix}.$

$R_iC_i = \rho_i$, for $i = 1, 2, ..., N$ represent the time constants and it signifies the convergence of $\mathfrak{c}_i$. If $\rho_i = \rho$ for each neuron, we should consider the values for ρ_i in such a manner that it adjust for these differences and maintains $\mathcal{R}_i$ the same manner for each neuron. Due to the high gain of the frequency response, the output $\mathcal{V}_i$ could converge faster. As a result, even if $\mathfrak{c}_i$ is already a long way from reaching its equilibrium, $\mathcal{V}_i$ could appear because in a minor fraction of ρ_i, the circuit has converged. Here, with the aforementioned range of parameters, we have the MJRNNs defined by (4.2).

$$\mathcal{R}_i = \mathcal{C}_i, i = 1, 2$$
$$S = \begin{bmatrix} 1 & 1.5 \\ -1.5 & -1 \end{bmatrix}, \hbar_1(t) = tanh(t).$$

Making use of MATLAB LMI toolbox, the LMI obtained using Corollary (3.3) are solved which yields a range of feasible solutions. In Figure 4.7, the state response of (4.3) is depicted.

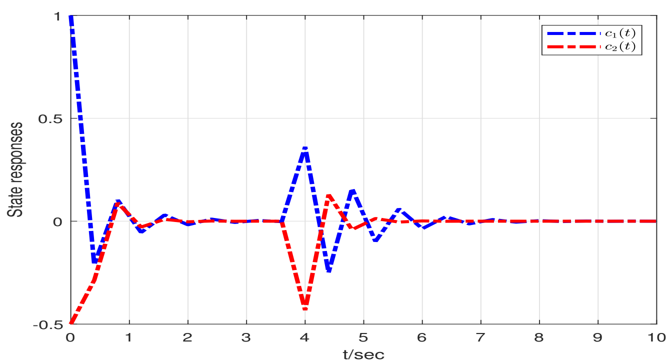

Figure 4.7: State responses for Example 4.2.

5. Conclusion

The extended dissipativity and asymptotic stability of uncertain MJRNNs with multiple time-varying delays are discussed and the results which use memory sampled-data control and one of its applications to circuit theory have been discussed. Using a suitable Lyapunov functional, the results were obtained. Eventually, the neural networks were synchronised with a memory sampled-data controller. By providing appropriate simulation results, the behaviour of such control design process has been evidenced. Future work subjects may include implementing the findings of this study to various fractional-order systems with various stability analysis and control techniques [29, 31, 35, 36, 37, 38].

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