Hindawi
Journal of Function Spaces
Volume 2021, Article ID 2571301, 8 pages
https://doi.org/10.1155/2021/2571301

Research Article

Pythagorean Nanogeneralized Closed Sets with Application in Decision-Making

D. Ajay,[1] **J. Joseline Charisma**,[1] **T. Petaratip**,[2] **P. Hammachukiattikul**,[3] **N. Boonsatit**,[4] **Jihad A. Younis**,[5] **and G. Rajchakit**[2]

[1]*Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur, 635 601 Tamilnadu, India*
[2]*Department of Mathematics, Faculty of Science, Maejo University, Chiang Mai 50290, Thailand*
[3]*Department of Mathematics, Faculty of Science and Technology, Phuket Rajabhat University, Phuket 83000, Thailand*
[4]*Department of Mathematics, Faculty of Science and Technology, Rajamangala University of Technology Suvarnabhumi, Nonthaburi 11000, Thailand*
[5]*Department of Mathematics, Aden University, Khormaksar, P.O. Box 6014, Yemen*

Correspondence should be addressed to Jihad A. Younis; jihadalsaqqaf@gmail.com

Received 16 October 2021; Accepted 8 December 2021; Published 26 December 2021

Academic Editor: Ganesh Ghorai

Topology is studying the objects which are considered to be equal if they may also be continually deformed through other shapes as bending and twisting without tearing or glueing them. Topology is similar in geometrical structures and quantitatively equivalent. Nanotopology is the study of set. The main goal of this article is to propose the idea of generalized closed sets in Pythagorean nanotopological spaces. In addition, the concept of semigeneralized closed sets is also defined, and their properties are investigated. An application to MADM using Pythagorean nanotopology has been proposed and illustrated using a numerical example.

1. Introduction

Topology is a discipline of mathematics in which two objects are regarded equal if they may be continously deformed into one another through motions in space such as bending, twisting, stretching, and shrinking without preventing tearing apart or glueing together sections. The qualities that stay constant by such continuous deformations are the core issues of interest in topology. While topology is similar to geometry, it varies in that geometrically equal things generally share numerically measured properties such as lengths or angles, whereas topologically analogous objects are qualitatively equivalent. General topology is the branch of topology that deals with the fundamental set-theoretic notions and constructs used in topology in mathematics. Most other fields of topology, such as differential topology, geometric topology, and algebraic topology, are built on it. Continuity, compactness, and connectedness are the three fundamental principles in point-set topology. Intuitively, continuous functions transport nearby points to nearby points. Compact sets can be covered by an infinite number of small sets. Connected sets are those that cannot be separated into two separate pieces.

Fuzzy set theory [1] plays a vital role in dealing with incomplete data and vagueness, and it is applied in a wide range of disciplines. Fuzzy set is an extension of the usual set holding elements with its membership grade in the interval [0,1]. Along with some conditions, there has been an advancement in fuzzy set (intuitionistic fuzzy set (IFS)) in the view of other human thinking options [2]. To each element in the IFS, it has membership and nonmembership grades which satisfy the condition that the sum of both the grades is lesser than or equal to 1. The Pythagorean fuzzy subset (PFS), an advancement of the fuzzy subset with various applications, was presented by Yager [3, 4]. PFS can be used in any situation where IFS is not appropriate.

Fuzziness was improved from intuitionistic and further extended to neutrosophic sets. Smarandache [5] presented neutrosophic sets, a crucial mathematical concept for dealing with indeterminate, and inconsistent data. The set that assigns truth, indeterminacy, and false membership grades for elements that assume values within the interval $]^{-}0, 1^{+}[$ characterizes a neutrosophic set (NS). Wang et al. instituted the generalization of intuitionistic sets and a sub of NS, single-valued NS in [6] which has elements with three membership grades holding the values in interval [0,1].

Chang defined fuzzy topology in [7] as a collection of fuzzy sets that satisfy the axioms of topological spaces. In topology, the fuzzy set theoretical concepts were applied and various notions of topological space were introduced as convergence and compactness [8–10]. Following this, intuitionistic topological spaces were developed into ideas as separation axioms, connectedness, and categorical property [11–15].

Using an equivalence relation in a subset of universe in terms of boundary region and approximations, nanotopological space was introduced. Subsequently, functions on nanotopological spaces, namely, nanocontinuous functions and their characterizations in forms of nanoclosed sets, closure and interior were derived [16]. Weak forms of open sets as nanoaplha open, semiopen, and preopen sets with various form of nano-α-open and semiopen sets corresponding to various case of approximations were derived in [17]. In [18, 19], the concept of nanocompactness and connectedness, generalized closed sets were developed with their properties. The nanosemipreneighbourhoods, semipreinterior, semiprefrontier, semipreexterior, nanogeneralized preregular closed sets were defined, and relations between the existing sets have been examined in [7, 8].

The notion of intuitionistic fuzzy nanotopological space was introduced, and the weak forms of intuitionistic fuzzy nanoopen sets and properties of intuitionistic fuzzy nanocontinuous functions are investigated in [20]. Intuitionistic fuzzy nanogeneralized continuous mappings and closed sets were defined, and their properties were examined in [21, 22]. Thivagar et al. presented the idea of nanotopology neutrosophic units in [23]. [24] introduced the Pythagorean fuzzy topological space by following Chang. By making a fusion of the concepts Pythagorean topological space and nanotopological space, Pythagorean nanotopological spaces (PNTSs) were developed in [25–28].

Multicriteria decision-making is a branch of operation research. Decision-making often involves vagueness which can be effectively handled by fuzzy sets and fuzzy decision-making techniques. In recent years, a great deal of research has been carried out on the theoretical and application aspects of MCDM and fuzzy MCDM. The algorithms of the popular MCDM processes are AHP and TOPSIS. Subsequently, fuzzy MCDM techniques are introduced, and their applications in different disciplines are more effective nowadays. MDCM in general is as follows: problem formulation, identification of the requirements, goal setting, identification of various alternatives, development of criteria, and identification and application of decision-making technique. Various mathematical techniques can be used for this process, and the choice of techniques is made based on the nature of the problem and the level of complexity assigned to the decision-making process. All methods have their own pros and cons. According to a recent literature review by [29], there were more than hundreds of research articles published in the last two decades showing the application of MCDM. The development of the fuzzy decision-making and its tremendous growth is discussed in detail in the review by Mardani et al. [30]. As the fuzzy set has been developed into many fuzzy sets, the MCDM has also been evolved around those sets and transformed into a usable tool in the application for different disciplines. Recently, the MCDM has been developed and used in applications as in [31–35]. Our motivation for the work is that this is still a developing area in fuzzy mathematics, and we want to produce more theoretical concepts and show the application of the work in some real-life situations by combining it with wide-area decision-making. There are many existing models which are still developing in this particular area but when we deal with more fuzzified data, this method is more useful without reducing the constraint when compared to the other concepts. The proposed concepts and model have the more fuzzified values as information but still hold the same condition as the other models, which has a great advantage in dealing with the more vague details of the problem.

The following is how the article is organized: In Section 2, we define generalized closed sets of PNTS along with its characterizations. Sections 3 discusses the generalized semiclosed sets of PNTS. In Section 4, we present an MADM algorithm by using Pythagorean nanotopology, illustrate with the help of numerical example, and conclude in Section 5.

2. Pythagorean Nanogeneralized Closed Sets

$\mathfrak{PNTS}$ has been defined in [28], the weak forms of open sets of $\mathfrak{PNTS}$ have been defined, and their properties were investigated in [26, 28]. In this section, as an extension of these ideas of $\mathfrak{PNTS}$, the generalized closed sets have been developed and various characterizations of these sets have been examined. Throughout this paper, Pythagorean nano is denoted by $\mathfrak{PN}$.

Definition 1 (see [36]). Let the Universe be $\mathfrak{U}$ and equivalence relation on $\mathfrak{U}$ be R, and if $\tau_R(\mathfrak{Y}) = \{\varnothing, \mathfrak{U}, \mathfrak{PNL}_R(\mathfrak{Y}), \mathfrak{PNU}_R(\mathfrak{Y}), \mathfrak{PNB}_R(\mathfrak{Y})\}$ where $\mathfrak{Y} \subseteq \mathscr{U}$, $\tau_R(\mathfrak{Y})$ holds the following axioms:

(1) $\varnothing, \mathfrak{U} \in \tau_{\mathfrak{R}}(\mathfrak{Y})$

(2) If $\mathfrak{T}_{\mathfrak{k}} \in \tau_{\mathfrak{R}}(\mathfrak{Y})$ for $\mathfrak{k} = 1, 2, \cdots$, then $\bigcup_{\mathfrak{k}=1}^{\infty} \mathfrak{T}_{\mathfrak{k}} \in \tau_R(\mathfrak{Y})$

(3) If $\mathfrak{T}_{\mathfrak{k}} \in \tau_R(\mathfrak{Y})$ for $\mathfrak{k} = 1, 2, \cdots, n$, then $\bigcap_{\mathfrak{k}=1}^{n} \mathfrak{T}_{\mathfrak{k}} \in \tau_R(\mathfrak{Y})$

Then, $\tau_R(\mathfrak{Y})$ is termed as $\mathfrak{PN}$ topology ($\mathfrak{PNT}$) on $\mathscr{U}$ *w.r.t* $\mathfrak{Y}$ whereas $\varnothing = \{\mathfrak{w}, 0, 1 | \forall \mathfrak{w} \in \mathfrak{U}\}$, $\mathfrak{U} = \{\mathfrak{w}, 1, 0 | \forall \mathfrak{w} \in \mathscr{U}\}$, $\mathfrak{PNL}_R(\mathfrak{Y}) = \{\langle \mathfrak{w}, \theta_{L\mathfrak{Y}}(\mathfrak{w}), \omega_{L\mathfrak{Y}}(\mathfrak{w}) | \mathfrak{z} \in [\mathfrak{w}]_R, \mathfrak{w} \in \mathfrak{U} \rangle\}$, $\mathfrak{PN}$ $\mathfrak{U}_R(\mathfrak{Y}) = \{\mathfrak{w}, \theta_{R\mathfrak{Y}}(\mathfrak{w}), \omega_{R\mathfrak{Y}}(\mathfrak{w}) | \mathfrak{z} \in [\mathfrak{w}]_R, \mathfrak{w} \in \mathfrak{U}\}$, and $\mathfrak{PNB}_R(\mathfrak{Y}) = \mathfrak{PNU}_R(\mathfrak{Y}) - \mathfrak{PNL}_R(\mathfrak{Y})$ where $\theta_{L\mathfrak{Y}}(\mathfrak{w}) = \wedge_{\mathfrak{k} \in [\mathfrak{w}]_R} \theta_{\mathfrak{Y}}(\mathfrak{k})$

, $\omega_{L\mathfrak{Y}}(\mathfrak{w}) = \vee_{y\in[\mathfrak{w}]_R}\omega_{L\mathfrak{Y}}(\mathfrak{k})$ and $\theta_{R\mathfrak{Y}}(\mathfrak{w}) = \vee_{\mathfrak{k}\in[\mathfrak{w}]_R}\theta_{\mathfrak{Y}}(\mathfrak{k})$, $\omega_{R\mathfrak{Y}}(\mathfrak{w}) = \wedge_{\mathfrak{k}\in[\mathfrak{w}]_R}\omega_{\mathfrak{Y}}(\mathfrak{k})$.

We call $(\mathfrak{U}, \tau_{\mathfrak{R}}(\mathfrak{Y}))$ as Pythagorean nanotopological spaces ($\mathfrak{PNTS}$). The elements of $\tau_R(\mathfrak{Y})$ are called $\mathfrak{PN}$ open ($\mathfrak{PNO}$) sets.

Definition 2. A subset $\mathfrak{T}$ of $\mathfrak{PNTS}(\mathfrak{U}, \tau_R(\mathfrak{Y}))$ is $\mathfrak{PN}$ generalized closed ($\mathfrak{PNgC}$) if $\mathfrak{PNcl}(\mathfrak{T}) \subseteq \mathfrak{G}$ whenever $\mathfrak{T} \subseteq \mathfrak{G}$, $\mathfrak{G}$ is $\mathfrak{PNO}$ in $\mathfrak{U}$.

Theorem 3. *$\mathfrak{T}$ is $\mathfrak{PNgC}$ in a $\mathfrak{PNTU}$ iff $\mathfrak{PNcl}(\mathfrak{T}) - \mathfrak{T}$ has no nonvoid $\mathfrak{PNC}$ set.*

Proof. Let $\mathfrak{T}$ be a $\mathfrak{PNgC}$ and $\mathfrak{F}$ be $\mathfrak{PNC}$ subset of $\mathfrak{PNcl}(\mathfrak{T}) - \mathfrak{T}$. Then, $\mathfrak{T} \subseteq \mathfrak{F}^c$, and $\mathfrak{F}^c$ is $\mathfrak{PNO}$. Since $\mathfrak{T}$ is $\mathfrak{PNgC}$, $\mathfrak{PNcl}(\mathfrak{T}) \subseteq \mathfrak{F}^c$. Therefore, $\mathfrak{F} \subseteq [\mathfrak{PNcl}(\mathfrak{T})]^c$ and $\mathfrak{F} \subseteq \mathfrak{PNcl}(\mathfrak{T}) \cap [\mathfrak{PNcl}(\mathfrak{T})]^c = \varnothing$. That is, $\mathfrak{F} = \varnothing$; thus, $\mathfrak{PNcl}(\mathfrak{T}) - \mathfrak{T}$ cannot have any nonempty $\mathfrak{PNC}$ set.

Conversely, let $\mathfrak{PNcl}(\mathfrak{T}) - \mathfrak{T}$ has no nonvoid $\mathfrak{PNC}$ set. $\mathfrak{G}$ be a $\mathfrak{PNO}$ such that $\mathfrak{T} \subseteq \mathfrak{G}$. Then, $\mathfrak{PNcl}(\mathfrak{T}) \cap \mathfrak{G}^c$ is a $\mathfrak{PNC}$ subset of $\mathfrak{PNcl}(\mathfrak{T}) - \mathfrak{T}$, since $\mathfrak{PNcl}(\mathfrak{T}) \cap \mathfrak{G}^c \leq \mathfrak{PNcl}(\mathfrak{T}) \cap \mathfrak{T}^c$, as $\mathfrak{T} \subseteq \mathfrak{G}$. Therefore, $\mathfrak{PNcl}(\mathfrak{T}) \cap \mathfrak{G}^c = \varnothing$. Since $\mathfrak{PNcl}(\mathfrak{T}) - \mathfrak{T}$ does not have any nonempty $\mathfrak{PNC}$ set. Hence, $\mathfrak{PNcl}(\mathfrak{T}) \subseteq \mathfrak{G}$. Therefore, $\mathfrak{T}$ is $\mathfrak{PNgC}$ in $\mathfrak{U}$.□ □

Theorem 4. *Every $\mathfrak{PNC}$ set is $\mathfrak{PNgC}$.*

Proof. If $\mathfrak{T}$ is $\mathfrak{PNC}$, then $\mathfrak{PNcl}(\mathfrak{T}) = \mathfrak{T}$. Therefore, for every $\mathfrak{PNO}$ set $\mathfrak{G}$ such that $\mathfrak{T} \subseteq \mathfrak{G}$,$\mathfrak{PNcl}(\mathfrak{T}) \subseteq \mathfrak{G}$ and hence $\mathfrak{T}$ is $\mathfrak{PNgC}$.□ □

Theorem 5. *A $\mathfrak{PNgC}$ set $\mathfrak{H}$ is $\mathfrak{PNC}$ in $\mathfrak{U}$ if and only if $\mathfrak{PNcl}(\mathfrak{H}) - \mathfrak{H}$ is $\mathfrak{PNC}$.*

Proof. If $\mathfrak{H}$ is $\mathfrak{PNC}$ in $\mathfrak{U}$, then $\mathfrak{PNcl}(\mathfrak{H}) = \mathfrak{H}$ and hence $\mathfrak{PNcl}(\mathfrak{H}) - \mathfrak{H} = \varnothing$ which is $\mathfrak{PNC}$. Conversely, $\mathfrak{PNcl}(\mathfrak{H}) - \mathfrak{H}$ is $\mathfrak{PNC}$, and $\mathfrak{PNcl}(\mathfrak{H}) - \mathfrak{H} = \varnothing$. Therefore, $\mathfrak{PNcl}(\mathfrak{H}) = \mathfrak{H}$, and hence, $\mathfrak{H}$ is $\mathfrak{PNC}$.□ □

Theorem 6. *If $\mathfrak{H}$ and $\mathfrak{T}$ are $\mathfrak{PNgC}$, then $\mathfrak{H} \bigcup \mathfrak{T}$ is $\mathfrak{PNgC}$.*

Proof. Let $\mathfrak{G}$ be $\mathfrak{PNO}$ in $\mathfrak{U}$ such that $\mathfrak{H} \bigcup \mathfrak{T} \subseteq \mathfrak{G}$. Then, $\mathfrak{H}$ and $\mathfrak{T} \subseteq \mathfrak{G}$, since $\mathfrak{H}$ is $\mathfrak{PNgC}$ and $\mathfrak{G}$ is a $\mathfrak{PNO}$ set having $\mathfrak{PNcl}(\mathfrak{H}) \subseteq \mathfrak{G}$.

Similarly, $\mathfrak{PNcl}(\mathfrak{T}) \subseteq \mathfrak{G}$. $\mathfrak{PNcl}(\mathfrak{H} \bigcup \mathfrak{T}) = \mathfrak{PNcl}(\mathfrak{H}) \bigcup \mathfrak{PNcl}(\mathfrak{T}) \subseteq \mathfrak{G}$.$\mathfrak{H} \bigcup \mathfrak{T}$ is $\mathfrak{PNgC}$.□ □

Theorem 7. *Let $\mathfrak{H}$ be a $\mathfrak{PNgC}$ subset of $\mathfrak{U}$ and $\mathfrak{T} \subseteq \mathfrak{H}$ be a $\mathfrak{PNgC}$ set to $\mathfrak{H}$. Then, $\mathfrak{T}$ is $\mathfrak{PNgC}$ to $\mathfrak{U}$.*

Proof. Let $\mathfrak{G}$ be a $\mathfrak{PNO}$ set in $\mathfrak{U}$ such that $\mathfrak{T} \subseteq \mathfrak{G}$. Then, $\mathfrak{T} \subseteq \mathfrak{H} \cap \mathfrak{G}$.$\mathfrak{H} \cap \mathfrak{G}$ is a $\mathfrak{PNO}$ to $\mathfrak{H}$ containing $\mathfrak{T}$. Since $\mathfrak{T}$ is $\mathfrak{PNgC}$ to $\mathfrak{H}$,$\mathfrak{PNcl}_{\mathfrak{H}}(\mathfrak{T}) = \mathfrak{H} \cap \mathfrak{G}$, where $\mathfrak{PNcl}_{\mathfrak{H}}(\mathfrak{T})$ is the $\mathfrak{PN}$ closure of $\mathfrak{T}$ in $\mathfrak{H}$. Then, $\mathfrak{PNcl}(\mathfrak{T}) \subseteq \mathfrak{H} \cap \mathfrak{G}$ and $\mathfrak{H} \subseteq \mathfrak{G} \bigcup [\mathfrak{PNcl}(\mathfrak{T})]^c$. Also, $\mathfrak{G} \bigcup [\mathfrak{PNcl}(\mathfrak{T})]^c$ is $\mathfrak{PNO}$. Since $\mathfrak{H}$ is $\mathfrak{PNgC}$ in $\mathfrak{U}$,$\mathfrak{PNcl}(\mathfrak{H}) \subseteq \mathfrak{G} \bigcup [\mathfrak{PNcl}(\mathfrak{T})]^c$. Therefore, $\mathfrak{PNcl}(\mathfrak{T}) \subseteq \mathfrak{PNcl}(\mathfrak{H}) \subseteq \mathfrak{G} \bigcup [\mathfrak{PNcl}(\mathfrak{T})]^c$, since $\mathfrak{T} \subseteq \mathfrak{H}$. Therefore, $\mathfrak{PNcl}(\mathfrak{T}) \subseteq \mathfrak{G}$. Thus, for every $\mathfrak{PNO}$ set $\mathfrak{G}$ in $\mathfrak{U}$ containing $\mathfrak{T}$, $\mathfrak{PNcl}(\mathfrak{T}) \subseteq \mathfrak{G}$. Therefore, $\mathfrak{T}$ is $\mathfrak{PNgC}$ in $\mathfrak{U}$.□ □

Now, we prove the intersection of a $\mathfrak{PNgC}$ and a $\mathfrak{PNC}$ is $\mathfrak{PNgC}$.

Corollary 8. *If $\mathfrak{H}$ is $\mathfrak{PNgC}$ and $\mathfrak{F}$ is $\mathfrak{PNC}$ in $\mathfrak{U}$, then $\mathfrak{H} \cap \mathfrak{F}$ is $\mathfrak{PNgC}$.*

Proof. Since $\mathfrak{H} \cap \mathfrak{F}$ is $\mathfrak{PNC}$ to $\mathfrak{H}$, it is $\mathfrak{PNgC}$ to $\mathfrak{H}$. Since $\mathfrak{H} \cap \mathfrak{F} \subseteq \mathfrak{H}$ where $\mathfrak{H}$ is $\mathfrak{PNgC}$ in $\mathfrak{U}$ and $\mathfrak{H} \cap \mathfrak{F}$ is $\mathfrak{PNgC}$ to $\mathfrak{H}$, $\mathfrak{H} \cap \mathfrak{F}$ is $\mathfrak{PNgC}$ in $\mathfrak{U}$ by the above theorem.□ □

Theorem 9. *If $\mathfrak{H}$ is $\mathfrak{PNgC}$ and $\mathfrak{H} \subseteq \mathfrak{T} \subseteq \mathfrak{PNcl}(\mathfrak{H})$, then $\mathfrak{T}$ is $\mathfrak{PNgC}$.*

Proof. Since $\mathfrak{H}$ is $\mathfrak{PNgC}$, $\mathfrak{PNcl}(\mathfrak{H}) - \mathfrak{H}$ has no nonempty $\mathfrak{PNC}$ subset. Since $\mathfrak{PNcl}(\mathfrak{T}) - \mathfrak{T} \subseteq \mathfrak{PNcl}(\mathfrak{H}) - \mathfrak{H}$, $\mathfrak{PNcl}(\mathfrak{T}) - \mathfrak{T}$ also does not have any nonempty $\mathfrak{PNC}$ set. Therefore, $\mathfrak{T}$ is $\mathfrak{PNgC}$.□ □

Theorem 10. *Let $\mathfrak{U}$ and $\mathfrak{B}$ be two $\mathfrak{PNT}$ spaces and $\mathfrak{H} \subseteq \mathfrak{B} \subseteq \mathfrak{U}$ and $\mathfrak{H}$ be $\mathfrak{PNgC}$ in $\mathfrak{U}$. Then, $\mathfrak{H}$ is $\mathfrak{PNgC}$ in $\mathfrak{B}$.*

Proof. Let $\mathfrak{G}_1$ be $\mathfrak{PNO}$ in $\mathfrak{B}$ such that $\mathfrak{H} \subseteq \mathfrak{G}_1$. Then, $\mathfrak{G}_1 = \mathfrak{B} \cap \mathfrak{G}$ where $\mathfrak{G}$ is $\mathfrak{PNO}$ in $\mathfrak{U}$ and $\mathfrak{H} = \mathfrak{B} \cap \mathfrak{G}$. Therefore, $\mathfrak{H} \subseteq \mathfrak{G}$. That is, $\mathfrak{G}$ is $\mathfrak{PNO}$ containing $\mathfrak{H}$. Since $\mathfrak{H}$ is $\mathfrak{PNgC}$ in $\mathfrak{U}$,$\mathfrak{PNcl}(\mathfrak{H}) \subseteq \mathfrak{G}$. Therefore, $\mathfrak{B} \cap \mathfrak{PNcl}(\mathfrak{H}) \subseteq \mathfrak{B} \cap \mathfrak{G}$. That is, $\mathfrak{PNcl}(\mathfrak{H}) \subseteq \mathfrak{G}_1$, for every $\mathfrak{PNO}\mathfrak{G}_1$ in $\mathfrak{B}$ such that $\mathfrak{G}_1 \supseteq \mathfrak{H}$. Therefore, $\mathfrak{H}$ is $\mathfrak{PNgC}$ in $\mathfrak{B}$.□ □

Theorem 11. *Every subset of $\mathfrak{PNTU}$ is $\mathfrak{PNgC}$ iff $\mathfrak{PNU}_R(\mathfrak{U}) = \mathfrak{U}$.*

Proof. Let $\mathfrak{PNU}_R(\mathfrak{U}) = \mathfrak{U}$ and $\mathfrak{H} \subseteq \mathfrak{U}$. Let $\mathfrak{G}$ be $\mathfrak{PNO}$ in $\mathfrak{U}$ such that $\mathfrak{H} \subseteq \mathfrak{G}$. Then, $\mathfrak{PNcl}(\mathfrak{H}) \subseteq \mathfrak{PNcl}(\mathfrak{G}) \subseteq \mathfrak{G}$, since $\mathfrak{U}$, $\varnothing$, $\mathfrak{PNL}_R(\mathfrak{U})$, and $\mathfrak{PNB}_R(\mathfrak{U})$ are the only sets which are $\mathfrak{PNO}$ as well as $\mathfrak{PNC}$ in $\mathfrak{U}$, when $\mathfrak{PNU}_R(\mathfrak{U}) = \mathfrak{U}$. Thus, $\mathfrak{PNcl}(\mathfrak{H}) \subseteq \mathfrak{G}$ whenever $\mathfrak{G}$ is $\mathfrak{PNO}$ and $\mathfrak{H} \subseteq \mathfrak{G}$.$\mathfrak{H}$ is $\mathfrak{PNgC}$, if $\mathfrak{PNU}_R(\mathfrak{U}) = \mathfrak{U}$.□ □

Conversely, assume that every subset of $\mathfrak{U}$ is $\mathfrak{PNgC}$. Let $\mathfrak{G} \in \tau_R(\mathfrak{U})$. Then, $\mathfrak{G}$ is $\mathfrak{PNgC}$. Since $\mathfrak{G}$ is $\mathfrak{PNO}$ and $\mathfrak{G} \subseteq \mathfrak{G}$, $\mathfrak{PNcl}(\mathfrak{G}) \subseteq \mathfrak{G}$; hence, $\mathfrak{G}$ is $\mathfrak{PNC}$. Thus, whenever $\mathfrak{G}$ is $\mathfrak{PNO}$ in $\mathfrak{U}$, $\mathfrak{PNcl}(\mathfrak{G}) \subseteq \mathfrak{G}$ and $\mathfrak{PNcl}(\mathfrak{G})$ is $\mathfrak{PNO}$ in $\mathfrak{U}$. The $\mathfrak{PN}$ closure of each $\mathfrak{PNO}$ set in $\mathfrak{U}$ is $\mathfrak{PNO}$. $\mathfrak{U}$ is extremally disconnected, and hence, $\mathfrak{PNU}_R(\mathfrak{U}) = \mathfrak{U}$.

Definition 12. A set $\mathfrak{H}$ in a $\mathfrak{PNTSU}$ is $\mathfrak{PN}$ generalized open ($\mathfrak{PNgO}$) if $\mathfrak{H}^c$ is $\mathfrak{PNgC}$.

Theorem 13. *Every $\mathfrak{PNO}$ set is $\mathfrak{PNgO}$.*

Proof. We know that every $\mathfrak{PNC}$ is $\mathfrak{PNgC}$. Thus, if we take complement, we get every $\mathfrak{PNO}$ is $\mathfrak{PNgO}$. But the converse need not to be true. That is, every $\mathfrak{PNgO}$ need not be $\mathfrak{PNO}$.□ □

Theorem 14. *$\mathfrak{H}$ is $\mathfrak{PNgO}$ iff $\mathfrak{F} \subseteq \mathfrak{PNint}(\mathfrak{H})$ whenever $\mathfrak{F}$ is $\mathfrak{PNC}$ and $\mathfrak{F} \subseteq \mathfrak{H}$.*

Proof. Let $\mathfrak{H}$ be a $\mathfrak{PNgO}$ in $\mathfrak{U}$. Then, $\mathfrak{H}^c$ is $\mathfrak{PNgC}$. Therefore, $\mathfrak{PNcl}(\mathfrak{H}^c) \subseteq \mathfrak{G}$ whenever $\mathfrak{G}$ is $\mathfrak{PNO}$ and $\mathfrak{H}^c \subseteq \mathfrak{G}$. Let $\mathfrak{F}$ be $\mathfrak{PNC}$ and $\mathfrak{F} \subseteq \mathfrak{H}$. Then, $\mathfrak{H}^c \subseteq \mathfrak{F}^c$, and $\mathfrak{F}^c$ is $\mathfrak{PNO}$. Therefore, $\mathfrak{PNcl}(\mathfrak{H}^c) \subseteq \mathfrak{F}^c$. That is, $\mathfrak{U} - \mathfrak{PNint}(\mathfrak{H}) \subseteq \mathfrak{U} - \mathfrak{F}$. Hence, $\mathfrak{F} \subseteq \mathfrak{PNint}(\mathfrak{H})$.

Conversely, let $\mathfrak{F} \subseteq \mathfrak{PNint}(\mathfrak{H})$ for every $\mathfrak{PNC}$ set such that $\mathfrak{F} \subseteq \mathfrak{H}$, and let $\mathfrak{G}$ be $\mathfrak{PNO}$ such that $\mathfrak{H}^c \subseteq \mathfrak{G}$. Then, $\mathfrak{G}^c \subseteq \mathfrak{H}$ where $\mathfrak{G}^c$ is $\mathfrak{PNC}$. Therefore, $\mathfrak{G}^c \subseteq \mathfrak{PNint}(\mathfrak{H})$. Therefore, $\mathfrak{U} - \mathfrak{PNint}(\mathfrak{H}) \subseteq \mathfrak{G}$. Therefore, $\mathfrak{PNcl}(\mathfrak{U} - \mathfrak{H}) \subseteq \mathfrak{G}$. That is, $\mathfrak{PNcl}(\mathfrak{H}^c) \subseteq \mathfrak{G}$. Therefore, $\mathfrak{H}^c$ is $\mathfrak{PNgC}$, and hence, $\mathfrak{H}$ is $\mathfrak{PNgO}$ in $\mathfrak{U}$.□ □

Definition 15. If $\mathfrak{H}$ and $\mathfrak{T}$ are subset of $\mathfrak{PNTSU}$, then $\mathfrak{H}$ and $\mathfrak{T}$ are said to be $\mathfrak{PN}$ separated ($\mathfrak{PNS}$), if $\mathfrak{H} \bigcap \mathfrak{PNcl}(\mathfrak{T}) = \varnothing$ and $\mathfrak{T} \bigcap \mathfrak{PNcl}(\mathfrak{H}) = \varnothing$.

Theorem 16. *If $\mathfrak{H}$ and $\mathfrak{T}$ are $\mathfrak{PNS}$ and $\mathfrak{PNgO}$, then $\mathfrak{H} \bigcup \mathfrak{T}$ is $\mathfrak{PNgO}$.*

Proof. Let $\mathfrak{F}$ be $\mathfrak{PNC}$ in $\mathfrak{U}$ such that $\mathfrak{F} \subseteq \mathfrak{H} \bigcup \mathfrak{T}$. Since $\mathfrak{H}$ and $\mathfrak{T}$ are $\mathfrak{PNS}$, $\mathfrak{PNcl}(\mathfrak{H}) \bigcap \mathfrak{T} = \varnothing$. Therefore, no element of $\mathfrak{PNcl}(\mathfrak{H})$ belongs to $\mathfrak{T}$. Thus, no element of $\mathfrak{F} \bigcap \mathfrak{PNcl}(\mathfrak{H})$ belongs to $\mathfrak{T}$. Hence, every element of $\mathfrak{F} \bigcap \mathfrak{PNcl}(\mathfrak{H})$ belongs to $\mathfrak{H}$, since $\mathfrak{F} \subseteq \mathfrak{H} \bigcup \mathfrak{T}$. That is, $\mathfrak{F} \bigcap \mathfrak{PNcl}(\mathfrak{H}) \subseteq \mathfrak{H}$. Thus, $\mathfrak{F} \bigcap \mathfrak{PNcl}(\mathfrak{H})$ is $\mathfrak{PNC}$ subset of $\mathfrak{H}$. Since $\mathfrak{H}$ is $\mathfrak{PNgO}$, $\mathfrak{F} \bigcap \mathfrak{PNcl}(\mathfrak{H}) \subseteq \mathfrak{PNint}(\mathfrak{H})$.

Similarly, $\mathfrak{F} \bigcap \mathfrak{PNcl}(\mathfrak{T}) \subseteq \mathfrak{PNint}(\mathfrak{T})$. Since $\mathfrak{F} \subseteq \mathfrak{H} \bigcup \mathfrak{H}$, $\mathfrak{F} = \mathfrak{F} \bigcap (\mathfrak{H} \bigcup \mathfrak{T}) = (\mathfrak{F} \bigcap \mathfrak{H}) \bigcup (\mathfrak{F} \bigcap \mathfrak{T}) \subseteq (\mathfrak{F} \bigcap \mathfrak{PNcl}(\mathfrak{H})) \bigcup (\mathfrak{F} \bigcap \mathfrak{PNcl}(\mathfrak{T}) \subseteq \mathfrak{PNint}(\mathfrak{H}) \bigcup \mathfrak{PNint}(\mathfrak{T}) \subseteq \mathfrak{PNint}(\mathfrak{H} \bigcup \mathfrak{T})$. Thus, $\mathfrak{F} \subseteq \mathfrak{PNint}(\mathfrak{H} \bigcup \mathfrak{T})$, whenever $\mathfrak{F}$ is $\mathfrak{PNC}$ and $\mathfrak{F} \subseteq \mathfrak{H} \bigcup \mathfrak{T}$.□ □

Therefore, $\mathfrak{H} \bigcup \mathfrak{T}$ is $\mathfrak{PNgO}$.

Theorem 17. *If $\mathfrak{H}$ and $\mathfrak{T}$ are $\mathfrak{PNgO}$ in $\mathfrak{U}$, then $\mathfrak{H} \bigcap \mathfrak{T}$ is $\mathfrak{PNgO}$.*

Proof. $\mathfrak{H}^c$ and $\mathfrak{T}^c$ are $\mathfrak{PNgC}$ and hence $\mathfrak{H}^c \bigcup \mathfrak{T}^c = (\mathfrak{H} \bigcap \mathfrak{T})^c$ is $\mathfrak{PNgC}$ and hence $(\mathfrak{H} \bigcap \mathfrak{T})$ is $\mathfrak{PNgO}$.□ □

Theorem 18. *If $\mathfrak{H}$ and $\mathfrak{T}$ are $\mathfrak{PNgC}$ sets such that $\mathfrak{H}^c$ and $\mathfrak{T}^c$ are $\mathfrak{PNS}$, then $\mathfrak{H} \bigcap \mathfrak{T}$ is $\mathfrak{PNgC}$.*

Proof. $\mathfrak{H}^c$ and $\mathfrak{T}^c$ are $\mathfrak{PNS}$ and $\mathfrak{PNgO}$, and hence, $\mathfrak{H}^c \bigcup \mathfrak{T}^c$ is $\mathfrak{PNgO}$. Therefore, $\mathfrak{H} \bigcap \mathfrak{T}$ is $\mathfrak{PNgC}$.□ □

Theorem 19. *$\mathfrak{H}$ is $\mathfrak{PNgO}$ iff $\mathfrak{G} = \mathfrak{U}$ where $\mathfrak{G}$ is $\mathfrak{PNO}$ and $\mathfrak{PNint}(\mathfrak{H})$.*

$$\bigcup \mathfrak{H}^c \subseteq \mathfrak{G}. \tag{1}$$

Proof. Let $\mathfrak{H}$ be $\mathfrak{PNgO}$. Let $\mathfrak{G}$ be $\mathfrak{PNO}$ such that $\mathfrak{PNint}(\mathfrak{H}) \bigcup \mathfrak{H}^c \subseteq \mathfrak{G}$. Then, $\mathfrak{G}^c \subseteq \mathfrak{H} \bigcap \mathfrak{PNcl}(\mathfrak{H}^c) \subseteq \mathfrak{PNcl}(\mathfrak{H}^c) - \mathfrak{H}^c$. Since $\mathfrak{H}^c$ is $\mathfrak{PNgC}$, $\mathfrak{PNcl}(\mathfrak{H}^c) - \mathfrak{H}^c$ cannot have any nonempty $\mathfrak{PNC}$ set. But $\mathfrak{G}^c$ is $\mathfrak{PNC}$ subset of $\mathfrak{PNcl}(\mathfrak{H}^c) - \mathfrak{H}^c$. Therefore, $\mathfrak{G}^c \subseteq \varnothing$. That is, $\mathfrak{G} = \mathfrak{U}$.

Conversely, assume that whenever $\mathfrak{G}$ is $\mathfrak{PNO}$ and $\mathfrak{PNint}(\mathfrak{H}) \bigcup \mathfrak{H}^c \subseteq \mathfrak{G}$, then $\mathfrak{G} \subseteq \mathfrak{U}$. Let $\mathfrak{F}$ be $\mathfrak{PNC}$ such that $\mathfrak{F} \subseteq \mathfrak{H}$. Then, $\mathfrak{PNint}(\mathfrak{H}) \bigcup \mathfrak{H}^c \subseteq \mathfrak{PNint}(\mathfrak{H}) \bigcup \mathfrak{F}^c$ which is $\mathfrak{PNO}$. Therefore, $\mathfrak{PNint}(\mathfrak{H}) \bigcup \mathfrak{H}^c = \mathfrak{U}$. That is, $\mathfrak{F} \subseteq \mathfrak{PNint}(\mathfrak{H})$, since every $\mathfrak{x} \in \mathfrak{F}$, belongs to $\mathfrak{PNint}(\mathfrak{H})$. Thus, $\mathfrak{F} \subseteq \mathfrak{PNint}(\mathfrak{H})$ whenever $\mathfrak{F}$ is $\mathfrak{PNC}$ and $\mathfrak{F} \subseteq \mathfrak{H}$. Therefore, $\mathfrak{H}$ is $\mathfrak{PNgO}$.□ □

Theorem 20. *If $\mathfrak{PNint}(\mathfrak{H}) \subseteq \mathfrak{T} \subseteq \mathfrak{H}$ and if $\mathfrak{H}$ is $\mathfrak{PNgO}$, then $\mathfrak{T}$ is also $\mathfrak{PNgO}$.*

Proof. $\mathfrak{H}^c \subseteq \mathfrak{T}^c \subseteq \mathfrak{PNcl}(\mathfrak{H}^c)$ where $\mathfrak{H}^c$ is $\mathfrak{PNgC}$, and hence, $\mathfrak{T}^c$ is $\mathfrak{PNgC}$. Therefore, $\mathfrak{T}$ is $\mathfrak{PNgO}$.□ □

Theorem 21. *$\mathfrak{H}$ is $\mathfrak{PNgC}$ if and only if $\mathfrak{PNcl}(\mathfrak{H}) - \mathfrak{H}$ is $\mathfrak{PNgO}$.*

Proof. Let $\mathfrak{H}$ be $\mathfrak{PNgC}$. Let $\mathfrak{F}$ be a $\mathfrak{PNC}$ such that $F \subseteq \mathfrak{PNcl}(\mathfrak{H}) - \mathfrak{H}$. Then, $\mathfrak{F} = \varnothing$, since $\mathfrak{PNcl}(\mathfrak{H}) - \mathfrak{H}$ cannot have any nonempty closed set. Therefore, $\mathfrak{F} \subseteq \mathfrak{PNint}(\mathfrak{PNcl}(\mathfrak{H}) - \mathfrak{H})$, and hence, $\mathfrak{PNcl}(\mathfrak{H}) - \mathfrak{H}$ is $\mathfrak{PNgO}$.

Conversely, if $\mathfrak{PNcl}(\mathfrak{H}) - \mathfrak{H}$ is $\mathfrak{PNgO}$ and $\mathfrak{G}$ is $\mathfrak{PNO}$ such that $\mathfrak{H} \subseteq \mathfrak{G}$, then $\mathfrak{PNcl}(\mathfrak{H}) \bigcap \mathfrak{G}^c \subseteq \mathfrak{PNcl}(\mathfrak{H}) \bigcap \mathfrak{H}^c = \mathfrak{PNcl}(\mathfrak{H}) - \mathfrak{H}$ where $\mathfrak{PNcl}(\mathfrak{H}) \bigcap \mathfrak{G}^c$ is $\mathfrak{PNC}$. Since $\mathfrak{PNcl}(\mathfrak{H}) - \mathfrak{H}$ is $\mathfrak{PNgO}$, $\mathfrak{PNcl}(\mathfrak{H}) \bigcap \mathfrak{G}^c \subseteq \mathfrak{PNint}[\mathfrak{PNcl}(\mathfrak{H}) - \mathfrak{H}] = \varnothing$. Therefore, $\mathfrak{PNcl}(\mathfrak{H}) \bigcap \mathfrak{G}^c = \varnothing$, and hence, $\mathfrak{PNcl}(\mathfrak{H}) \subseteq \mathfrak{G}$. Thus, whenever $\mathfrak{G}$ is $\mathfrak{PNO}$ and $\mathfrak{H} \subseteq \mathfrak{G}$, $\mathfrak{PNcl}(\mathfrak{H}) \subseteq \mathfrak{G}$.$\mathfrak{H}$ is $\mathfrak{PNgC}$.□ □

Theorem 22. *In a $\mathfrak{PNTS}$ $(\mathfrak{U}, \tau_R(\mathfrak{X}))$, if $\mathfrak{PNL}_R(\mathfrak{X}) = \mathfrak{PNU}_R(\mathfrak{X})$, then any set $\mathfrak{H}$ such that $\mathfrak{H}\mathfrak{U}\mathfrak{PNL}_R(\mathfrak{X})$ is the only $\mathfrak{PNgC}$ set in $\mathfrak{U}$.*

Proof. When $\mathfrak{PNL}_R(\mathfrak{X}) = \mathfrak{U}$, $\mathfrak{U}$ and $\varnothing$ are the only $\mathfrak{PNO}$ sets and hence for any subset $\mathfrak{H}$ of $\mathfrak{U}$, $\mathfrak{U}$ is the only $\mathfrak{PNO}$ set holding it. Therefore, $\mathfrak{PNcl}(\mathfrak{H}) \subseteq \mathfrak{G}$ for every $\mathfrak{PNO}$ set $\mathfrak{G}$ having $\mathfrak{H}$. Thus, every subset $\mathfrak{H}$ of $\mathfrak{U}$ is PNg-C, if $\mathfrak{PNL}_R(\mathfrak{X}) = \mathfrak{PNU}_R(\mathfrak{X}) = \mathfrak{U}$. When $\mathfrak{PNL}_R(\mathfrak{X}) \neq \mathfrak{U}$, the $\mathfrak{PNO}$ sets in $\mathfrak{U}$ are $\mathfrak{U}$, $\varnothing$, and $\mathfrak{PNL}_R(\mathfrak{X})$. If $\mathfrak{H} \subseteq \mathfrak{PNL}_R(\mathfrak{X})$, then $\mathfrak{PNO}$ sets having $\mathfrak{H}$ are $\mathfrak{PNL}_R(\mathfrak{X})$ and $\mathfrak{U}$. Also, $\mathfrak{PNL}_R(\mathfrak{X}) \neq \mathfrak{U}$. And $\mathfrak{PNcl}(\mathfrak{H})\mathfrak{U}\mathfrak{PNL}_R(\mathfrak{X})$. Therefore, $\mathfrak{H}$ is not $\mathfrak{PNgC}$. If $\mathfrak{H}\mathfrak{U}\mathfrak{PNL}_R(\mathfrak{X})$, then $\mathfrak{U}$ is the only $\mathfrak{PNO}$ set holding $\mathfrak{H}$ and hence $\mathfrak{PNcl}(\mathfrak{H}) \subseteq \mathfrak{G}$ for every $\mathfrak{PNO}$ set $\mathfrak{G} \supseteq \mathfrak{H}$. Therefore, $\mathfrak{H}$ is $\mathfrak{PNgC}$. Thus, only those sets $\mathfrak{H}$ such that $\mathfrak{H}\mathfrak{U}\mathfrak{PNL}_R(\mathfrak{X})$ are $\mathfrak{PNgC}$, if $\mathfrak{PNL}_R(\mathfrak{X}) = \mathfrak{PNU}_R(\mathfrak{X})$.□ □

Theorem 23. *If $\mathfrak{PNL}_R(\mathfrak{X}) = \varnothing$ and $\mathfrak{PNU}_R(\mathfrak{X}) \neq \mathfrak{U}$ in a $\mathfrak{PNTS}$, then those sets $\mathfrak{H}$ for which $\mathfrak{H}\mathfrak{U}\mathfrak{PNU}_R(\mathfrak{X})$ are the only $\mathfrak{PNgC}$ sets.*

Proof. $\tau_R(\mathfrak{X}) = \{\varnothing, \mathfrak{U}, \mathfrak{PNU}_R(\mathfrak{X})\}$. If $\mathfrak{H} \subseteq \mathfrak{PNU}_R(\mathfrak{X})$, then $\mathfrak{U}$ and $\mathfrak{PNU}_R(\mathfrak{X})$ are the $\mathfrak{PNO}$ set containing $\mathfrak{H}$. $\mathfrak{PNcl}(\mathfrak{H}) = \mathfrak{U}$; hence, $\mathfrak{PNcl}(\mathfrak{H})\mathfrak{U}\mathfrak{PNU}_R(\mathfrak{X})$. Thus, $\mathfrak{PNcl}(\mathfrak{H})\mathfrak{U}\mathfrak{G}$ when $\mathfrak{G} = \mathfrak{PNU}_R(\mathfrak{X})$. Therefore, $\mathfrak{H}$ is not $\mathfrak{PNgC}$. But, if $\mathfrak{H}\mathfrak{U}\mathfrak{PNU}_R(\mathfrak{X})$, then $\mathfrak{U}$ is the only $\mathfrak{PNO}$ set that contains $\mathfrak{H}$ and hence $\mathfrak{PNcl}(A) \subseteq \mathfrak{G}$ whenever $\mathfrak{G}$ is $\mathfrak{PNO}$ and $\mathfrak{G} \supseteq \mathfrak{H}$. Therefore, $\mathfrak{H}$ is $\mathfrak{PNgC}$. Thus, only those subsets $\mathfrak{H}$ of $\mathfrak{U}$ such that $\mathfrak{H}\mathfrak{U}\mathfrak{PNU}_R(\mathfrak{X})$ are $\mathfrak{PNgC}$ in $\mathfrak{U}$, if $\mathfrak{PNL}_R(\mathfrak{X}) = \varnothing$ and $\mathfrak{PNU}_R(\mathfrak{X}) \neq \mathfrak{U}$.□ □

Theorem 24. *If $\mathfrak{PNL}_R(\mathfrak{X}) \neq \varnothing$ and $\mathfrak{PNU}_R(\mathfrak{X}) = \mathfrak{U}$, then every subset $\mathfrak{H}$ of $\mathfrak{U}$ is $\mathfrak{PNgC}$.*

Theorem 25. *If $\mathfrak{PNL}_R(\mathfrak{X}) \neq \mathfrak{PNU}_R(\mathfrak{X})$ and $\mathfrak{PNL}_R(\mathfrak{X}) \neq \varnothing$, $\mathfrak{PNU}_R(\mathfrak{X}) \neq \mathfrak{U}$, and only those subsets $\mathfrak{H}$ of $\mathfrak{U}$ such that $\mathfrak{H}\mathfrak{U}\mathfrak{PNU}_R(\mathfrak{X})$ are $\mathfrak{PNgC}$ in $\mathfrak{U}$.*

Proof. $\tau_R(\mathfrak{X}) = \{\varnothing, \mathfrak{U}, \mathfrak{PNL}_R(\mathfrak{X}), \mathfrak{PNU}_R(\mathfrak{X}), \mathfrak{PNB}_R(\mathfrak{X})\}$. If $\mathfrak{H} \subseteq \mathfrak{PNL}_R(\mathfrak{X})$, then $\mathfrak{PNO}$ sets containing $\mathfrak{H}$ are $\mathfrak{PNL}_R(\mathfrak{X})$, $\mathfrak{PNU}_R(\mathfrak{X})$, and $\mathfrak{U}$. But $\mathfrak{PNcl}(\mathfrak{H}) = [\mathfrak{PNB}_R(\mathfrak{X})]^c = \mathfrak{PNL}_R(\mathfrak{X}) \bigcup [\mathfrak{PNU}_R(\mathfrak{X})]^c \mathfrak{U}\mathfrak{PNL}_R(\mathfrak{X})$, since $[\mathfrak{PNU}_R(\mathfrak{X})]^c \neq \varnothing$. Therefore, $\mathfrak{H}$ is not $\mathfrak{PNgC}$. If $\mathfrak{H} \subseteq \mathfrak{PNB}_R(\mathfrak{X})$, then $\mathfrak{PNB}_R(\mathfrak{X})$, $\mathfrak{PNU}_R(\mathfrak{X})$, and $\mathfrak{U}$ are the $\mathfrak{PNO}$ sets containing $\mathfrak{H}$ and $\mathfrak{PNcl}(\mathfrak{H}) = [\mathfrak{PNL}_R(X)]^c = \mathfrak{PNB}_R(\mathfrak{X}) \bigcup [\mathfrak{PNU}_R(\mathfrak{X})]^c \mathfrak{U}\mathfrak{PNB}_R(\mathfrak{X})$. Therefore, $\mathfrak{H}$ is not $\mathfrak{PNgC}$. If $\mathfrak{H} \subseteq \mathfrak{PNU}_R(\mathfrak{X})$, neither a subset of $\mathfrak{PNL}_R(\mathfrak{X})$ is a $\mathfrak{PNO}$ set containing $\mathfrak{H}$ for which $\mathfrak{PNcl}(\mathfrak{H})\mathfrak{U}\mathfrak{PNU}_R(\mathfrak{X})$. Therefore, $\mathfrak{H}$ is not $\mathfrak{PNgC}$. If $\mathfrak{H}\mathfrak{U}\mathfrak{PNU}_R(\mathfrak{X})$, then $\mathfrak{U}$ is the only $\mathfrak{PNO}$ set containing $\mathfrak{H}$ and hence $\mathfrak{PNcl}(\mathfrak{H}) \subseteq \mathfrak{G}$ for every $\mathfrak{PNO}\mathfrak{G} \supseteq \mathfrak{H}$. Therefore, $\mathfrak{H}$ is $\mathfrak{PNgC}$. Thus, only those $\mathfrak{H} \subseteq \mathfrak{U}$ for which $\mathfrak{H}\mathfrak{U}\mathfrak{PNU}_R(\mathfrak{X})$ are $\mathfrak{PNgC}$ in $\mathfrak{U}$.□ □

3. Pythagorean Nanosemigeneralized Closed Sets

As we have defined in the last section, we have extended the concept of $\mathfrak{PN}$ generalized closed sets to $\mathfrak{PN}$ semigeneralized closed sets and investigated their properties.

Definition 26. A subset $\mathfrak{H}$ of $\mathfrak{PNTSU}$ is said to be $\mathfrak{PN}$ semigeneralized closed ($\mathfrak{PNsgC}$), if $\mathfrak{PNscl}(\mathfrak{H}) \subseteq \mathfrak{G}$ whenever $\mathfrak{G}$ is $\mathfrak{PNSO}$ and $\mathfrak{H} \subseteq \mathfrak{G}$. The set $\mathfrak{H}$ is named as $\mathfrak{PN}$ semigeneralized open ($\mathfrak{PNsgO}$) if $\mathfrak{H}^c$ is $\mathfrak{PNsgC}$.

Definition 27. If $\mathfrak{H} \subseteq \mathfrak{U}$, then the $\mathfrak{PN}$ semigeneralized closure represented by $\mathfrak{PNsgcl}(\mathfrak{H})$ is defined as the smallest $\mathfrak{PNsgC}$ set having $\mathfrak{H}$.

The $\mathfrak{PN}$ semigeneralized interior of A, symbolized by $\mathfrak{PNsgint}(\mathfrak{H})$, is defined as the largest $\mathfrak{PNsgO}$ set in $\mathfrak{H}$.

Remark 28. For subsets $\mathfrak{H}$ and $\mathfrak{T}$ of a $\mathfrak{PNTS}(u, \tau_R(\mathfrak{X}))$,

(1) $\mathfrak{U} - \mathfrak{PNsgint}(\mathfrak{H}) = \mathfrak{PNsgcl}(\mathfrak{U} - \mathfrak{H})$

(2) $\mathfrak{U} - \mathfrak{PNsgcl}(\mathfrak{H}) = \mathfrak{PNsgint}(\mathfrak{U} - \mathfrak{H})$

Theorem 29. *Every $\mathfrak{PNSC}$ set is $\mathfrak{PNsgC}$.*

Theorem 30. *A set $\mathfrak{H}$ is $\mathfrak{PNsgC}$ in $\mathfrak{U}$ iff $\mathfrak{PNscl}(\mathfrak{H}) - \mathfrak{H}$ has no nonempty, $\mathfrak{PNSC}$ set.*

Proof. Let $\mathfrak{H}$ be a $\mathfrak{PNsgC}$ and $\mathfrak{F}$ be a $\mathfrak{PNSC}$ subset of $\mathfrak{PNscl}(\mathfrak{H}) - \mathfrak{H}$. Then, $(\mathfrak{PNscl}(\mathfrak{H}) - \mathfrak{H})^c \subseteq \mathfrak{F}^c$, and $\mathfrak{F}^c$ is $\mathfrak{PNSO}$. That is, $(\mathfrak{PNscl}(\mathfrak{H}) \bigcap \mathfrak{H}^c)^c \subseteq \mathfrak{F}^c$. Therefore, $\mathfrak{H} \bigcup (\mathfrak{PNsint}(\mathfrak{H}))^c \subseteq \mathfrak{F}^c$. Thus, $\mathfrak{F}^c$ is $\mathfrak{PNSO}$, and $\mathfrak{H} \subseteq \mathfrak{F}^c$. Since $\mathfrak{H}$ is $\mathfrak{PNsgC}$, $\mathfrak{PNscl}(\mathfrak{H}) \subseteq \mathfrak{F}^c$. That is, $\mathfrak{F} \subseteq (\mathfrak{PNscl}(\mathfrak{H}))^c$. Thus, $\mathfrak{F} \subseteq (\mathfrak{PNscl}(\mathfrak{H}.) \bigcap (\mathfrak{PNscl}(\mathfrak{H}))^c = \varnothing$. Therefore, $\mathfrak{F} = \varnothing$.

Conversely, let $\mathfrak{PNscl}(\mathfrak{H}) - \mathfrak{H}$ have no nonempty, $\mathfrak{PNSC}$ set. Let $\mathfrak{G}$ be $\mathfrak{PNSO}$ in $\mathfrak{U}$ such that $\mathfrak{H} \subseteq \mathfrak{G}$. If $\mathfrak{PNscl}(\mathfrak{H})\mathfrak{U}\mathfrak{G}$, then $\mathfrak{PNscl}(\mathfrak{H}) \bigcap \mathfrak{G}^c \neq \varnothing$. And $\mathfrak{PNscl}(\mathfrak{H}) \bigcap \mathfrak{G}^c \subseteq \mathfrak{PNscl}(\mathfrak{H}) - \mathfrak{H}$, since $\mathfrak{H} \subseteq \mathfrak{G}$. Thus, $\mathfrak{PNscl}(\mathfrak{H}) \bigcap \mathfrak{G}^c$ is a nonvoid $\mathfrak{PNSC}$ subset of $\mathfrak{PNscl}(\mathfrak{H}) - \mathfrak{H}$, which is contradiction. Therefore, $\mathfrak{PNscl}(\mathfrak{H}) \subseteq \mathfrak{G}$ whenever $\mathfrak{G}$ is $\mathfrak{PNSO}$ and $\mathfrak{H} \subseteq \mathfrak{G}$. That is, $\mathfrak{H}$ is $\mathfrak{PNsgC}$ in $\mathfrak{U}$.□ □

Theorem 31. *Let $\mathfrak{H}$ be $\mathfrak{PNsgC}$. Then, $\mathfrak{H}$ is $\mathfrak{PNSC}$ iff $\mathfrak{PNscl}(\mathfrak{H}) - \mathfrak{H}$ is $\mathfrak{PNSC}$.*

Proof. Let $\mathfrak{H}$ be $\mathfrak{PNsgC}$. If $\mathfrak{H}$ if $\mathfrak{PNSC}$, $\mathfrak{PNscl}(\mathfrak{H}) = \mathfrak{H}$ and hence $\mathfrak{PNscl}(\mathfrak{H}) - \mathfrak{H} = \varnothing$ which is $\mathfrak{PNSC}$. Conversely, let $\mathfrak{PNscl}(\mathfrak{H}) - \mathfrak{H}$ be $\mathfrak{PNSC}$. Then, $\mathfrak{PNscl}(\mathfrak{H}) - \mathfrak{H}$ is $\mathfrak{PNsgC}$. Then, $\mathfrak{PNscl}(\mathfrak{H}) - \mathfrak{H}$ does not contain any nonempty, $\mathfrak{PNSC}$ set. Therefore, $\mathfrak{PNscl}(\mathfrak{H}) - \mathfrak{H} = \varnothing$. That is, $\mathfrak{PNscl}(\mathfrak{H}) = \mathfrak{H}$. Therefore, $\mathfrak{H}$ is $\mathfrak{PNSC}$.

Now, we derive the forms of $\mathfrak{PN}$ semigeneralized closed sets for various cases of approximations.□ □

Theorem 32. *If $\mathfrak{PNL}_R(\mathfrak{X}) = \mathfrak{PNU}_R(\mathfrak{X})$ in a $\mathfrak{PNTSU}$, then any $\mathfrak{H} \subseteq [\mathfrak{PNL}_R(\mathfrak{X})]^c$ and $[\mathfrak{PNL}_R(\mathfrak{X})]^c \bigcup \mathfrak{T}$ where $\mathfrak{T} \subseteq \mathfrak{PNL}_R(\mathfrak{X})$ are the only $\mathfrak{PNsgC}$ sets in $\mathfrak{U}$.*

Proof. When $\mathfrak{PNL}_R(\mathfrak{X}) = \mathfrak{PNU}_R(\mathfrak{X})$, $\tau_R(\mathfrak{X}) = \{\varnothing, \mathfrak{U}, \mathfrak{PNL}_R(\mathfrak{X})\}$. Also, $\varnothing$ and any $\mathfrak{H} \subseteq \mathfrak{PNL}_R(\mathfrak{X})$ are the only $\mathfrak{PNSO}$ sets in $\mathfrak{U}$. If $\mathfrak{H} \subseteq \mathfrak{PNL}_R(\mathfrak{X})$, then $\mathfrak{PNscl}(\mathfrak{H}) = \mathfrak{U}$ and the $\mathfrak{PNSO}$ sets containing $\mathfrak{H}$ are those sets $\mathfrak{T}$ for which $\mathfrak{PNL}_R(\mathfrak{X}) \subseteq \mathfrak{T}$. Thus, $\mathfrak{PNscl}(\mathfrak{H}) \subseteq G$, not for every $\mathfrak{PNSO}\mathfrak{G}$ such that $\mathfrak{H} \subseteq \mathfrak{G}$.

Therefore, $\mathfrak{H}$ is not $\mathfrak{PNsgC}$. If $\mathfrak{H} \subseteq [\mathfrak{PNL}_R(\mathfrak{X})]^c$, then $\mathfrak{PNscl}(\mathfrak{H}) - \mathfrak{H}$, since any subset of $[\mathfrak{PNL}_R(\mathfrak{X})]^c$ is $\mathfrak{PNSC}$ in $\mathfrak{U}$. Thus, $\mathfrak{PNscl}(\mathfrak{H}) = \mathfrak{H} \subseteq \mathfrak{G}$ whenever $\mathfrak{G}$ is $\mathfrak{PNSO}$ and $\mathfrak{H} \subseteq \mathfrak{G}$. Therefore, $\mathfrak{H}$ is $\mathfrak{PNsgC}$. If $\mathfrak{PNL}_R(\mathfrak{X}) \subseteq \mathfrak{G}$ and $\mathfrak{H} \neq \mathfrak{U}$, $\mathfrak{PNscl}(\mathfrak{H}) = \mathfrak{U}$ and the $\mathfrak{PNSO}$ sets containing $\mathfrak{H}$ are $\mathfrak{H}$ and $\mathfrak{U}$. Therefore, $\mathfrak{PNscl}(\mathfrak{H})\mathfrak{U}\mathfrak{H}$. Therefore, any $\mathfrak{H} \supseteq \mathfrak{PNL}_R(\mathfrak{X})$ and $\mathfrak{H} \neq \mathfrak{U}$ are not $\mathfrak{PNsgC}$. If $[\mathfrak{PNL}_R(\mathfrak{X})]^c \subseteq \mathfrak{H}$, then $\mathfrak{PNscl}(\mathfrak{H}) \subseteq \mathfrak{G}$ whenever $\mathfrak{G}$ is $\mathfrak{PNSO}$ and $\mathfrak{G} \subseteq \mathfrak{H}$, since $\mathfrak{U}$ is the only $\mathfrak{PNSO}$ set containing $\mathfrak{H}$. Therefore, if $[\mathfrak{PNL}_R(\mathfrak{X})]^c \subseteq \mathfrak{H}$, then $\mathfrak{H}$ is $\mathfrak{PNsgC}$. When $\mathfrak{H}$ has at least one element of $\mathfrak{PNL}_R(\mathfrak{X})$ and exactly one element of $[\mathfrak{PNL}_R(\mathfrak{X})]^c$ where $\mathfrak{PNL}_R(\mathfrak{X})$ is not a singleton set, $\mathfrak{PNscl}(\mathfrak{H}) = \mathfrak{U}$. But union of that element and $\mathfrak{PNL}_R(\mathfrak{X})$ is a $\mathfrak{PNSO}$ set containing $\mathfrak{H}$ and $\mathfrak{PNscl}(\mathfrak{H}) = \mathfrak{U}\mathfrak{U}\mathfrak{PNL}_R(\mathfrak{X})$ union with that element. Therefore, $\mathfrak{H}$ is not $\mathfrak{PNsgC}$. Thus, the only $\mathfrak{PNsgC}$ sets in $\mathfrak{U}$ are subsets of $[\mathfrak{PNL}_R(\mathfrak{X})]^c$ and any $\mathfrak{H} \supset [\mathfrak{PNL}_R(\mathfrak{X})]^c$.□ □

Theorem 33. *If $\mathfrak{PNL}_R(\mathfrak{X}) = \varnothing$ and $\mathfrak{PNU}_R(\mathfrak{X}) \neq \mathfrak{U}$, then the only $\mathfrak{PNsgC}$ sets in $\mathfrak{U}$ are subsets of $[\mathfrak{PNU}_R(\mathfrak{X})]^c$ and any $\mathfrak{H} \supset [\mathfrak{PNU}_R(\mathfrak{X})]^c$.*

Proof. If $\mathfrak{PNL}_R(\mathfrak{X}) = \varnothing$ and $\mathfrak{PNU}_R(\mathfrak{X}) = \mathfrak{U}$, then $\tau_R(\mathfrak{X}) = \{\varnothing, \mathfrak{U}, \mathfrak{PNU}_R(\mathfrak{X})\}$. Also, $\varnothing$ and those sets $\mathfrak{H}$ for which $\mathfrak{H} \supseteq \mathfrak{PNU}_R(\mathfrak{X})$ are the only $\mathfrak{PNSO}$ sets in $\mathfrak{U}$. Therefore, the sets $\mathfrak{H}$ for which $\mathfrak{H} \subseteq [\mathfrak{PNU}_R(\mathfrak{X})]^c$ are the only $\mathfrak{PNS}$ $\mathfrak{C}$ sets in $\mathfrak{U}$. If $\mathfrak{H} \subseteq \mathfrak{PNU}_R(\mathfrak{X})$, then $\mathfrak{PNscl}(\mathfrak{H}) = \mathfrak{U}$. But $\mathfrak{PNU}_R(\mathfrak{X})$ is a $\mathfrak{PNSO}$ set containing $\mathfrak{H}$, for which $\mathfrak{PNsc}$ $\mathfrak{l}(\mathfrak{H})\mathfrak{U}\mathfrak{G}$. Therefore, $\mathfrak{H}$ is not $\mathfrak{PNsgC}$. If $\mathfrak{PNU}_R(\mathfrak{X}) \subseteq \mathfrak{H}$ and $\mathfrak{H} \neq \mathfrak{U}$, then $\mathfrak{PNscl}(\mathfrak{H}) = \mathfrak{U}$. But, for $\mathfrak{G} = \mathfrak{H}$, which is $\mathfrak{PNSO}$ set containing itself, $\mathfrak{PNscl}(\mathfrak{H})\mathfrak{U}\mathfrak{G}$. Therefore, $\mathfrak{H}$ is not $\mathfrak{PNsgC}$. If $\mathfrak{H} \subseteq [\mathfrak{PNU}_R(\mathfrak{X})]^c$, then $\mathfrak{PNscl}(\mathfrak{H}) = \mathfrak{H}$ and hence for every $\mathfrak{PNSO}$ set $\mathfrak{G}$ such that $\mathfrak{H} \subseteq \mathfrak{G}$, $\mathfrak{PNsc}$ $\mathfrak{l}(\mathfrak{H}) \subseteq \mathfrak{G}$. Therefore, $\mathfrak{H}$ is $\mathfrak{PNsgC}$. If $[\mathfrak{PNU}_R(\mathfrak{X})]^c \subseteq \mathfrak{H}$, then $\mathfrak{U}$ is the only $\mathfrak{PNSO}$ set holding $\mathfrak{H}$ and hence $\mathfrak{PNs}$ $\mathfrak{cl}(\mathfrak{H}) \subseteq \mathfrak{G}$ whenever $\mathfrak{G}$ is $\mathfrak{PNSO}$ and $\mathfrak{H} \subseteq \mathfrak{G}$. Therefore, $\mathfrak{H}$ is $\mathfrak{PNsgC}$. If $\mathfrak{H}$ has one element of $\mathfrak{PNU}_R(\mathfrak{X})$ and at least one element of $(\mathfrak{PNU}_R(\mathfrak{X}))^c$, then $\mathfrak{PNscl}(\mathfrak{H}) = \mathfrak{U}$. Since any set having $\mathfrak{PNU}_R(\mathfrak{X})$ is $\mathfrak{PNSO}$ in $\mathfrak{U}$,

$\mathfrak{H} \bigcup (\mathfrak{PNU}_R(\mathfrak{X}))$ and any set having $\mathfrak{H} \bigcup (\mathfrak{PNU}_R(\mathfrak{X}))$ are $\mathfrak{PNSO}$ sets containing $\mathfrak{H}$. But, $\mathfrak{PNscl}(\mathfrak{H}) = \mathfrak{UUH} \bigcup (\mathfrak{PNU}_R(\mathfrak{X}))$. Therefore, $\mathfrak{H}$ is not $\mathfrak{PNsgC}$ in $\mathfrak{U}$. Thus, only subsets of $(\mathfrak{PNU}_R(\mathfrak{X}))^c$ and any $\mathfrak{H} \supset [\mathfrak{PNU}_R(\mathfrak{X})]^c$ are $\mathfrak{P}$ $\mathfrak{NsgC}$ in $\mathfrak{U}$ when $\mathfrak{PNL}_R(\mathfrak{X}) = \varnothing$ and $\mathfrak{PNU}_R(\mathfrak{X}) \neq \mathfrak{U}$.□ □

Theorem 34. *If $\mathfrak{PNU}_R(\mathfrak{X}) = \mathfrak{U}$ and $\mathfrak{PNL}_R(\mathfrak{X}) \neq \varnothing$ in a $\mathfrak{P}$ $\mathfrak{NTSU}$, then every subset of $\mathfrak{U}$ is $\mathfrak{PNsgC}$.*

Proof. $\varnothing$, $\mathfrak{U}$, $\mathfrak{PNL}_R(\mathfrak{X})$, and $\mathfrak{PNB}_R(\mathfrak{X})$ are the only sets in $\mathfrak{U}$ which are $\mathfrak{PNO}$, $\mathfrak{PNSO}$, and $\mathfrak{PNSC}$ in $\mathfrak{U}$. If $\mathfrak{H} \subseteq \mathfrak{P}$ $\mathfrak{NL}_R(\mathfrak{X})$, then $\mathfrak{PNL}_R(\mathfrak{X})$ and $\mathfrak{U}$ are the only $\mathfrak{PNSO}$ sets containing $\mathfrak{H}$ and $\mathfrak{PNscl}(\mathfrak{H}) = \mathfrak{PNL}_R(\mathfrak{X})$. Therefore, $\mathfrak{PN}$ $\mathfrak{scl}(\mathfrak{H}) \subseteq \mathfrak{G}$ whenever $\mathfrak{G}$ is $\mathfrak{PNSO}$ and $\mathfrak{H} \subseteq \mathfrak{G}$. Thus, $\mathfrak{H}$ is $\mathfrak{PNsgC}$. If $\mathfrak{H} \subseteq \mathfrak{PNB}_R(\mathfrak{X})$, then $\mathfrak{PNB}_R(\mathfrak{X})$ and $\mathfrak{U}$ are the only $\mathfrak{PNSO}$ sets containing $\mathfrak{H}$ and $\mathfrak{PNscl}(\mathfrak{H}) = \mathfrak{PNB}_R(\mathfrak{X})$. Therefore, $\mathfrak{H}$ is PNsg-C. If $\mathfrak{PNL}_R(\mathfrak{X}) \subset \mathfrak{H}$ or $\mathfrak{PNB}_R(\mathfrak{X}) \subset \mathfrak{H}$, then $\mathfrak{U}$ is the only $\mathfrak{PNSO}$ set containing $\mathfrak{H}$ and hence $\mathfrak{H}$ is $\mathfrak{PNsgC}$. If $\mathfrak{H}$ contains atleast one element of $\mathfrak{PNL}_R(\mathfrak{X})$ and at least one element of $\mathfrak{PNB}_R(\mathfrak{X})$, then $\mathfrak{U}$ is the only $\mathfrak{PNSO}$ set containing $\mathfrak{H}$. Therefore, $\mathfrak{H}$ is $\mathfrak{PN}$ $\mathfrak{sgC}$. Thus, every subset of $\mathfrak{U}$ is $\mathfrak{PNsgC}$, if $\mathfrak{PNU}_R(\mathfrak{X}) = \mathfrak{U}$ and $\mathfrak{PNL}_R(\mathfrak{X}) \neq \varnothing$.□ □

Definition 35. Let $(\mathfrak{U}, \tau_R(\mathfrak{X}))$ and $(\mathfrak{V}, \tau'_R(\mathfrak{Y}))$ be two PNTSs. Then, a function $f : \mathfrak{U} \longrightarrow \mathfrak{V}$ is named as

(1) $\mathfrak{PN}$ generalized continuous ($\mathfrak{PNgCN}$), if the inverse image of every $\mathfrak{PNC}$ set in $\mathfrak{V}$ is $\mathfrak{PNgC}$ in $\mathfrak{U}$

(2) $\mathfrak{PN}$ semigeneralized continuous ($\mathfrak{PNsgCN}$), if the inverse image of every $\mathfrak{PNC}$ set in $\mathfrak{V}$ is $\mathfrak{PNsgC}$ in $\mathfrak{U}$

(3) $\mathfrak{PN}$ semigeneralized closed ($\mathfrak{PNsgC}$) if the image of every $\mathfrak{PNC}$ set in $\mathfrak{U}$ is $\mathfrak{PNsgC}$ in $\mathfrak{V}$

(4) $\mathfrak{PN}$ semigeneralized open ($\mathfrak{PNsgO}$) if the image of every $\mathfrak{PNO}$ set in $\mathfrak{U}$ is $\mathfrak{PNsgO}$ in $\mathfrak{V}$

Theorem 36. *Every $\mathfrak{PN}$ continuous ($\mathfrak{PNCN}$) function is $\mathfrak{PNsgCN}$.*

Proof. If f : $\mathfrak{U} \longrightarrow \mathfrak{V}$ is $\mathfrak{PNCN}$ on $\mathfrak{U}$ and if $\mathfrak{G}$ is $\mathfrak{PNO}$ in $\mathfrak{V}$, then $f^{-1}(\mathfrak{G})$ is $\mathfrak{PNO}$ in $\mathfrak{U}$. Therefore, $f^{-1}(\mathfrak{G})$ is $\mathfrak{PNsgO}$, since any $\mathfrak{PNO}$ set is $\mathfrak{PNSO}$ and any $\mathfrak{PNSO}$ is PNSgO. Therefore, f is $\mathfrak{PNsgCN}$.□ □

4. Pythagorean Nanotopology in Multiple Attribute Decision-Making

MADM is a method for selecting the best solution with the highest level of satisfaction from a set of alternatives. Multiple attributes are used to represent these types of MADM problems, which occur in most real-time situations. When it comes to dealing with real-life problems, collecting vague details is done with the help of attributes for the particular object and the decision-making technique is applied for the list of objects considered. Many models already exist for the decision-making problems, but the proposed algorithm deals with membership and nonmembership which has more advantage in fuzziness than intuitionistic fuzzy set and fuzzy set theory. Many types of models exist for the different developments of topological spaces, but for the different category of topological spaces, this method is proposed. The proposed algorithm describes how $\mathfrak{PN}$ topology influences decision-making.

A new decision-making approach using $\mathfrak{PN}$ topology and a methodological approach for selecting the right alternatives is proposed.

4.1. Algorithm

Step 1. Consider the universe $\mathfrak{D}$ and attributes $\mathfrak{E}$.

Step 2. Make a fuzzy Pythagorean matrix of attributes versus objects.

Step 3. Define $\mathfrak{R}$ on $\mathfrak{D}$ to represent the indiscernibility relation.

Step 4. Build the Pythagorean fuzzy nanotopology τ.

Step 5. Find the score values by using the score function $1/\mathfrak{k}\sum_{i=1}^{\mathfrak{k}}[1/2\{1 + m - \text{n.m}\}]$ (where m means membership, n.m means nonmembership, and $\mathfrak{k}$ is the number of values in the corresponding topology) of each of the entries of Pythagorean fuzzy nanotopological spaces.

Step 6. Arrange the score values of the alternatives in decreasing order and select the maximum as the optimal decision.

The pieces of information for the object are collected for the particular object and formed the table, and after that, using the relation, the PN topology is being framed. Using score function, the optimal values are calculated and the decision is made upon the maximal value.

4.2. Numerical Example. The proposed algorithm helps to find the suitable choice among all the options (set of objects). We choose any random situation for this decision-making process. As in Algorithm, using the $\mathfrak{PNTS}$ method, the problem is solved. Let us consider the decision-making situation where a company in a tourist hotspot desires to select and draw a contract with a hotel for certain years. Let us consider the set of objects as the hotels which were considered to have a contract. That is, $\mathfrak{D} = \{\mathfrak{d}_1, \mathfrak{d}_2, \mathfrak{d}_3, \mathfrak{d}_4, \mathfrak{d}_5\}$ where $\mathfrak{d}_i (i = 1, 2, 3, 4, 5)$. Consider the criteria for deciding to pick a hotel. The attributes are $\mathfrak{E} = \{\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3, \mathfrak{t}_4, \mathfrak{t}_5\}$ where $\mathfrak{t}_i, (i = 1, 2, 3, 4, 5)$ stands for criteria clean and tidy, good food, reasonable price, customer driven, and location, respectively.

Step 1. Let $\mathfrak{D} = \{\mathfrak{d}_1, \mathfrak{d}_2, \mathfrak{d}_3, \mathfrak{d}_4, \mathfrak{d}_5\}$ be the set of objects and $\mathfrak{E} = \{\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3, \mathfrak{t}_4, \mathfrak{t}_5\}$ be the set of attributes for the objects.

Step 2. In the matrix of Pythagorean fuzzy relationship between hotels, attributes are developed as in Table 1.

Step 3. The indiscernibility relation for the objects is constructed as

$$R = \{\{\mathfrak{t}_1, \mathfrak{t}_2\}, \{\mathfrak{t}_3, \mathfrak{t}_4, \mathfrak{t}_5\}\}. \tag{2}$$

Step 4. Build the $\mathfrak{PNTS}$ for each hotel $(\mathfrak{d}_i)$ with respect to the attributes.

$$\begin{gathered}
\tau(\mathfrak{d}_1) = \{\phi, \mathfrak{D}, <.6,.4>, <.3,.3>, <.7,.3>, <.5,.1>, <.4,.6>, <.5,.7>\}, \\
\tau(\mathfrak{d}_2) = \{\phi, \mathfrak{D}, <.5,.6>, <.5,.5>, <.6,.3>, <.7,.2>, <.5,.4>\}, \\
\tau(\mathfrak{d}_3) = \{\phi, \mathfrak{D}, <.3,.5>, <.5,.7>, <.6,.4>, <.8,.4>, <.6,.5>, <.5,.4>\}, \\
\tau(\mathfrak{d}_4) = \{\phi, \mathfrak{D}, <.3,.5>, <.4,.5>, <.7,.4>, <.9,.2>, <.7,.5>, <.6,.5>\}, \\
\tau(\mathfrak{d}_5) = \{\phi, \mathfrak{D}, <.7,.7>, <.3,.7>, <.7,.6>, <.5,.3>, <.3,.6>\}.
\end{gathered} \tag{3}$$

Step 5. Computation of Pythagorean fuzzy score functions for the hotels $(\mathfrak{d}_i)$ as in algorithm are as follows:

Score values of hotels $(\mathfrak{d}_i)(i = 1, 2, \cdots 5)$ are

$$S(\mathfrak{d}_1) = .5375, S(\mathfrak{d}_2) = .557, S(\mathfrak{d}_3) = .525, S(\mathfrak{d}_4) = .5625, S(\mathfrak{d}_5) = .471. \tag{4}$$

Step 6. Organizing the score values, we get the sequence of the hotels as $\mathfrak{d}_4 > \mathfrak{d}_2 > \mathfrak{d}_1 > \mathfrak{d}_3 > \mathfrak{d}_5$. Thus, the hotel with maximum value and in the first position is chosen as the optimal decision (i.e., $\mathfrak{d}_4$).

4.3. Comparison Analysis. To check the effectualness of the presented decision-making approach, a comparison analysis is performed with Pythagorean fuzzy decision-making model used in [36]. Though the ranking principle and method are different, the ranking order results are consistent with the result obtained in [36] for the selection of the best alternative. The computation may seem hard, but the calculation is too easy to compute, while when compared to the three-valued sets, this possesses a little lack in the indeterminacy part. When compared to the other sets and models, this plays the upper hand.

Table 1: Pythagorean fuzzy system of relationship between hotels and attributes.

$\mathfrak{R}$	$\mathfrak{t}_1$	$\mathfrak{t}_2$	$\mathfrak{t}_3$	$\mathfrak{t}_4$	$\mathfrak{t}_5$
$\mathfrak{d}_1$	<.7,.4>	<.6,.3>	<.3,.2>	<.5,.1>	<.4,.3>
$\mathfrak{d}_2$	<.5,.3>	<.6,.6>	<.7,.2>	<.6,.5>	<.5,.2>
$\mathfrak{d}_3$	<.6,.4>	<.3,.5>	<.8,.5>	<.5,.4>	<.6,.7>
$\mathfrak{d}_4$	<.3,.4>	<.7,.5>	<.5,.5>	<.9,.2>	<.4,.5>
$\mathfrak{d}_5$	<.7,.6>	<.7,.7>	<.3,.3>	<.4,.3>	<.5,.7>

5. Conclusion

$\mathfrak{PNTS}$ is a newly defined space by combining the concepts of nanotopology and Pythagorean fuzzy topological spaces. The topological space has been developed, and as an extension, the concepts of the weak open sets, namely, nanoalpha, semiopen sets, have been developed and their characterizations were examined. In this article, the idea of generalized closed sets in Pythagorean nanotopology has been introduced along with its characteristics. The notion of semigeneralized closed sets has also been defined, and their properties were investigated. An application in MADM using $\mathfrak{PNTS}$ has been proposed and illustrated using a numerical example. Further, the proposed concept can be extended to strong open sets in $\mathfrak{PNTS}$ and applied to real-life problems.

Data Availability

No data were used to support this study.

Disclosure

The statements made and views expressed are solely the responsibility of the authors.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[2] K. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 20, no. 1, pp. 87–96, 1986.

[3] R. R. Yager and A. M. Abbasov, "Pythagorean membership grades, complex numbers, and decision making,"

International Journal of Intelligent Systems, vol. 28, no. 5, pp. 436–452, 2013.

[4] R. R. Yager, “Pythagorean membership grades in multicriteria decision making,” *IEEE Transactions on Fuzzy Systems*, vol. 22, no. 4, pp. 958–965, 2014.

[5] F. Smarandache, *A Unifying Field in Logics. Neutrosophy: Neutrosophic Probability, Set and Logic*, American Research Press, 1999.

[6] H. Wang, F. Smarandache, R. Sunderraman, and Y. Q. Zhang, “Single valued neutrosophic sets,” *Multi-Space and Multi-Structure*, vol. 4, pp. 410–413, 2010.

[7] C. L. Chang, “Fuzzy topological spaces,” *Journal of Mathematical Analysis and Applications*, vol. 24, no. 1, pp. 182–190, 1968.

[8] T. E. Gantner, R. C. Steinlage, and R. H. Warren, “Compactness in fuzzy topological spaces,” *Journal of Mathematical Analysis and Applications*, vol. 62, no. 3, pp. 547–562, 1978.

[9] B. Hutton, “Products of fuzzy topological spaces,” *Topology and its Applications*, vol. 11, no. 1, pp. 59–67, 1980.

[10] R. Lowen, “Convergence in fuzzy topological spaces,” *General Topology and Its Applications*, vol. 10, no. 2, pp. 147–160, 1979.

[11] J. H. Kim, P. K. Lim, J. G. Lee, and K. Hur, *Intuitionistic Topological Spaces*, Infinite Study, 2018.

[12] D. Coker, “An introduction to intuitionistic fuzzy topological spaces,” *Fuzzy Sets and Systems*, vol. 88, no. 1, pp. 81–89, 1997.

[13] S. J. Lee and J. M. Chu, “Categorical property of intuitionistic topological spaces,” *Korean Mathematical Society*, vol. 24, no. 4, pp. 595–603, 2009.

[14] S. Bayhan and D. Coker, “On separation axioms in intuitionistic topological spaces,” *International Journal of Mathematics and Mathematical Sciences*, vol. 27, no. 10, 630 pages, 2001.

[15] N. Turanli and D. Coker, “Fuzzy connectedness in intuitionistic fuzzy topological spaces,” *Fuzzy Sets and Systems*, vol. 116, no. 3, pp. 369–375, 2000.

[16] M. Lellis Thivagar and C. Richard, “On nano continuity,” *Mathematical Theory and Modelling*, vol. 3, no. 7, pp. 32–37, 2013.

[17] M. L. Thivagar and C. Richard, “On nano forms of weakly open sets,” *International Journal of Mathematics and Statistics Invention*, vol. 1, no. 1, pp. 31–37, 2013.

[18] S. Krishnaprakash, R. Ramesh, and R. Suresh, “Nano-compactness and nano-connectedness in nano topological spaces,” *International Journal of Pure and Applied Mathematics*, vol. 119, no. 13, pp. 107–115, 2018.

[19] K. Bhuvaneswari and K. M. Gnanapriya, “Nano generalized closed sets in nano topological spaces,” *International Journal of Scientific and Research Publications*, vol. 4, no. 5, pp. 1–3, 2014.

[20] M. Ramachandran and A. Stephan Antony Raj, “Intuitionistic fuzzy Nano topological space: theory and applications,” *ScieXplore: International Journal of Research in Science*, vol. 4, no. 1, pp. 1–6, 2017.

[21] A. S. A. Raj and M. Ramachandran, “On intuitionistic fuzzy nano generalized continuous mappings,” *In Journal of Physics: Conference Series, IOP Publishing*, vol. 1139, no. 1, article 012059, 2018.

[22] A. S. A. Raj and M. Ramachandran, “Intuitionistic fuzzy nano generalized closed sets,” *Asian Research Journal of Mathematics*, vol. 9, no. 3, pp. 1–6, 2018.

[23] M. Lellis Thivagar, S. Jafari, V. Sutha Devi, and V. Antonysamy, “A novel approach to nano topology via neutrosophic sets,” *Neutrosophic Sets and Systems*, vol. 20, pp. 86–94, 2018.

[24] M. Olgun, M. Ünver, and Ş. Yardımcı, “Pythagorean fuzzy topological spaces,” *Complex and Intelligent Systems*, vol. 5, no. 2, pp. 177–183, 2019.

[25] A. H. Es, “Connectedness in Pythagorean fuzzy topological spaces,” *International Journal of Mathematics Trends and Technology*, vol. 65, no. 7, pp. 110–116, 2019.

[26] D. Ajay and J. J. Charisma, “On weak forms of Pythagorean nano open sets,” *Advances in Mathematics: Scientific Journal*, vol. 9, pp. 5953–5963, 2020.

[27] D. Ajay and J. J. Charisma, “Pythagorean nano continuity,” *Advances in Mathematics: Scientific Journal*, vol. 9, pp. 6291–6298, 2020.

[28] D. Ajay and J. J. Charisma, “Pythagorean nano topological space,” *International Journal of Recent Technology and Engineering*, vol. 8, pp. 3415–3419, 2020.

[29] H. C. Liu, X. Q. Chen, C. Y. Duan, and Y. M. Wang, “Failure mode and effect analysis using multi-criteria decision making methods: a systematic literature review,” *Computers & Industrial Engineering*, vol. 135, pp. 881–897, 2019.

[30] A. Mardani, A. Jusoh, and E. K. Zavadskas, “Fuzzy multiple criteria decision-making techniques and applications - two decades review from 1994 to 2014,” *Expert Systems with Applications*, vol. 42, no. 8, pp. 4126–4148, 2015.

[31] D. Ajay and J. J. Charisma, “An MCDM based on alpha open hypersoft sets and its application,” in *International Conference on Intelligent and Fuzzy Systems*, pp. 333–341, Izmir, Turkey, 2021.

[32] M. Riaz, F. Smarandache, F. Karaaslan, M. R. Hashmi, and I. Nawaz, “Neutrosophic soft rough topology and its applications to multi-criteria decision-making,” *Neutrosophic sets and systems*, vol. 35, pp. 198–219, 2020.

[33] D. Ajay, J. J. Charisma, N. Boonsatit, P. Hammachukiattikul, and G. Rajchakit, “Neutrosophic semiopen hypersoft sets with an application to MAGDM under the COVID-19 scenario,” *Journal of Mathematics*, vol. 2021, Article ID 5583218, 16 pages, 2021.

[34] M. Riaz, K. Naeem, M. Aslam, D. Afzal, F. A. A. Almahdi, and S. S. Jamal, “Multi-criteria group decision making with Pythagorean fuzzy soft topology,” *Journal of Intelligent & Fuzzy Systems*, vol. 39, no. 5, pp. 6703–6720, 2020.

[35] O. Yanmaz, Y. Turgut, E. N. Can, and C. Kahraman, “Interval-valued Pythagorean fuzzy EDAS method: an application to car selection problem,” *Journal of Intelligent & Fuzzy Systems*, vol. 38, no. 4, pp. 4061–4077, 2020.

[36] S. Naz, S. Ashraf, and M. Akram, “A novel approach to decision-making with Pythagorean fuzzy information,” *Mathematics*, vol. 6, no. 6, p. 95, 2018.