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Research article

Existence results for coupled system of nonlinear differential equations and inclusions involving sequential derivatives of fractional order

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Abstract: In this article, we investigate new results of existence and uniqueness for systems of nonlinear coupled differential equations and inclusions involving Caputo-type sequential derivatives of fractional order and along with new kinds of coupled discrete (multi-points) and fractional integral (Riemann-Liouville) boundary conditions. Our investigation is mainly based on the theorems of Schaefer, Banach, Covitz-Nadler, and nonlinear alternatives for Kakutani. The validity of the obtained results is demonstrated by numerical examples.

Keywords: Caputo derivatives; coupled system; fixed point; fractional differential equations(FDEs); inclusions; sequential derivatives
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1. Introduction

Over the last two decades, fractional calculus has gained a lot of attention. As a result, there is a growing interest in the theory and applications of fractional differential equations (FDEs) under various initial and boundary conditions (BCs); see, for example, [1–8], and the references cited therein. Fractional differentiation and integration have greatly aided the evaluation of mathematical models of numerous real-world situations in fractional contexts. It has been discovered that this subject has applications in a wide range of technical and physical sciences, including complex media electrodynamics, control theory ecology, viscoelasticity, biomathematics, electrical circuits, electroanalytical chemistry, aerodynamics, and blood flow phenomena [9–16]. The topic of consecutive fractional-order operators (i.e., sequential fractional derivative (SFD)) was described throughout the exceptional monograph [17]. Because it has been established that SFDs and non-SFDs are intimately related, some recent work on sequential fractional differential equations (SFDEs) has been published ([18–20]). Except for fractional-order boundary value problems (BVPs) for equations and inclusions [21, 22], the study of systems of coupled FDEs has advanced and engaged scholars. Disease models, Lorenz systems, ecological models, Duffing systems, synchronisation of chaotic systems, and so on are examples of coupled system applications [23–25]. We recommend readers to [26, 27] and the sources listed therein for the empirical research of coupled systems of FDEs. To demonstrate the effect of existence and uniqueness on the studied equations, sufficient conditions are used. Numerous mathematicians and applied researchers have attempted to use fractional calculus to model real-world processes. It has been deduced in biology that the membranes of biological organism cells have fractional-order electrical conductance [28] and thus, are classified in groups of non-integer-order models. Fractional derivatives are the most successful in the field of rheology because they embody essential features of cell rheological behaviour [29]. In most biological systems, such as HIV infection, hepatitis C virus (HCV) infection, and cancer spread, fractional-order ordinary differential equations are naturally related to systems with long-time memory. Additionally, they are related to fractals, which occur frequently in biological systems. Wang and Li [30] analyzed the global dynamics of HIV infection of CD4+ cells. Arafal et al. [31] studied fractional modeling dynamics of HIV and CD4+ T-cells during primary infection. As a result, fractional-order differential equations are thought to be a better tool than integer-order differential equations for describing hereditary properties of various materials and processes. Fractional-order models have become more realistic and practical than their classical integer-order counterparts as a result of this advantage, and their dynamics behaviour is also as stable as their integer-order counterparts. Due to the fact that theoretical results can aid in the development of a more complete understanding of the dynamic behaviour of biological processes, the study of abstract fractional dynamic models is becoming increasingly relevant and important in the modern era. On the other hand, the BCs in (1.2) are referred to as coupled BCs; they are encountered in the study of reaction-diffusion equations, Sturm-Liouville problems, and mathematical biology, among other fields. Recently in [32], authors discussed the existence and uniqueness of coupled system of FDEs with a novel class of coupled boundary conditions specified by

$$\begin{cases} {}^{C}\mathcal{D}^{\alpha}u(t) = f(t,u(t),v(t)), & t \in \mathcal{J} = [0,\mathcal{T}], \\ {}^{C}\mathcal{D}^{\beta}v(t) = g(t,u(t),v(t)), & t \in \mathcal{J} = [0,\mathcal{T}], \\ (u+v)(0) = -(u+v)(\mathcal{T}), & \int_{\xi}^{\eta}(u-v)(s)ds = A, \end{cases} \tag{1.1}$$

where ${}^{C}\mathcal{D}^{\chi}$ is the Caputo fractional derivatives (CFD) of order $\chi \in \{\alpha,\beta\}$, $\alpha,\beta \in (0,1]$, $f, g : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions, and A is non-negative constant. A fixed point equivalent problem is created by transforming the system (1.2) into a fixed point equivalent problem and solving it using conventional fixed point theorems. As far as we know, the single-valued and multi-valued maps for the solutions of nonlinear coupled SFDEs with coupled boundary conditions have been rarely investigated. Motivated by the HIV infection model and its application background, we study the consequences of existence for a system of nonlinear coupled Caputo-type SFDEs and inclusions subject to multi-point and fractional integral boundary conditions of the form

$$\begin{cases} \left({}^{C}\mathcal{D}^{\vartheta} + \varphi^{C}\mathcal{D}^{\vartheta-1}\right)u(t) = \mathcal{G}_1(t, u(t), v(t)), \;\; t \in \mathcal{J} = [0, \mathcal{T}] \\ \left({}^{C}\mathcal{D}^{\varpi} + \varphi^{C}\mathcal{D}^{\varpi-1}\right)v(t) = \mathcal{G}_2(t, u(t), v(t)), \;\; t \in \mathcal{J} = [0, \mathcal{T}] \\ (u+v)(0) = -(u+v)(\mathcal{T}), \\ \sum_{i=0}^{m} x_i(u-v)(\xi_i) + \mu \int_0^{\eta} \frac{(\eta - s)^{\delta-1}}{\Gamma(\delta)}(u-v)(s)ds = \mathcal{A}, \end{cases} \tag{1.2}$$

where ${}^{C}\mathcal{D}^{\varsigma}$ denotes CFD of order ς is defined by

$${}^{C}\mathcal{D}^{\varsigma}v(t) = \frac{1}{\Gamma(n-\varsigma)} \int_0^t (t-s)^{n-\varsigma-1}\Big(\frac{d}{ds}\Big)^n v(s)ds, \;\; n-1 < \varsigma < n, \;\; n = [\varsigma] + 1,$$

and the Riemann-Liouville integral of fractional order ς defined by

$${}^{RL}\mathcal{I}^{\varsigma}v(t) = \frac{1}{\Gamma(\varsigma)} \int_0^t (t-s)^{\varsigma-1}v(s)ds, \;\; \varsigma > 0,$$

where $\varsigma \in \{\vartheta, \varpi\}$, $\vartheta, \varpi \in (0, 1]$, given constants $\varphi, \mathcal{A}, x_i$ $(i = 0, 1, ..., m) \in \mathbb{R}$, and $\mathcal{G}_1, \mathcal{G}_2 : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$, $\mathcal{G}_1, \mathcal{G}_2 : [0, T] \times \mathbb{R}^2 \to \mathcal{U}(\mathbb{R})$ are continuous functions, $\mathcal{U}(\mathbb{R})$ is the collection of non-empty subsets of $\mathbb{R}$. We should point out that the term "sequential" is used in this context in the sense that the operator ${}^{C}\mathcal{D}^{\vartheta} + \varphi^{C}\mathcal{D}^{\vartheta-1}$ can be written as the composition of the operators ${}^{C}\mathcal{D}^{\vartheta-1}(\mathcal{D} + \varphi)$. Under the same assumptions,

$$\begin{cases} \left({}^{C}\mathcal{D}^{\vartheta} + \varphi^{C}\mathcal{D}^{\vartheta-1}\right)u(t) \in \mathcal{G}_1(t, u(t), v(t)), \;\; t \in \mathcal{J} = [0, \mathcal{T}] \\ \left({}^{C}\mathcal{D}^{\varpi} + \varphi^{C}\mathcal{D}^{\varpi-1}\right)v(t) \in \mathcal{G}_2(t, u(t), v(t)), \;\; t \in \mathcal{J} = [0, \mathcal{T}], \\ (u+v)(0) = -(u+v)(\mathcal{T}), \\ \sum_{i=0}^{m} x_i(u-v)(\xi_i) + \mu \int_0^{\eta} \frac{(\eta - s)^{\delta-1}}{\Gamma(\delta)}(u-v)(s)ds = \mathcal{A}, \end{cases} \tag{1.3}$$

the existence of the following nonlinear coupled differential inclusion is investigated further. Unlike the [32], the main results of this article are entirely different. Because we consider the problems in the context of SFD, employ unique techniques based on Schaefer's and Banach's, Covitz- Nadler's, and nonlinear alternatives for Kakutani fixed point theorems, and investigate the nonlinear coupled differential inclusion (1.3), which was not considered in [32]. Furthermore, to the best of our knowledge, there are no published results relating to the system (1.3). Additionally, the boundary conditions (1.2) establish that the sum of the unknown functions u and v at the interval $[0, T]$'s equals zero. The second is the sum of the unknown function's Riemann-Liouville fractional integral values on the strip $(\eta, 0)$, while the unknown function's multi-point values at ξ_i $(i = 0, 1, ..., m)$ remain constant. On the other hand, Section 2 gives important preliminaries and an auxiliary lemma that are required to solve the given problem. The key conclusions are discussed in Section 3, where we look at the existence and uniqueness of solutions for systems (1.2) and (1.3), respectively. Section 4 provides particular examples that are consistent with the investigated systems and the fundamental theorems.

2. Preliminaries

The purpose of this part is to recall some definitions of multi-valued maps and lemmas that are important for establishing fundamental results [11, 33, 34].

Let $(\mathcal{S}, \|\cdot\|)$ be a normed space and that $\mathcal{U}_{cl}(\mathcal{S}) = \{\mathcal{P} \in \mathcal{U}(\mathcal{S}) : \mathcal{P}$ is closed $\}$, $\mathcal{U}_{c,cp}(\mathcal{S}) = \{\mathcal{P} \in \mathcal{U}(\mathcal{S}) : \mathcal{P}$ is convex and compact $\}$.

A multi-valued map $Q : \mathcal{S} \to \mathcal{U}(\mathcal{S})$ is

(a) convex valued if $Q(s)$ is convex $\forall s \in \mathcal{S}$;
(b) upper semi-continuous (U.S.C.) on $\mathcal{S}$ if, for each $s_0 \in \mathcal{S}$; the set $Q(s_0)$ is a non-empty closed subset of $\mathcal{S}$ and if, for each open set $\mathcal{V}$ of $\mathcal{S}$ containing $Q(s_0)$, there exists an open neighborhood $\mathcal{V}_0$ of s_0 such that $Q(\mathcal{V}_0) \subset \mathcal{V}$;
(c) lower semi-continuous (L.S.C.) if the set $\{s \in \mathcal{S} : Q(s) \cap \mathcal{E} \neq \oslash\}$ is open for any open set $\mathcal{E}$ in $\mathcal{H}$;
(d) completely continuous (C.C) if $Q(\mathcal{E})$ is relatively compact (r.c) for every $\mathcal{E} \in \mathcal{U}_b(\mathcal{S}) = \{\mathcal{P} \in \mathcal{U}(\mathcal{S}) : \mathcal{P}$ is bounded$\}$.

A map $Q : [c, d] \to \mathcal{U}_{cl}(\mathbb{R})$ of multi-valued is said to be measurable if, for every $s \in \mathbb{R}$, the function $t \longmapsto d(s, Q(t)) = \inf\{|s - l| : l \in Q(t)\}$ is measurable.

A multi-valued map $Q : [c, d] \times \mathbb{R} \to \mathcal{U}(\mathbb{R})$ is said to be Caratheodory if

(i) $t \longmapsto Q(t, q, s)$ is measurable for each $q, s \in \mathbb{R}$;
(ii) $(q, s) \longmapsto Q(t, q, s)$ is U.S.C for almost all $t \in [c, d]$.

Further a Caratheodory function Q is called $\mathcal{L}^1$-Caratheodory if

(i) for each $\varepsilon > 0$, $\exists\ \phi_\varepsilon \mathcal{L}^1([c, d], \mathbb{R}^+) \ni \|Q(t, q, s)\| = \sup\{|q| : q \in Q(t, q, s) \leq \phi_\varepsilon(t)\}\ \forall\ q, s \in \mathbb{R}$ with $\|q\|, \|s\| \leq \varepsilon$ and for a.e. $t \in [c, d]$.

Lemma 2.1. *[35] Let $\mathcal{M}$ a closed convex subset of a Banach space $\mathcal{S}$ and $\mathcal{W}$ be an open subset of $\mathcal{K}$ with $0 \in \mathcal{W}$. In addition, $\mathcal{H} : \hat{\mathcal{W}} \to \mathcal{Z}_{c,cp}(\mathcal{K})$ is an u.s.c compact map. Then either*

- *$\mathcal{H}$ has fixed point in $\hat{\mathcal{W}}$ or*
- *$\exists\ w \in \partial\mathcal{W}$ and $\lambda \in (0, 1)$ such that $w \in \lambda\mathcal{H}(w)$.*

Lemma 2.2. *[36] Let $(\mathcal{S}, d)$ be a complete metric space. If $Q : \mathcal{S} \to \mathcal{Z}_{cl}(\mathcal{S})$ is a contraction, then Fix $Q \neq 0$.*

Lemma 2.3. *[37] Let $\mathcal{Y} : \mathcal{S} \to \mathcal{S}$ be a completely continous operator in Banach Space $\mathcal{S}$ and the set $\Psi = \{s \in \mathcal{S} | s = \delta\mathcal{Y}s, 0 < \delta < 1\}$ is bounded. Then $\mathcal{Y}$ has a fixed point in $\mathcal{S}$.*

Lemma 2.4. *[11] Let $\varsigma > 0$. Then for $u \in C(0, T) \cap \mathcal{L}(0, T)$ it holds*

$$\mathcal{I}^{\varsigma}({}^{C}\mathcal{D}^{\varsigma}u)(t) = u(t) + d_0 + d_1 t + \cdots + d_{n-1}t^{n-1},$$

where $d_i \in \mathbb{R}$, $i = 1, 2, \cdots, n-1$ and $n = [\varsigma] + 1$.

Lemma 2.5. *Let $\hat{\mathcal{G}}_1, \hat{\mathcal{G}}_2 \in C[0,\mathcal{T}]$ and $u, v \in AC(\mathcal{S})$. The solution of the linear system of FDEs:*

$$\begin{cases} \left({}^C\mathcal{D}^{\vartheta} + \varphi\, {}^C\mathcal{D}^{\vartheta-1}\right)u(t) = \hat{\mathcal{G}}_1(t), \quad t \in \mathcal{J} := [0,\mathcal{T}], \\ \left({}^C\mathcal{D}^{\varpi} + \varphi\, {}^C\mathcal{D}^{\varpi-1}\right)v(t) = \hat{\mathcal{G}}_2(t), \quad t \in \mathcal{J} := [0,\mathcal{T}], \\ (u+v)(0) = -(u+v)(\mathcal{T}), \\ \sum_{i=0}^{m} x_i(u-v)(\xi_i) + \mu \int_0^{\eta} \frac{(\eta-s)^{\delta-1}}{\Gamma(\delta)}(u-v)(s)ds = \mathcal{A}, \end{cases} \tag{2.1}$$

is given by

$$\begin{aligned} u(t) = \frac{e^{-\varphi(t)}}{2}\Bigg[&\frac{1}{\Delta_2}\Bigg\{\mathcal{A} - \Bigg(\sum_{i=1}^{m} x_i \left\{\int_0^{\xi_i} e^{-\varphi(\xi_i - s)}\left(\int_0^s \frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}\hat{\mathcal{G}}_1(a)da\right)ds\right\} \\ &- \sum_{i=1}^{m} x_i \left\{\int_0^{\xi_i} e^{-\varphi(\xi_i - s)}\left(\int_0^s \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}\hat{\mathcal{G}}_2(a)da\right)ds\right\} \\ &+ \Bigg\{\mu \int_0^{\eta} \frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\left(\int_0^s e^{-\varphi(s-a)}\left(\int_0^a \frac{(a-m)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}\hat{\mathcal{G}}_1(m)dm\right)da\right)ds \\ &- \mu \int_0^{\eta} \frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\left(\int_0^s e^{-\varphi(s-a)}\left(\int_0^a \frac{(a-m)^{(\varpi-2)}}{\Gamma(\varpi-1)}\hat{\mathcal{G}}_2(m)dm\right)da\right)ds\Bigg\}\Bigg)\Bigg\} \\ &- \frac{1}{\Delta_1}\Bigg\{\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\left(\int_0^s \frac{(s-a)^{\vartheta-2}}{\Gamma(\vartheta-1)}\hat{\mathcal{G}}_1(a)da\right)ds \\ &+ \int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\left(\int_0^s \frac{(s-a)^{\varpi-2}}{\Gamma(\varpi-1)}\hat{\mathcal{G}}_2(a)da\right)ds\Bigg\}\Bigg] \\ &+ \int_0^{t} e^{-\varphi(t-s)}\left(\int_0^s \frac{(s-a)^{\vartheta-2}}{\Gamma(\vartheta-1)}\hat{\mathcal{G}}_1(a)da\right)ds, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} v(t) = \frac{e^{-\varphi(t)}}{2}\Bigg[&-\frac{1}{\Delta_2}\Bigg\{\mathcal{A} - \Bigg(\sum_{i=1}^{m} x_i \left\{\int_0^{\xi_i} e^{-\varphi(\xi_i - s)}\left(\int_0^s \frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}\hat{\mathcal{G}}_1(a)da\right)ds\right\} \\ &- \sum_{i=1}^{m} x_i \left\{\int_0^{\xi_i} e^{-\varphi(\xi_i - s)}\left(\int_0^s \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}\hat{\mathcal{G}}_2(a)da\right)ds\right\} \\ &+ \Bigg\{\mu \int_0^{\eta} \frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\left(\int_0^s e^{-\varphi(s-a)}\left(\int_0^a \frac{(a-m)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}\hat{\mathcal{G}}_1(m)dm\right)da\right)ds \\ &- \mu \int_0^{\eta} \frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\left(\int_0^s e^{-\varphi(s-a)}\left(\int_0^a \frac{(a-m)^{(\varpi-2)}}{\Gamma(\varpi-1)}\hat{\mathcal{G}}_2(m)dm\right)da\right)ds\Bigg\}\Bigg)\Bigg\} \\ &- \frac{1}{\Delta_1}\Bigg\{\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\left(\int_0^s \frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}\hat{\mathcal{G}}_1(a)da\right)ds \\ &+ \int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\left(\int_0^s \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}\hat{\mathcal{G}}_2(a)da\right)ds\Bigg\}\Bigg] \\ &+ \int_0^{t} e^{-\varphi(t-s)}\left(\int_0^s \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}\hat{\mathcal{G}}_2(a)da\right)ds, \end{aligned} \tag{2.3}$$

where

$$\Delta_1 = (1 + e^{-\varphi(\mathcal{T})}), \quad \Delta_2 = \left(\sum_{i=1}^{m} x_i e^{-\varphi(\xi_i)} + \mu \int_0^{\eta} \frac{(\eta - s)^{\delta-1}}{\Gamma(\delta)} e^{-\varphi s} ds\right). \tag{2.4}$$

3. Main results

The main results are stated and supported by the facts in this section. The results are obtained individually for the systems (1.2) and (1.3).

3.1. Existence solutions for the system (1.2)

Define $\mathcal{S} = C(\mathcal{J}, \mathbb{R}) \times C(\mathcal{J}, \mathbb{R})$ as the Banach space endowed with norm $\|(u, v)\| = \sup_{t\in\mathcal{J}} |u(t)| + \sup_{t\in\mathcal{J}} |v(t)|$, for $(u, v) \in \mathcal{S}$. Using Lemma 2.5, we convert system (1.2) into a fixed point problem as $u = \Omega u$, the following operator $\Omega : \mathcal{S} \to \mathcal{S}$ is defined by:

$$\Omega(u, v)(t) = (\Omega_1(u, v)(t), \Omega_2(u, v)(t)), \tag{3.1}$$

where

$$\begin{aligned}
&(\Omega_1(u, v))(t) \\
&= \frac{e^{-\varphi(t)}}{2}\left[\frac{1}{\Delta_2}\left\{\mathcal{A} - \left(\sum_{i=1}^{m} x_i \left\{\int_0^{\xi_i} e^{-\varphi(\xi_i - s)}\left(\int_0^{s} \frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}\mathcal{G}_1(a, u(a), v(a))da\right)ds\right\}\right.\right.\right. \\
&- \sum_{i=1}^{m} x_i \left\{\int_0^{\xi_i} e^{-\varphi(\xi_i - s)}\left(\int_0^{s} \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}\mathcal{G}_2(a, u(a), v(a))da\right)ds\right\} \\
&+ \left\{\mu \int_0^{\eta} \frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\left(\int_0^{s} e^{-\varphi(s-a)}\left(\int_0^{a} \frac{(a-m)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}\mathcal{G}_1(m, u(m), v(m))dm\right)da\right)ds\right. \\
&\left.\left.\left.-\mu \int_0^{\eta} \frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\left(\int_0^{s} e^{-\varphi(s-a)}\left(\int_0^{a} \frac{(a-m)^{(\varpi-2)}}{\Gamma(\varpi-1)}\mathcal{G}_2(m, u(m), v(m))dm\right)da\right)ds\right\}\right)\right\} \\
&-\frac{1}{\Delta_1}\left\{\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\left(\int_0^{s} \frac{(s-a)^{\vartheta-2}}{\Gamma(\vartheta-1)}\mathcal{G}_1(a, u(a), v(a))da\right)ds\right. \\
&\left.\left.+\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\left(\int_0^{s} \frac{(s-a)^{\varpi-2}}{\Gamma(\varpi-1)}\mathcal{G}_2(a, u(a), v(a))da\right)ds\right\}\right] \\
&+\int_0^{t} e^{-\varphi(t-s)}\left(\int_0^{s} \frac{(s-a)^{\vartheta-2}}{\Gamma(\vartheta-1)}\mathcal{G}_1(a, u(a), v(a))da\right)ds,
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
&(\Omega_2(u, v))(t) \\
&= \frac{e^{-\varphi(t)}}{2}\left[-\frac{1}{\Delta_2}\left\{\mathcal{A} - \left(\sum_{i=1}^{m} x_i \left\{\int_0^{\xi_i} e^{-\varphi(\xi_i - s)}\left(\int_0^{s} \frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}\mathcal{G}_1(a, u(a), v(a))da\right)ds\right\}\right.\right.\right. \\
&- \sum_{i=1}^{m} x_i \left\{\int_0^{\xi_i} e^{-\varphi(\xi_i - s)}\left(\int_0^{s} \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}\mathcal{G}_2(a, u(a), v(a))da\right)ds\right\}
\end{aligned}$$

$$+\left\{\mu\int_0^\eta \frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\left(\int_0^s e^{-\varphi(s-a)}\left(\int_0^a \frac{(a-m)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}\mathcal{G}_1(m,u(m),v(m))dm\right)da\right)ds\right.$$
$$\left.-\mu\int_0^\eta \frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\left(\int_0^s e^{-\varphi(s-a)}\left(\int_0^a \frac{(a-m)^{(\varpi-2)}}{\Gamma(\varpi-1)}\mathcal{G}_2(m,u(m),v(m))dm\right)da\right)ds\right\}\Bigg)\Bigg\}$$
$$-\frac{1}{\Delta_1}\left\{\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\left(\int_0^s \frac{(s-a)^{\vartheta-2}}{\Gamma(\vartheta-1)}\mathcal{G}_1(a,u(a),v(a))da\right)ds\right.$$
$$\left.+\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\left(\int_0^s \frac{(s-a)^{\varpi-2}}{\Gamma(\varpi-1)}\mathcal{G}_2(a,u(a),v(a))da\right)ds\right\}\Bigg]$$
$$+\int_0^t e^{-\varphi(t-s)}\left(\int_0^s \frac{(s-a)^{\varpi-2}}{\Gamma(\varpi-1)}\mathcal{G}_2(a,u(a),v(a))da\right)ds. \tag{3.3}$$

Next, the following assumptions will be used to demonstrate the paper's study results.

Let $\mathcal{G}_1,\mathcal{G}_2:\mathcal{J}\times\mathbb{R}^2\to\mathbb{R}$ be continuous functions.

$(\mathcal{W}_1)$ There exist continuous non-negative function $\gamma_i,k_i\in C(\mathcal{J},\mathbb{R}^+)$, $i=1,2,3$, such that

$$|\mathcal{G}_1(t,u,v)|\le\gamma_1(t)+\gamma_2(t)|u|+\gamma_3(t)|v| \quad \text{for all } (t,u,v)\in\mathcal{J}\times\mathbb{R}^2,$$
$$|\mathcal{G}_2(t,u,v)|\le k_1(t)+k_2(t)|u|+k_3(t)|v| \quad \text{for all } (t,u,v)\in\mathcal{J}\times\mathbb{R}^2;$$

$(\mathcal{W}_2)$ There exist non-negative constents $C_1,C_2,\mathcal{K}_1$ and $\mathcal{K}_2$ such that, $\forall\ t\in\mathcal{J}\ u_i,v_i\in\mathbb{R}, i=1,2.$

$$|\mathcal{G}_1(t,u_1,v_1)-\mathcal{G}_1(t,u_2,v_2)|\le C_1(|u_1-u_2|+C_2|v_1-v_2|),\ \text{for all}\ t\in\mathcal{J},$$
$$|\mathcal{G}_2(t,u_1,v_1)-\mathcal{G}_2(t,u_2,v_2)|\le \mathcal{K}_1(|u_1-u_2|+\mathcal{K}_2|v_1-v_2|),\ \text{for all}\ t\in\mathcal{J}.$$

To facilitate the computation, we introduce the notation:

$$p=\frac{e^{-\varphi T}}{2},$$

$$\begin{aligned}\Upsilon_1 &= p\left[\frac{1}{\Delta_2}\left\{\sum_{i=1}^m x_i\left(\frac{\xi_i^{\vartheta-1}}{\varphi\Gamma(\vartheta)}(1-e^{-\varphi(\xi_i)})\right)\right\}+\mu\left(\frac{\eta^{\vartheta+\delta-2}}{\varphi^2\Gamma(\delta)\Gamma(\vartheta)}(\eta\varphi+e^{-\varphi\eta}-1)\right)\right.\\ &\quad\left.+\frac{1}{\Delta_1}\left(\frac{\mathcal{T}^{(\vartheta-1)}}{\varphi\Gamma(\vartheta)}(1-e^{-\varphi\mathcal{T}})\right)\right],\end{aligned} \tag{3.4}$$

$$\begin{aligned}\Upsilon_2 &= p\left[\frac{1}{\Delta_2}\left\{\sum_{i=1}^m x_i\left(\frac{\xi_i^{\varpi-1}}{\varphi\Gamma(\varpi)}(1-e^{-\varphi(\xi_i)})\right)\right\}+\mu\left(\frac{\eta^{\varpi+\delta-2}}{\varphi^2\Gamma(\delta)\Gamma(\varpi)}(\eta\varphi+e^{-\varphi\eta}-1)\right)\right.\\ &\quad\left.+\frac{1}{\Delta_1}\left(\frac{\mathcal{T}^{(\varpi-1)}}{\varphi\Gamma(\varpi)}(1-e^{-\varphi\mathcal{T}})\right)\right],\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\Phi=\min\Bigg\{&1-\left[\|\gamma_2\|\left(2\Upsilon_1+\frac{T^{\vartheta-1}}{\varphi\Gamma(\vartheta)}(1-e^{-\varphi T})\right)+\|k_2\|\left(2\Upsilon_2+\frac{T^{\varpi-1}}{\varphi\Gamma(\varpi)}(1-e^{-\varphi T})\right)\right],\\ &1-\left[\|\gamma_3\|\left(2\Upsilon_1+\frac{T^{\vartheta-1}}{\varphi\Gamma(\vartheta)}(1-e^{-\varphi T})\right)+\|k_3\|\left(2\Upsilon_2+\frac{T^{\varpi-1}}{\varphi\Gamma(\varpi)}(1-e^{-\varphi T})\right)\right]\Bigg\}.\end{aligned}$$

In this part, we prove the existence of a solution to the BVP (1.2) via fixed point theorem of Schaefer's [37].

Theorem 3.1. *Assume that* $(\mathcal{W}_1)$ *holds. In addition, the assumption is that*

$$\|\gamma_2\|\left(2\Upsilon_1+\frac{T^{\vartheta-1}}{\varphi\Gamma(\vartheta)}(1-e^{-\varphi T})\right)+\|k_2\|\left(2\Upsilon_2+\frac{T^{\varpi-1}}{\varphi\Gamma(\varpi)}(1-e^{-\varphi T})\right)<1,$$
$$\|\gamma_3\|\left(2\Upsilon_1+\frac{T^{\vartheta-1}}{\varphi\Gamma(\vartheta)}(1-e^{-\varphi T})\right)+\|k_3\|\left(2\Upsilon_2+\frac{T^{\varpi-1}}{\varphi\Gamma(\varpi)}(1-e^{-\varphi T})\right)<1. \tag{3.6}$$

where $\Upsilon_i(i=1,2)$ *are defined by (3.4) and (3.5). Then the problem (1.2) has at least one solution on* $\mathcal{J}$

Proof. In the first part, we demonstrate that the operator $\Omega:\mathcal{S}\to\mathcal{S}$ is c.c. The $\mathcal{G}_1$ and $\mathcal{G}_2$, functions demonstrate that the Ω_1 and Ω_2 operators are both continuous. Thus, Ω is a continuous operator. Following that, we show that the Ω operator is continuously bounded. Let $\Pi_{\bar{r}}\subset\mathcal{S}$ be bounded. Then $\exists$ non-negative $\mathcal{L}_{\mathcal{G}_1}$ and $\mathcal{L}_{\mathcal{G}_2}$ constants, which means

$$|\mathcal{G}_1(t,u(t),v(t))|\le\mathcal{L}_{\mathcal{G}_1},$$
$$|\mathcal{G}_2(t,u(t),v(t))|\le\mathcal{L}_{\mathcal{G}_2},$$

for all $(u,v)\in\Pi_{\bar{r}}, t\in\mathcal{J}$, we have

$$|\Omega_1(u,v)(t)|\le\mathcal{L}_{\mathcal{G}_1}\left(\Upsilon_1+\frac{T^{\vartheta-1}}{\varphi\Gamma(\vartheta)}(1-e^{-\varphi T})\right)+\mathcal{L}_{\mathcal{G}_2}\Upsilon_2+\frac{\mathcal{A}}{\Delta_2},$$
$$|\Omega_2(u,v)(t)|\le\mathcal{L}_{\mathcal{G}_1}\Upsilon_1+\mathcal{L}_{\mathcal{G}_2}\left(\Upsilon_2+\frac{T^{\varpi-1}}{\varphi\Gamma(\varpi)}(1-e^{-\varphi T})\right)+\frac{\mathcal{A}}{\Delta_2}.$$

Thus,

$$\begin{aligned}\|\Omega(u,v)\|&=\|\Omega_1(u,v)\|+\|\Omega_2(u,v)\|\\&\le\mathcal{L}_{\mathcal{G}_1}\left(2\Upsilon_1+\frac{T^{\vartheta-1}}{\varphi\Gamma(\vartheta)}(1-e^{-\varphi T})\right)+\mathcal{L}_{\mathcal{G}_2}\left(2\Upsilon_2+\frac{T^{\varpi-1}}{\varphi\Gamma(\varpi)}(1-e^{-\varphi T})\right)+\frac{2\mathcal{A}}{\Delta_2}.\end{aligned}$$

It implies that the Ω operator has uniformly bounded from the inequality shown above.

In order to show that Ω maps bounded sets into equi-continuous sets of $\mathcal{S}$, let $t_1,t_2\in[0,\mathcal{T}]$, $t_1<t_2$, and $(u,v\in\Pi_{\bar{r}})$. Following that,

$$\begin{aligned}&|(\Omega_1(u,v)(t_2)-\Omega_1(u,v)(t_1))|\\&\le\left|\left(\frac{e^{-\varphi(t_2)}-e^{-\varphi(t_1)}}{2}\right)\right|\left[\frac{1}{\Delta_2}\left\{\mathcal{A}+\left(\sum_{i=1}^{m}x_i\left\{\int_0^{\xi_i}e^{-\varphi(\xi_i-s)}\left(\int_0^s\frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}\mathcal{L}_{\mathcal{G}_1}da\right)ds\right\}\right.\right.\right.\\&\quad+\sum_{i=1}^{m}x_i\left\{\int_0^{\xi_i}e^{-\varphi(\xi_i-s)}\left(\int_0^s\frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}\mathcal{L}_{\mathcal{G}_2}da\right)ds\right\}\\&\quad+\left\{\mu\int_0^{\eta}\frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\left(\int_0^s e^{-\varphi(s-a)}\left(\int_0^a\frac{(a-m)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}\mathcal{L}_{\mathcal{G}_1}dm\right)da\right)ds\right.\\&\quad\left.\left.\left.+\mu\int_0^{\eta}\frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\left(\int_0^s e^{-\varphi(s-a)}\left(\int_0^a\frac{(a-m)^{(\varpi-2)}}{\Gamma(\varpi-1)}\mathcal{L}_{\mathcal{G}_2}dm\right)da\right)ds\right\}\right)\right\}\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Delta_1}\left\{\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\left(\int_0^s \frac{(s-a)^{\vartheta-2}}{\Gamma(\vartheta-1)}\mathcal{L}_{\mathcal{G}_1}da\right)ds\right.\\
&\left.+\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\left(\int_0^s \frac{(s-a)^{\varpi-2}}{\Gamma(\varpi-1)}\mathcal{L}_{\mathcal{G}_2}da\right)ds\right\}\Bigg]\\
&+\int_0^{t_1}\left(e^{-\varphi(t_2-s)}-e^{-\varphi(t_1-s)}\right)\left(\int_0^s \frac{(s-a)^{\vartheta-2}}{\Gamma(\vartheta-1)}\mathcal{L}_{\mathcal{G}_1}da\right)ds\\
&+\int_{t_1}^{t_2} e^{-\varphi(t_2-s)}\left(\int_0^s \frac{(s-a)^{\vartheta-2}}{\Gamma(\vartheta-1)}\mathcal{L}_{\mathcal{G}_1}da\right)ds\Bigg|.
\end{aligned}
$$

Similarly, we can obtain

$$
\begin{aligned}
&|(\Omega_2(u,v)(t_2)-\Omega_2(u,v)(t_1))|\\
\le&\Bigg|\left(\frac{e^{-\varphi(t_2)}-e^{-\varphi(t_1)}}{2}\right)\Bigg[\frac{1}{\Delta_2}\Bigg\{\mathcal{A}+\Bigg(\sum_{i=1}^m x_i\left\{\int_0^{\xi_i} e^{-\varphi(\xi_i-s)}\left(\int_0^s \frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}\mathcal{L}_{\mathcal{G}_1}da\right)ds\right\}\\
&+\sum_{i=1}^m x_i\left\{\int_0^{\xi_i} e^{-\varphi(\xi_i-s)}\left(\int_0^s \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}\mathcal{L}_{\mathcal{G}_2}da\right)ds\right\}\\
&+\Bigg\{\mu\int_0^{\eta}\frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\left(\int_0^s e^{-\varphi(s-a)}\left(\int_0^a \frac{(a-m)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}\mathcal{L}_{\mathcal{G}_1}dm\right)da\right)ds\\
&+\mu\int_0^{\eta}\frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\left(\int_0^s e^{-\varphi(s-a)}\left(\int_0^a \frac{(a-m)^{(\varpi-2)}}{\Gamma(\varpi-1)}\mathcal{L}_{\mathcal{G}_2}dm\right)da\right)ds\Bigg\}\Bigg)\Bigg\}\\
&+\frac{1}{\Delta_1}\left\{\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\left(\int_0^s \frac{(s-a)^{\vartheta-2}}{\Gamma(\vartheta-1)}\mathcal{L}_{\mathcal{G}_1}da\right)ds\right.\\
&\left.+\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\left(\int_0^s \frac{(s-a)^{\varpi-2}}{\Gamma(\varpi-1)}\mathcal{L}_{\mathcal{G}_2}da\right)ds\right\}\Bigg]\\
&+\int_0^{t_1}\left(e^{-\varphi(t_2-s)}-e^{-\varphi(t_1-s)}\right)\left(\int_0^s \frac{(s-a)^{\varpi-2}}{\Gamma(\varpi-1)}\mathcal{L}_{\mathcal{G}_2}da\right)ds\\
&+\int_{t_1}^{t_2} e^{-\varphi(t_2-s)}\left(\int_0^s \frac{(s-a)^{\varpi-2}}{\Gamma(\varpi-1)}\mathcal{L}_{\mathcal{G}_2}da\right)ds\Bigg|.
\end{aligned}
$$

In the limit of $t_1 \to t_2$ notice that the right-hand sides of the above inequalities tend to zero independently of $(u,v)\in\Pi_{\bar{r}}$. Thus, it follows by the Arzela-Ascoli theorem that the operator $\Omega:\mathcal{E}\to\mathcal{E}$ is c.c. Next, we consider the set $\Theta=\{(u,v)\in\mathcal{S}|(u,v)=\gamma\Omega(u,v), 0<\gamma<1\}$. For any $t\in\mathcal{J}$, we have

$$
u(t)=\gamma\Omega_1(u,v)(t),\quad v(t)=\gamma\Omega_2(u,v)(t).
$$

Using $\Upsilon_i (i=1,2)$ given by (3.4) and (3.5), we find that

$$
\begin{aligned}
|u(t)|=&\gamma|\Omega_1(u,v)(t)|\le(\|\gamma_1\|+\|\gamma_2\|\|u\|+\|\gamma_3\|\|v\|)\left(\Upsilon_1+\frac{T^{\vartheta-1}}{\varphi\Gamma(\vartheta)}(1-e^{-\varphi T})\right)\\
&+(\|k_1\|+\|k_2\|\|u\|+\|k_3\|\|v\|)\,\Upsilon_2+\frac{\mathcal{A}}{\Delta_2},
\end{aligned}
$$

$$|v(t)| = \gamma|\Omega_2(u,v)(t)| \leq (\|\gamma_1\| + \|\gamma_2\|\|u\| + \|\gamma_3\|\|v\|)\,\Upsilon_1 \\ + (\|k_1\| + \|k_2\|\|u\| + \|k_3\|\|v\|)\left(\Upsilon_2 + \frac{T^{\varpi-1}}{\varphi\Gamma(\varpi)}(1-e^{-\varphi T})\right) + \frac{\mathcal{A}}{\Delta_2}.$$

In consequence, we get

$$\|u\| + \|v\| \leq \|\gamma_1\|\left(2\Upsilon_1 + \frac{T^{\vartheta-1}}{\varphi\Gamma(\vartheta)}(1-e^{-\varphi T})\right) + \|k_1\|\left(2\Upsilon_2 + \frac{T^{\varpi-1}}{\varphi\Gamma(\varpi)}(1-e^{-\varphi T})\right) + \frac{2\mathcal{A}}{\Delta_2} \\ + \left[\|\gamma_2\|\left(2\Upsilon_1 + \frac{T^{\vartheta-1}}{\varphi\Gamma(\vartheta)}(1-e^{-\varphi T})\right) + \|k_2\|\left(2\Upsilon_2 + \frac{T^{\varpi-1}}{\varphi\Gamma(\varpi)}(1-e^{-\varphi T})\right)\right]\|u\| \\ + \left[\|\gamma_3\|\left(2\Upsilon_1 + \frac{T^{\vartheta-1}}{\varphi\Gamma(\vartheta)}(1-e^{-\varphi T})\right) + \|k_3\|\left(2\Upsilon_2 + \frac{T^{\varpi-1}}{\varphi\Gamma(\varpi)}(1-e^{-\varphi T})\right)\right]\|v\|.$$

Thus, using (3.6) the above assigned equations, we can get

$$\|(u,v)\| \leq \frac{\|\gamma_1\|\left(2\Upsilon_1 + \frac{T^{\vartheta-1}}{\varphi\Gamma(\vartheta)}(1-e^{-\varphi T})\right) + \|k_1\|\left(2\Upsilon_2 + \frac{T^{\varpi-1}}{\varphi\Gamma(\varpi)}(1-e^{-\varphi T})\right) + \frac{2\mathcal{A}}{\Delta_2}}{\Phi},$$

It shows that $\|(u,v)\|$ is bounded for $t \in \mathcal{J}$. The set Θ is bounded. Therefore, the fixed point theorem of Schaefer's concludes and, therefore, the Ω operator has at least one fixed point, which is a solution for the problem (1.2). □

If $\gamma_2(t) = \gamma_3(t) \equiv 0$ and $k_2(t) = k_2(t) \equiv 0$ are valid, then Theorem (3.1) has the special case form shown below.

Remark 3.1. *There exist positive functions $\gamma_1, k_1 \in C(\mathcal{J}, \mathbb{R}^+)$ and $\mathcal{G}_1, \mathcal{G}_2 : \mathcal{J} \times \mathbb{R}^2 \to \mathbb{R}$ which are continuous functions such that*

$$|\mathcal{G}_1(t,u,v)| \leq \gamma_1(t), \quad |\mathcal{G}_2(t,u,v)| \leq k_1(t) \quad \textit{for all } (t,u,v) \in \mathcal{J} \times \mathbb{R}^2;$$

Then system (1.2) has at least one solution on $\mathcal{J}$.

Remark 3.2. *According to the assumptions of Theorem 3.1, if $\gamma_i(t) = \delta_i, k_i(t) = \varepsilon_i, i = 1,2,3$ (ε_i and δ_i) non-negative constants, and the criteria of the functions $\mathcal{G}_1$ and $\mathcal{G}_2$ have the following form:*

($\hat{W}_1$) there are real constants $\delta_i, \varepsilon_i > 0, i = 1,2,3,$ so

$$|\mathcal{G}_1(t,u,v)| \leq \delta_1 + \delta_2|u| + \delta_3|v| \quad \textit{for all } (t,u,v) \in \mathcal{J} \times \mathbb{R}^2, \\ |\mathcal{G}_2(t,u,v)| \leq \varepsilon_1 + \varepsilon_2|u| + \varepsilon_3|v| \quad \textit{for all } (t,u,v) \in \mathcal{J} \times \mathbb{R}^2;$$

and (3.6) becomes

$$\delta_2\left(2\Upsilon_1 + \frac{T^{\vartheta-1}}{\varphi\Gamma(\vartheta)}(1-e^{-\varphi T})\right) + \varepsilon_2\left(2\Upsilon_2 + \frac{T^{\varpi-1}}{\varphi\Gamma(\varpi)}(1-e^{-\varphi T})\right) < 1, \\ \delta_3\left(2\Upsilon_1 + \frac{T^{\vartheta-1}}{\varphi\Gamma(\vartheta)}(1-e^{-\varphi T})\right) + \varepsilon_3\left(2\Upsilon_2 + \frac{T^{\varpi-1}}{\varphi\Gamma(\varpi)}(1-e^{-\varphi T})\right) < 1.$$

Theorem 3.2. *Assume that* $(\mathcal{W}_2)$ *holds and that*

$$\mathcal{C}\left(2\Upsilon_1 + \frac{T^{\vartheta-1}}{\varphi\Gamma(\vartheta)}(1-e^{-\varphi T})\right) + \mathcal{K}\left(2\Upsilon_2 + \frac{T^{\varpi-1}}{\varphi\Gamma(\varpi)}(1-e^{-\varphi T})\right) < 1, \tag{3.7}$$

where $\mathcal{C} = \max\{\mathcal{C}_1, \mathcal{C}_2\}$*,* $\mathcal{K} = \max\{\mathcal{K}_1, \mathcal{K}_2\}$ *and* $\Upsilon_i, i = 1, 2$ *are defined by (3.4) and (3.5). Then the problem (1.2) has a unique solution on* $\mathcal{J}$

Proof. Consider the operator $\Omega : \mathcal{S} \to \mathcal{S}$ defined by (3.1) and fix

$$r > \frac{\mathcal{M}_1\left(2\Upsilon_1 + \frac{T^{\vartheta-1}}{\varphi\Gamma(\vartheta)}(1-e^{-\varphi T})\right) + \mathcal{M}_2\left(2\Upsilon_2 + \frac{T^{\varpi-1}}{\varphi\Gamma(\varpi)}(1-e^{-\varphi T})\right)}{1 - \left(\mathcal{C}\left(2\Upsilon_1 + \frac{T^{\vartheta-1}}{\varphi\Gamma(\vartheta)}(1-e^{-\varphi T})\right) + \mathcal{K}\left(2\Upsilon_2 + \frac{T^{\varpi-1}}{\varphi\Gamma(\varpi)}(1-e^{-\varphi T})\right)\right)},$$

where $\mathcal{M}_1 = \sup_{t\in\mathcal{J}} |\mathcal{G}_1(t, 0, 0)|$, and $\mathcal{M}_2 = \sup_{t\in\mathcal{J}} |\mathcal{G}_2(t, 0, 0)|$. Then we show that $\Omega\mathcal{B}_r \subset \mathcal{B}_r$, where $\mathcal{B}_r = \{(u, v) \in \mathcal{S} : \|(u, v)\| \le r\}$, we have

$$\begin{aligned}
\|\Omega_1(u,v)\| \le& \left(\mathcal{C}\left(\Upsilon_1 + \frac{T^{\vartheta-1}}{\varphi\Gamma(\vartheta)}(1-e^{-\varphi T})\right) + \mathcal{K}\Upsilon_2\right)(\|u\| + \|v\|) \\
&+ \mathcal{M}_1\left(\Upsilon_1 + \frac{T^{\vartheta-1}}{\varphi\Gamma(\vartheta)}(1-e^{-\varphi T})\right) + \mathcal{M}_2\Upsilon_2,
\end{aligned}$$

when the norm for $t \in \mathcal{J}$. In the same way, for $(u, v) \in \mathcal{B}_r$, one can obtain

$$\begin{aligned}
\|\Omega_2(u,v)\| \le& \left(\mathcal{K}\left(\Upsilon_2 + \frac{T^{\varpi-1}}{\varphi\Gamma(\varpi)}(1-e^{-\varphi T})\right) + \mathcal{C}\Upsilon_1\right)(\|u\| + \|v\|) \\
&+ \mathcal{M}_2\left(\Upsilon_2 + \frac{T^{\varpi-1}}{\varphi\Gamma(\varpi)}(1-e^{-\varphi T})\right) + \mathcal{M}_1\Upsilon_1.
\end{aligned}$$

Therefore, for any $(u, v) \in \mathcal{B}_r$, we have

$$\begin{aligned}
&\|\Omega(u,v)\| = \|\Omega_1(u,v)\| + \|\Omega_2(u,v)\| \\
&\le \left(\mathcal{C}\left(2\Upsilon_1 + \frac{T^{\vartheta-1}}{\varphi\Gamma(\vartheta)}(1-e^{-\varphi T})\right) + \mathcal{K}\left(2\Upsilon_2 + \frac{T^{\vartheta-1}}{\varphi\Gamma(\vartheta)}(1-e^{-\varphi T})\right)\right)(\|u\| + \|v\|) \\
&+ \mathcal{M}_1\left(2\Upsilon_1 + \frac{T^{\vartheta-1}}{\varphi\Gamma(\vartheta)}(1-e^{-\varphi T})\right) + \mathcal{M}_2\left(2\Upsilon_2 + \frac{T^{\varpi-1}}{\varphi\Gamma(\varpi)}(1-e^{-\varphi T})\right) < r.
\end{aligned}$$

which shows that Ω maps $\mathcal{B}_r$ into itself.

In order to demonstrate that the Ω operator is a contraction, let $(u_1, v_1), (u_2, v_2) \in \mathcal{S}, t \in [0, 1]$ in view of $(\mathcal{W}_2)$, we obtain

$$\|(\Omega_1(u_1,v_1)) - (\Omega_1(u_2,v_2))\| \le \left(\mathcal{C}\left(\Upsilon_1 + \frac{T^{\vartheta-1}}{\varphi\Gamma(\vartheta)}(1-e^{-\varphi T})\right) + \mathcal{K}\Upsilon_2\right)(\|u\| + \|v\|),$$

and

$$\|(\Omega_2(u_1,v_1)) - (\Omega_2(u_2,v_2))\| \le \left(\mathcal{K}\left(\Upsilon_2 + \frac{T^{\varpi-1}}{\varphi\Gamma(\varpi)}(1-e^{-\varphi T})\right) + \mathcal{C}\Upsilon_1\right)(\|u\| + \|v\|).$$

Clearly, the preceding inequalities imply that

$$\begin{aligned}&\|(\Omega(u_1,v_1))-(\Omega(u_2,v_2))\|\\&=\|(\Omega_1(u_1,v_1))-(\Omega_1(u_2,v_2))\|+\|(\Omega_2(u_1,v_1))-(\Omega_2(u_2,v_2))\|\\&\le\left(C\left(2\Upsilon_1+\frac{T^{\vartheta-1}}{\varphi\Gamma(\vartheta)}(1-e^{-\varphi T})\right)+\mathcal{K}\left(2\Upsilon_2+\frac{T^{\varpi-1}}{\varphi\Gamma(\varpi)}(1-e^{-\varphi T})\right)\right)\|(u_1-u_2,v_1-v_2)\|.\end{aligned}$$

As a result, the operator Ω is a contraction in light of assumption (3.7). The mapping theorem has a unique fixed point as a result of Banach's contraction. It indicates that system (1.2) has a solution that is unique on $\mathcal{J}$. □

3.2. Existence results for system (1.3)

Definition 3.3. *A function $(u,v)\in C(\mathcal{J},\mathbb{R})\times C(\mathcal{J},\mathbb{R})$ satisfying the BCs and for which $\exists$ function $g_1,g_2\in L^1(\mathcal{J},\mathbb{R})\ni g_1(t)\in\mathcal{G}_1(t,u(t),v(t))$, $g_2(t)\in\mathcal{G}_2(t,u(t),v(t))$ a.e. on $t\in\mathcal{J}$ and*

$$\begin{aligned}u(t)=\frac{e^{-\varphi(t)}}{2}\Bigg[&\frac{1}{\Delta_2}\Bigg\{\mathcal{A}-\Bigg(\sum_{i=1}^{m}x_i\left\{\int_0^{\xi_i}e^{-\varphi(\xi_i-s)}\left(\int_0^s\frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}g_1(a)da\right)ds\right\}\\&-\sum_{i=1}^{m}x_i\left\{\int_0^{\xi_i}e^{-\varphi(\xi_i-s)}\left(\int_0^s\frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}g_2(a)da\right)ds\right\}\\&+\Bigg\{\mu\int_0^{\eta}\frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\left(\int_0^s e^{-\varphi(s-a)}\left(\int_0^a\frac{(a-m)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}g_1(m)dm\right)da\right)ds\\&-\mu\int_0^{\eta}\frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\left(\int_0^s e^{-\varphi(s-a)}\left(\int_0^a\frac{(a-m)^{(\varpi-2)}}{\Gamma(\varpi-1)}g_2(m)dm\right)da\right)ds\Bigg\}\Bigg)\Bigg\}\\&-\frac{1}{\Delta_1}\left\{\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-s)}\left(\int_0^s\frac{(s-a)^{\vartheta-2}}{\Gamma(\vartheta-1)}g_1(a)da\right)ds\right.\\&\left.+\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-s)}\left(\int_0^s\frac{(s-a)^{\varpi-2}}{\Gamma(\varpi-1)}g_2(a)da\right)ds\right\}\Bigg]\\&+\int_0^t e^{-\varphi(t-s)}\left(\int_0^s\frac{(s-a)^{\vartheta-2}}{\Gamma(\vartheta-1)}g_1(a)da\right)ds,\end{aligned}\tag{3.8}$$

and

$$\begin{aligned}v(t)=\frac{e^{-\varphi(t)}}{2}\Bigg[&-\frac{1}{\Delta_2}\Bigg\{\mathcal{A}-\Bigg(\sum_{i=1}^{m}x_i\left\{\int_0^{\xi_i}e^{-\varphi(\xi_i-s)}\left(\int_0^s\frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}g_1(a)da\right)ds\right\}\\&-\sum_{i=1}^{m}x_i\left\{\int_0^{\xi_i}e^{-\varphi(\xi_i-s)}\left(\int_0^s\frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}g_2(a)da\right)ds\right\}\\&+\Bigg\{\mu\int_0^{\eta}\frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\left(\int_0^s e^{-\varphi(s-a)}\left(\int_0^a\frac{(a-m)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}g_1(m)dm\right)da\right)ds\\&-\mu\int_0^{\eta}\frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\left(\int_0^s e^{-\varphi(s-a)}\left(\int_0^a\frac{(a-m)^{(\varpi-2)}}{\Gamma(\varpi-1)}g_2(m)dm\right)da\right)ds\Bigg\}\Bigg)\Bigg\}\\&-\frac{1}{\Delta_1}\left\{\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-s)}\left(\int_0^s\frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}g_1(a)da\right)ds\right.\end{aligned}\tag{3.9}$$

$$+\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\left(\int_0^s \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)} g_2(a)da\right)ds\Bigg\}\Bigg]$$
$$+\int_0^{t} e^{-\varphi(t-s)}\left(\int_0^s \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)} g_2(a)da\right)ds.$$

is referred to as a coupled solution for system (1.3). Let

$$\mathcal{W}_{\mathcal{G}_1(u,v)} = \{g_1 \in L^1(\mathcal{J},\mathbb{R}) : g_1(t) \in \mathcal{G}_1(t,u(t),v(t)) \text{ for a.e. } t \in \mathcal{J}\}$$

and

$$\mathcal{W}_{G_2(u,v)} = \{g_2 \in L^1(\mathcal{J},\mathbb{R}) : g_2(t) \in \mathcal{G}_2(t,u(t),v(t)) \text{ for a.e. } t \in \mathcal{J}\},$$

define the sets of $\mathcal{G}_1, \mathcal{G}_2$ selections for each $(u,v) \in \mathcal{S}\times\mathcal{S}$. Using Lemma 2.5, the following operators $K_1 \times K_2 : \mathcal{S}\times\mathcal{S} \to \mathcal{U}(\mathcal{S}\times\mathcal{S})$ are defined by:

$$\begin{aligned} K_1(u,v)(t) =\{h_1 \in \mathcal{S}\times\mathcal{S} :\ & \text{there exist } g_1 \in \mathcal{W}_{\mathcal{G}_1(u,v)}, g_2 \in \mathcal{W}_{\mathcal{G}_2(u,v)} \text{ such that} \\ & h_1(u,v)(t) = Q_1(u,v)(t), \forall\ t \in \mathcal{J}\} \end{aligned} \tag{3.10}$$

$$\begin{aligned} K_2(u,v)(t) =\{h_2 \in \mathcal{S}\times\mathcal{S} :\ & \text{there exist } g_1 \in \mathcal{W}_{\mathcal{G}_1(u,v)}, g_2 \in \mathcal{W}_{\mathcal{G}_2(u,v)} \text{ such that} \\ & h_2(u,v)(t) = Q_2(u,v)(t), \forall\ t \in \mathcal{J}\} \end{aligned} \tag{3.11}$$

where

$$\begin{aligned} Q_1(u,v)(t) = \frac{e^{-\varphi(t)}}{2}\Bigg[\frac{1}{\Delta_2}\Bigg\{\mathcal{A} - \Bigg(&\sum_{i=1}^m x_i\Bigg\{\int_0^{\xi_i} e^{-\varphi(\xi_i-s)}\left(\int_0^s \frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)} g_1(a)da\right)ds\Bigg\} \\ &-\sum_{i=1}^m x_i\Bigg\{\int_0^{\xi_i} e^{-\varphi(\xi_i-s)}\left(\int_0^s \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)} g_2(a)da\right)ds\Bigg\} \\ &+\Bigg\{\mu\int_0^{\eta} \frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\left(\int_0^s e^{-\varphi(s-a)}\left(\int_0^a \frac{(a-m)^{(\vartheta-2)}}{\Gamma(\vartheta-1)} g_1(m)dm\right)da\right)ds \\ &-\mu\int_0^{\eta} \frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\left(\int_0^s e^{-\varphi(s-a)}\left(\int_0^a \frac{(a-m)^{(\varpi-2)}}{\Gamma(\varpi-1)} g_2(m)dm\right)da\right)ds\Bigg\}\Bigg)\Bigg\} \\ &-\frac{1}{\Delta_1}\Bigg\{\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\left(\int_0^s \frac{(s-a)^{\vartheta-2}}{\Gamma(\vartheta-1)} g_1(a)da\right)ds \\ &+\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\left(\int_0^s \frac{(s-a)^{\varpi-2}}{\Gamma(\varpi-1)} g_2(a)da\right)ds\Bigg\}\Bigg] \\ &+\int_0^{t} e^{-\varphi(t-s)}\left(\int_0^s \frac{(s-a)^{\vartheta-2}}{\Gamma(\vartheta-1)} g_1(a)da\right)ds, \end{aligned} \tag{3.12}$$

and

$$Q_2(u,v)(t) = \frac{e^{-\varphi(t)}}{2}\Bigg[-\frac{1}{\Delta_2}\Bigg\{\mathcal{A} - \Bigg(\sum_{i=1}^m x_i\Bigg\{\int_0^{\xi_i} e^{-\varphi(\xi_i-s)}\left(\int_0^s \frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)} g_1(a)da\right)ds\Bigg\}$$

$$
\begin{aligned}
&-\sum_{i=1}^{m} x_i \left\{ \int_0^{\xi_i} e^{-\varphi(\xi_i - s)} \left(\int_0^s \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)} g_2(a)da \right) ds \right\} \\
&+ \left\{ \mu \int_0^{\eta} \frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)} \left(\int_0^s e^{-\varphi(s-a)} \left(\int_0^a \frac{(a-m)^{(\vartheta-2)}}{\Gamma(\vartheta-1)} g_1(m)dm \right) da \right) ds \right. \\
&\left. -\mu \int_0^{\eta} \frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)} \left(\int_0^s e^{-\varphi(s-a)} \left(\int_0^a \frac{(a-m)^{(\varpi-2)}}{\Gamma(\varpi-1)} g_2(m)dm \right) da \right) ds \right\} \Bigg) \Bigg\} \\
&-\frac{1}{\Delta_1} \left\{ \int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)} \left(\int_0^s \frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)} g_1(a)da \right) ds \right. \\
&\left. + \int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)} \left(\int_0^s \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)} g_2(a)da \right) ds \right\} \Bigg] \\
&+ \int_0^t e^{-\varphi(t-s)} \left(\int_0^s \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)} g_2(a)da \right) ds.
\end{aligned} \tag{3.13}
$$

Following that, we define the operator $K : \mathcal{S} \times \mathcal{S} \to \mathcal{U}(\mathcal{S} \times \mathcal{S})$ by

$$
K(u, v)(t) = \begin{pmatrix} K_1(u, v)(t) \\ K_2(u, v)(t) \end{pmatrix},
$$

where K_1 and K_2 are defined by 3.10 and 3.11.

Here we demonstrate the existence of solutions for the BVP (1.3) using the nonlinear alternative of Leray-Schauder [36] to verify the existence of solutions. Following that, we begin to develop the hypotheses that will be used to support the main results presented in the paper.

(B_1) $\mathcal{G}_1, \mathcal{G}_2 : \mathcal{J} \times \mathbb{R}^2 \to \mathcal{U}(\mathbb{R})$ are L^1 - Caratheodory and have convex values;

(B_2) There exist continuous non-decreasing functions $\psi_1, \psi_2, \phi_1, \phi_2 : [0, \infty) \to [0, \infty)$ and functions $p_1, p_2 \in C(\mathcal{J}, \mathbb{R}_+)$ such that

$$
\|\mathcal{G}_1(t, u, v)\|_{\mathcal{U}} := \sup\{|g_1| : g_1 \in \mathcal{G}_1(t, u, v)\} \le p_1(t)[\psi_1(\|u\|) + \phi_1(\|v\|)]
$$

and

$$
\|\mathcal{G}_2(t, u, v)\|_{\mathcal{U}} := \sup\{|g_2| : g_2 \in \mathcal{G}_2(t, u, v)\} \le p_2(t)[\psi_1(\|u\|) + \phi_2(\|v\|)]
$$

for each $(t, u, v) \in \mathcal{J} \times \mathbb{R}^2$;

(B_3) there exists a number $\mathcal{N} > 0$ such that

$$
\frac{\mathcal{N}}{(2\Upsilon_1)\|p_1\|(\psi_1(\mathcal{N}) + \phi_1(\mathcal{N})) + (2\Upsilon_2)\|p_2\|(\psi_2(\mathcal{N}) + \phi_2(\mathcal{N}))} > 1,
$$

where $\Upsilon_i (i = 1, 2)$ are defined by (3.4) and (3.5).

(B_4) $\mathcal{G}_1, \mathcal{G}_2 : \mathcal{J} \times \mathbb{R}^2 \to \mathcal{U}_{cp}(\mathbb{R})$ are such that $\mathcal{G}_1(., u, v) : \mathcal{J} \to \mathcal{U}_{cp}(\mathbb{R})$ and $\mathcal{G}_2(., u, v) : \mathcal{J} \to \mathcal{U}_{cp}(\mathbb{R})$ are measurable for each $u, v \in \mathbb{R}$;

(B_5)

$$
\mathcal{H}_d(\mathcal{G}_1(t, u, v), \mathcal{G}_1(t, \bar{u}, \bar{v})) \le m_1(t)(|u - \bar{u}| + |v - \bar{v}|)
$$

and

$$\mathcal{H}_d(\mathcal{G}_2(t,u,v),\mathcal{G}_2(t,\bar{u},\bar{v}))\leq m_2(t)(|u-\bar{u}|+|v-\bar{v}|)$$

$\forall\ t\in\mathcal{J}$ and $u,v,\bar{u},\bar{v}\in\mathbb{R}$ with $m_1,m_2\in C(\mathcal{J},\mathbb{R}^+)$ and $d(0,\mathcal{G}_1(t,0,0))\leq m_1(t), d(0,\mathcal{G}_2(t,0,0))\leq m_2(t)$ for almost $t\in\mathcal{J}$.

Theorem 3.4. *Suppose that* (B_1), (B_2), *and* (B_3) *hold. Then coupled system (1.3) has at least one solution on* $\mathcal{J}$.

Proof. Consider $K_1{\times}K_2:\mathcal{S}{\times}\mathcal{S}\to\mathcal{U}(\mathcal{S}{\times}\mathcal{S})$ the operators which is given by 3.10 and 3.11, respectively. Using the assumption (B_1), the sets $\mathcal{W}_{\mathcal{G}_1(u,v)}$ and $\mathcal{W}_{\mathcal{G}_2(u,v)}$ are non-empty for each $(u,v)\in\mathcal{S}\times\mathcal{S}$. Then, for $g_1\in\mathcal{W}_{\mathcal{G}_1(u,v)}, g_2\in\mathcal{W}_{\mathcal{G}_2(u,v)}$ for $(u,v)\in\mathcal{S}\times\mathcal{S}$, we have

$$\begin{aligned}
h_1(u,v)(t)=\frac{e^{-\varphi(t)}}{2}\Bigg[&\frac{1}{\Delta_2}\Bigg\{\mathcal{A}-\Bigg(\sum_{i=1}^{m}x_i\Bigg\{\int_0^{\xi_i}e^{-\varphi(\xi_i-s)}\Bigg(\int_0^s\frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}g_1(a)da\Bigg)ds\Bigg\}\\
&-\sum_{i=1}^{m}x_i\Bigg\{\int_0^{\xi_i}e^{-\varphi(\xi_i-s)}\Bigg(\int_0^s\frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}g_2(a)da\Bigg)ds\Bigg\}\\
&+\Bigg\{\mu\int_0^{\eta}\frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\Bigg(\int_0^s e^{-\varphi(s-a)}\Bigg(\int_0^a\frac{(a-m)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}g_1(m)dm\Bigg)da\Bigg)ds\\
&-\mu\int_0^{\eta}\frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\Bigg(\int_0^s e^{-\varphi(s-a)}\Bigg(\int_0^a\frac{(a-m)^{(\varpi-2)}}{\Gamma(\varpi-1)}g_2(m)dm\Bigg)da\Bigg)ds\Bigg\}\Bigg)\Bigg\}\\
&-\frac{1}{\Delta_1}\Bigg\{\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-s)}\Bigg(\int_0^s\frac{(s-a)^{\vartheta-2}}{\Gamma(\vartheta-1)}g_1(a)da\Bigg)ds\\
&+\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-s)}\Bigg(\int_0^s\frac{(s-a)^{\varpi-2}}{\Gamma(\varpi-1)}g_2(a)da\Bigg)ds\Bigg\}\Bigg]\\
&+\int_0^{t}e^{-\varphi(t-s)}\Bigg(\int_0^s\frac{(s-a)^{\vartheta-2}}{\Gamma(\vartheta-1)}g_1(a)da\Bigg)ds,
\end{aligned}$$

and

$$\begin{aligned}
h_2(u,v)(t)=\frac{e^{-\varphi(t)}}{2}\Bigg[&-\frac{1}{\Delta_2}\Bigg\{\mathcal{A}-\Bigg(\sum_{i=1}^{m}x_i\Bigg\{\int_0^{\xi_i}e^{-\varphi(\xi_i-s)}\Bigg(\int_0^s\frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}g_1(a)da\Bigg)ds\Bigg\}\\
&-\sum_{i=1}^{m}x_i\Bigg\{\int_0^{\xi_i}e^{-\varphi(\xi_i-s)}\Bigg(\int_0^s\frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}g_2(a)da\Bigg)ds\Bigg\}\\
&+\Bigg\{\mu\int_0^{\eta}\frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\Bigg(\int_0^s e^{-\varphi(s-a)}\Bigg(\int_0^a\frac{(a-m)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}g_1(m)dm\Bigg)da\Bigg)ds\\
&-\mu\int_0^{\eta}\frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\Bigg(\int_0^s e^{-\varphi(s-a)}\Bigg(\int_0^a\frac{(a-m)^{(\varpi-2)}}{\Gamma(\varpi-1)}g_2(m)dm\Bigg)da\Bigg)ds\Bigg\}\Bigg)\Bigg\}\\
&-\frac{1}{\Delta_1}\Bigg\{\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-s)}\Bigg(\int_0^s\frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}g_1(a)da\Bigg)ds\\
&+\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-s)}\Bigg(\int_0^s\frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}g_2(a)da\Bigg)ds\Bigg\}\Bigg]
\end{aligned}$$

$$+\int_0^t e^{-\varphi(t-s)}\left(\int_0^s \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}g_2(a)da\right)ds.$$

Where $h_1 \in K_1(u,v), h_2 \in K_2(u,v)$, and $(h_1,h_2) \in K(u,v)$.

In a series of phases, will demonstrated that operator K satisfies the Leray-Schauder nonlinear alternative hypothesis. First, we show that $K(u,v)$ has a convex value. Let $(h,\bar{h}) \in (K_1,K_2), i = 1,2$. Then there exist $g_{1i} \in \mathcal{W}_{\mathcal{G}_1(u,v)}, g_{2i} \in \mathcal{W}_{\mathcal{G}_2(u,v)}, i = 1,2$, such that, for each $t \in \mathcal{J}$, we have

$$\begin{aligned}
h_i(t) = \frac{e^{-\varphi(t)}}{2}&\Bigg[\frac{1}{\Delta_2}\Bigg\{\mathcal{A} - \Bigg(\sum_{i=1}^m x_i\left\{\int_0^{\xi_i} e^{-\varphi(\xi_i-s)}\left(\int_0^s \frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}g_1(a)da\right)ds\right\}\\
&-\sum_{i=1}^m x_i\left\{\int_0^{\xi_i} e^{-\varphi(\xi_i-s)}\left(\int_0^s \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}g_2(a)da\right)ds\right\}\\
&+\Bigg\{\mu\int_0^\eta \frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\left(\int_0^s e^{-\varphi(s-a)}\left(\int_0^a \frac{(a-m)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}g_1(m)dm\right)da\right)ds\\
&-\mu\int_0^\eta \frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\left(\int_0^s e^{-\varphi(s-a)}\left(\int_0^a \frac{(a-m)^{(\varpi-2)}}{\Gamma(\varpi-1)}g_2(m)dm\right)da\right)ds\Bigg\}\Bigg)\Bigg\}\\
&-\frac{1}{\Delta_1}\Bigg\{\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\left(\int_0^s \frac{(s-a)^{\vartheta-2}}{\Gamma(\vartheta-1)}g_1(a)da\right)ds\\
&+\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\left(\int_0^s \frac{(s-a)^{\varpi-2}}{\Gamma(\varpi-1)}g_2(a)da\right)ds\Bigg\}\Bigg]\\
&+\int_0^t e^{-\varphi(t-s)}\left(\int_0^s \frac{(s-a)^{\vartheta-2}}{\Gamma(\vartheta-1)}g_1(a)da\right)ds,
\end{aligned}$$

and

$$\begin{aligned}
\bar{h}_i(t) = \frac{e^{-\varphi(t)}}{2}&\Bigg[-\frac{1}{\Delta_2}\Bigg\{\mathcal{A} - \Bigg(\sum_{i=1}^m x_i\left\{\int_0^{\xi_i} e^{-\varphi(\xi_i-s)}\left(\int_0^s \frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}g_1(a)da\right)ds\right\}\\
&-\sum_{i=1}^m x_i\left\{\int_0^{\xi_i} e^{-\varphi(\xi_i-s)}\left(\int_0^s \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}g_2(a)da\right)ds\right\}\\
&+\Bigg\{\mu\int_0^\eta \frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\left(\int_0^s e^{-\varphi(s-a)}\left(\int_0^a \frac{(a-m)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}g_1(m)dm\right)da\right)ds\\
&-\mu\int_0^\eta \frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\left(\int_0^s e^{-\varphi(s-a)}\left(\int_0^a \frac{(a-m)^{(\varpi-2)}}{\Gamma(\varpi-1)}g_2(m)dm\right)da\right)ds\Bigg\}\Bigg)\Bigg\}\\
&-\frac{1}{\Delta_1}\Bigg\{\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\left(\int_0^s \frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}g_1(a)da\right)ds\\
&+\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\left(\int_0^s \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}g_2(a)da\right)ds\Bigg\}\Bigg]\\
&+\int_0^t e^{-\varphi(t-s)}\left(\int_0^s \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}g_2(a)da\right)ds.
\end{aligned}$$

Let $0 \le \omega \le 1$. Then, for each $t \in \mathcal{J}$, we have

$$[\omega h_1 + (1-\omega)h_2](t)$$

$$
\begin{aligned}
&= \frac{e^{-\varphi(t)}}{2}\Bigg[\frac{1}{\Delta_2}\Bigg\{\mathcal{A}-\Bigg(\sum_{i=1}^{m}x_i\Bigg\{\int_0^{\xi_i}e^{-\varphi(\xi_i-s)}\Bigg(\int_0^{s}\frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}[\omega g_{11}(a)+(1-\omega)g_{12}(a)]da\Bigg)ds\Bigg\}\\
&-\sum_{i=1}^{m}x_i\Bigg\{\int_0^{\xi_i}e^{-\varphi(\xi_i-s)}\Bigg(\int_0^{s}\frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}[\omega g_{21}(a)+(1-\omega)g_{22}(a)]da\Bigg)ds\Bigg\}\\
&+\Bigg\{\mu\int_0^{\eta}\frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\Bigg(\int_0^{s}e^{-\varphi(s-a)}\Bigg(\int_0^{a}\frac{(a-m)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}[\omega g_{11}(m)+(1-\omega)g_{12}(m)]dm\Bigg)da\Bigg)ds\\
&-\mu\int_0^{\eta}\frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\Bigg(\int_0^{s}e^{-\varphi(s-a)}\Bigg(\int_0^{a}\frac{(a-m)^{(\varpi-2)}}{\Gamma(\varpi-1)}[\omega g_{21}(m)+(1-\omega)g_{22}(m)]dm\Bigg)da\Bigg)ds\Bigg\}\Bigg)\Bigg\}\\
&-\frac{1}{\Delta_1}\Bigg\{\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-s)}\Bigg(\int_0^{s}\frac{(s-a)^{\vartheta-2}}{\Gamma(\vartheta-1)}[\omega g_{11}(a)+(1-\omega)g_{12}(a)]da\Bigg)ds\\
&+\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-s)}\Bigg(\int_0^{s}\frac{(s-a)^{\varpi-2}}{\Gamma(\varpi-1)}[\omega g_{21}(a)+(1-\omega)g_{22}(a)]da\Bigg)ds\Bigg\}\Bigg]\\
&+\int_0^{t}e^{-\varphi(t-s)}\Bigg(\int_0^{s}\frac{(s-a)^{\vartheta-2}}{\Gamma(\vartheta-1)}[\omega g_{11}(a)+(1-\omega)g_{12}(a)]da\Bigg)ds,
\end{aligned}
$$

$$
\begin{aligned}
&[\omega\bar{h}_1+(1-\omega)\bar{h}_2](t)\\
&= \frac{e^{-\varphi(t)}}{2}\Bigg[-\frac{1}{\Delta_2}\Bigg\{\mathcal{A}-\Bigg(\sum_{i=1}^{m}x_i\Bigg\{\int_0^{\xi_i}e^{-\varphi(\xi_i-s)}\Bigg(\int_0^{s}\frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}[\omega g_{11}(a)+(1-\omega)g_{12}(a)]da\Bigg)ds\Bigg\}\\
&-\sum_{i=1}^{m}x_i\Bigg\{\int_0^{\xi_i}e^{-\varphi(\xi_i-s)}\Bigg(\int_0^{s}\frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}[\omega g_{21}(a)+(1-\omega)g_{22}(a)]da\Bigg)ds\Bigg\}\\
&+\Bigg\{\mu\int_0^{\eta}\frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\Bigg(\int_0^{s}e^{-\varphi(s-a)}\Bigg(\int_0^{a}\frac{(a-m)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}[\omega g_{11}(m)+(1-\omega)g_{12}(m)]dm\Bigg)da\Bigg)ds\\
&-\mu\int_0^{\eta}\frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\Bigg(\int_0^{s}e^{-\varphi(s-a)}\Bigg(\int_0^{a}\frac{(a-m)^{(\varpi-2)}}{\Gamma(\varpi-1)}[\omega g_{21}(m)+(1-\omega)g_{22}(m)]dm\Bigg)da\Bigg)ds\Bigg\}\Bigg)\Bigg\}\\
&-\frac{1}{\Delta_1}\Bigg\{\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-s)}\Bigg(\int_0^{s}\frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}[\omega g_{11}(a)+(1-\omega)g_{12}(a)]da\Bigg)ds\\
&+\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-s)}\Bigg(\int_0^{s}\frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}[\omega g_{21}(a)+(1-\omega)g_{22}(a)]da\Bigg)ds\Bigg\}\Bigg]\\
&+\int_0^{t}e^{-\varphi(t-s)}\Bigg(\int_0^{s}\frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}[\omega g_{21}(a)+(1-\omega)g_{22}(a)]da\Bigg)ds.
\end{aligned}
$$

We may deduce that $\mathcal{W}_{\mathcal{G}_1(u,v)}$ and $\mathcal{W}_{\mathcal{G}_2(u,v)}$ have convex values since $\mathcal{G}_1, \mathcal{G}_2$ have convex values. Obviously, $[\omega h_1 + (1-\omega)h_2] \in K_1, [\omega\bar{h}_1 + (1-\omega)\bar{h}_2] \in K_2$, and hence $[\omega(h_1, h_2) + (1-\omega)(\bar{h}_1, \bar{h}_2)] \in K$. For a positive number r, let $\mathcal{B}_r = \{(u,v) \in \mathcal{S}\times\mathcal{S} : \|(u,v)\| \le r\}$ be a bounded set in $\mathcal{S}\times\mathcal{S}$. Then there exist $g_1 \in \mathcal{W}_{\mathcal{G}_1(u,v)}$ and $g_2 \in \mathcal{W}_{\mathcal{G}_2(u,v)}$ such that

$$
h_1(u,v)(t) = \frac{e^{-\varphi(t)}}{2}\Bigg[\frac{1}{\Delta_2}\Bigg\{\mathcal{A}-\Bigg(\sum_{i=1}^{m}x_i\Bigg\{\int_0^{\xi_i}e^{-\varphi(\xi_i-s)}\Bigg(\int_0^{s}\frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}g_1(a)da\Bigg)ds\Bigg\}
$$

$$-\sum_{i=1}^{m} x_i \left\{\int_0^{\xi_i} e^{-\varphi(\xi_i-s)}\left(\int_0^s \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)} g_2(a)da\right)ds\right\}$$
$$+\left\{\mu\int_0^{\eta} \frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\left(\int_0^s e^{-\varphi(s-a)}\left(\int_0^a \frac{(a-m)^{(\vartheta-2)}}{\Gamma(\vartheta-1)} g_1(m)dm\right)da\right)ds\right.$$
$$\left.\left.\left.-\mu\int_0^{\eta} \frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\left(\int_0^s e^{-\varphi(s-a)}\left(\int_0^a \frac{(a-m)^{(\varpi-2)}}{\Gamma(\varpi-1)} g_2(m)dm\right)da\right)ds\right\}\right)\right\}$$
$$-\frac{1}{\Delta_1}\left\{\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\left(\int_0^s \frac{(s-a)^{\vartheta-2}}{\Gamma(\vartheta-1)} g_1(a)da\right)ds\right.$$
$$\left.\left.+\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\left(\int_0^s \frac{(s-a)^{\varpi-2}}{\Gamma(\varpi-1)} g_2(a)da\right)ds\right\}\right]$$
$$+\int_0^{t} e^{-\varphi(t-s)}\left(\int_0^s \frac{(s-a)^{\vartheta-2}}{\Gamma(\vartheta-1)} g_1(a)da\right)ds,$$

and

$$h_2(u,v)(t) = \frac{e^{-\varphi(t)}}{2}\left[-\frac{1}{\Delta_2}\left\{\mathcal{A}-\left(\sum_{i=1}^{m} x_i \left\{\int_0^{\xi_i} e^{-\varphi(\xi_i-s)}\left(\int_0^s \frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)} g_1(a)da\right)ds\right\}\right.\right.\right.$$
$$-\sum_{i=1}^{m} x_i \left\{\int_0^{\xi_i} e^{-\varphi(\xi_i-s)}\left(\int_0^s \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)} g_2(a)da\right)ds\right\}$$
$$+\left\{\mu\int_0^{\eta} \frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\left(\int_0^s e^{-\varphi(s-a)}\left(\int_0^a \frac{(a-m)^{(\vartheta-2)}}{\Gamma(\vartheta-1)} g_1(m)dm\right)da\right)ds\right.$$
$$\left.\left.\left.-\mu\int_0^{\eta} \frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\left(\int_0^s e^{-\varphi(s-a)}\left(\int_0^a \frac{(a-m)^{(\varpi-2)}}{\Gamma(\varpi-1)} g_2(m)dm\right)da\right)ds\right\}\right)\right\}$$
$$-\frac{1}{\Delta_1}\left\{\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\left(\int_0^s \frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)} g_1(a)da\right)ds\right.$$
$$\left.\left.+\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\left(\int_0^s \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)} g_2(a)da\right)ds\right\}\right]$$
$$+\int_0^{t} e^{-\varphi(t-s)}\left(\int_0^s \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)} g_2(a)da\right)ds.$$

Then we have

$$|h_1(u,v)(t)| \le \Upsilon_1\left(\|p_1\|(\psi_1(r)+\phi_1(r)) + \Upsilon_2\|p_2\|(\psi_2(r)+\phi_2(r))\right),$$

and

$|h_2(u,v)(t)| \le \Upsilon_1\left(\|p_1\|(\psi_1(r)+\phi_1(r)) + \Upsilon_2\|p_2\|(\psi_2(r)+\phi_2(r))\right).$

Hence we obtain

$$\|(h_1,h_2)\| = \|h_1(u,v)\| + \|h_2(u,v)\|$$

$$\le (2\Upsilon_1)\|p_1\|(\psi_1(\mathcal{N})+\phi_1(\mathcal{N})) + (2\Upsilon_2)\|p_2\|(\psi_2(\mathcal{N})+\phi_2(\mathcal{N})).$$

Next, we show that K is equicontinuous. Let $t_1, t_2 \in \mathcal{J}$ with $t_1 < t_2$. Then there exist $g_1 \in \mathcal{W}_{\mathcal{G}_1(u,v)}$ and $g_2 \in \mathcal{W}_{\mathcal{G}_2(u,v)}$ such that

$$
\begin{aligned}
&|h_1(u,v)(t_2) - h_1(u,v)(t_1)| \\
&\leq \left| \frac{e^{-\varphi t_2} - e^{-\varphi t_1}}{2} \left[\frac{1}{\Delta_2} \left\{ \mathcal{A} - \left(\sum_{i=1}^{m} x_i \left\{ \int_0^{\xi_i} e^{-\varphi(\xi_i - s)} \left(\int_0^s \frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)} \|p_1\|(\psi_1(r) + \phi_1(r)) da \right) ds \right\} \right.\right.\right.\right. \\
&- \sum_{i=1}^{m} x_i \left\{ \int_0^{\xi_i} e^{-\varphi(\xi_i - s)} \left(\int_0^s \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)} \|p_2\|(\psi_2(r) + \phi_2(r)) da \right) ds \right\} \\
&+ \left\{ \mu \int_0^{\eta} \frac{(\eta - s)^{(\delta-1)}}{\Gamma(\delta)} \left(\int_0^s e^{-\varphi(s-a)} \left(\int_0^a \frac{(a-m)^{(\vartheta-2)}}{\Gamma(\vartheta-1)} \|p_1\|(\psi_1(r) + \phi_1(r)) dm \right) da \right) ds \right. \\
&\left.\left.\left. - \mu \int_0^{\eta} \frac{(\eta - s)^{(\delta-1)}}{\Gamma(\delta)} \left(\int_0^s e^{-\varphi(s-a)} \left(\int_0^a \frac{(a-m)^{(\varpi-2)}}{\Gamma(\varpi-1)} \|p_2\|(\psi_2(r) + \phi_2(r)) dm \right) da \right) ds \right\} \right) \right\} \\
&- \frac{1}{\Delta_1} \left\{ \int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)} \left(\int_0^s \frac{(s-a)^{\vartheta-2}}{\Gamma(\vartheta-1)} \|p_1\|(\psi_1(r) + \phi_1(r)) da \right) ds \right. \\
&\left.\left. + \int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)} \left(\int_0^s \frac{(s-a)^{\varpi-2}}{\Gamma(\varpi-1)} \|p_2\|(\psi_2(r) + \phi_2(r)) da \right) ds \right\} \right] \\
&+ \int_0^{\sqcup_1} \left(e^{-\varphi(t_2 - s)} - e^{-\varphi(t_1 - s)} \right) \left(\int_0^s \frac{(s-a)^{\vartheta-2}}{\Gamma(\vartheta-1)} \|p_1\|(\psi_1(r) + \phi_1(r)) da \right) ds \\
&\left. - \int_{t_1}^{t_2} e^{-\varphi(t_2 - s)} \left(\int_0^s \frac{(s-a)^{\vartheta-2}}{\Gamma(\vartheta-1)} \|p_1\|(\psi_1(r) + \phi_1(r)) da \right) ds \right|.
\end{aligned}
$$

Similarly, we can obtain

$$
\begin{aligned}
&|h_2(u,v)(t_2) - h_2(u,v)(t_1)| \\
&\leq \left| \frac{e^{-\varphi t_2} - e^{-\varphi t_1}}{2} \left[-\frac{1}{\Delta_2} \left\{ \mathcal{A} - \left(\sum_{i=1}^{m} x_i \left\{ \int_0^{\xi_i} e^{-\varphi(\xi_i - s)} \left(\int_0^s \frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)} \|p_1\|(\psi_1(r) + \phi_1(r)) da \right) ds \right\} \right.\right.\right.\right. \\
&- \sum_{i=1}^{m} x_i \left\{ \int_0^{\xi_i} e^{-\varphi(\xi_i - s)} \left(\int_0^s \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)} \|p_2\|(\psi_2(r) + \phi_2(r)) da \right) ds \right\} \\
&+ \left\{ \mu \int_0^{\eta} \frac{(\eta - s)^{(\delta-1)}}{\Gamma(\delta)} \left(\int_0^s e^{-\varphi(s-a)} \left(\int_0^a \frac{(a-m)^{(\vartheta-2)}}{\Gamma(\vartheta-1)} \|p_1\|(\psi_1(r) + \phi_1(r)) dm \right) da \right) ds \right. \\
&\left.\left.\left. - \mu \int_0^{\eta} \frac{(\eta - s)^{(\delta-1)}}{\Gamma(\delta)} \left(\int_0^s e^{-\varphi(s-a)} \left(\int_0^a \frac{(a-m)^{(\varpi-2)}}{\Gamma(\varpi-1)} \|p_2\|(\psi_2(r) + \phi_2(r)) dm \right) da \right) ds \right\} \right) \right\} \\
&- \frac{1}{\Delta_1} \left\{ \int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)} \left(\int_0^s \frac{(s-a)^{\vartheta-2}}{\Gamma(\vartheta-1)} \|p_1\|(\psi_1(r) + \phi_1(r)) da \right) ds \right. \\
&\left.\left. + \int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)} \left(\int_0^s \frac{(s-a)^{\varpi-2}}{\Gamma(\varpi-1)} \|p_2\|(\psi_2(r) + \phi_2(r)) da \right) ds \right\} \right] \\
&+ \int_0^{\sqcup_1} \left(e^{-\varphi(t_2 - s)} - e^{-\varphi(t_1 - s)} \right) \left(\int_0^s \frac{(s-a)^{\varpi-2}}{\Gamma(\varpi-1)} \|p_2\|(\psi_2(r) + \phi_2(r)) da \right) ds \\
&\left. - \int_{t_1}^{t_2} e^{-\varphi(t_2 - s)} \left(\int_0^s \frac{(s-a)^{\varpi-2}}{\Gamma(\varpi-1)} \|p_2\|(\psi_2(r) + \phi_2(r)) da \right) ds \right|.
\end{aligned}
$$

As a result, the operator $K(u,v)$ is equicontinuous, hence the operator $K(u,v)$ is c.c according to the Ascoli-Arzela theorem. We know that [[34], Proposition 1.2] a C.C operator is U.S.C if it has a closed graph. As a consequence, we need to prove that K has a closed graph. Let $(u_n,v_n)\to(u_*,v_*)$, $(h_n,\bar{h}_n)\in K(u_n,v_n)$ and $(h_n,\bar{h}_n)\to(h_*,\bar{h}_*)$, then we need to show $(h_*,\bar{h}_*)\in K(u_*,v_*)$. Observe that $(h_n,\bar{h}_n)\in K(u_n,v_n)$ implies that exist $g_{1n}\in\mathcal{W}_{\mathcal{G}_1(u_n,v_n)}$ and $g_{2n}\in\mathcal{W}_{\mathcal{G}_2(u_n,v_n)}$ such that

$$\begin{aligned}
&h_n(u_n,v_n)(t)\\
&=\frac{e^{-\varphi(t)}}{2}\Bigg[\frac{1}{\Delta_2}\Bigg\{\mathcal{A}-\Bigg(\sum_{i=1}^{m}x_i\Bigg\{\int_0^{\xi_i}e^{-\varphi(\xi_i-s)}\Bigg(\int_0^{s}\frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}(g_{1n})(a)da\Bigg)ds\Bigg\}\\
&-\sum_{i=1}^{m}x_i\Bigg\{\int_0^{\xi_i}e^{-\varphi(\xi_i-s)}\Bigg(\int_0^{s}\frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}(g_{2n})(a)da\Bigg)ds\Bigg\}\\
&+\Bigg\{\mu\int_0^{\eta}\frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\Bigg(\int_0^{s}e^{-\varphi(s-a)}\Bigg(\int_0^{a}\frac{(a-m)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}(g_{1n})(m)dm\Bigg)da\Bigg)ds\\
&-\mu\int_0^{\eta}\frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\Bigg(\int_0^{s}e^{-\varphi(s-a)}\Bigg(\int_0^{a}\frac{(a-m)^{(\varpi-2)}}{\Gamma(\varpi-1)}(g_{2n})(m)dm\Bigg)da\Bigg)ds\Bigg\}\Bigg)\Bigg\}\\
&-\frac{1}{\Delta_1}\Bigg\{\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-s)}\Bigg(\int_0^{s}\frac{(s-a)^{\vartheta-2}}{\Gamma(\vartheta-1)}(g_{1n})(a)da\Bigg)ds\\
&+\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-s)}\Bigg(\int_0^{s}\frac{(s-a)^{\varpi-2}}{\Gamma(\varpi-1)}(g_{2n})(a)da\Bigg)ds\Bigg\}\Bigg]\\
&+\int_0^{t}e^{-\varphi(t-s)}\Bigg(\int_0^{s}\frac{(s-a)^{\vartheta-2}}{\Gamma(\vartheta-1)}(g_{1n})(a)da\Bigg)ds,
\end{aligned}$$

$$\begin{aligned}
&\bar{h}_n(u_n,v_n)(t)\\
&=\frac{e^{-\varphi(t)}}{2}\Bigg[-\frac{1}{\Delta_2}\Bigg\{\mathcal{A}-\Bigg(\sum_{i=1}^{m}x_i\Bigg\{\int_0^{\xi_i}e^{-\varphi(\xi_i-s)}\Bigg(\int_0^{s}\frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}(g_{1n})(a)da\Bigg)ds\Bigg\}\\
&-\sum_{i=1}^{m}x_i\Bigg\{\int_0^{\xi_i}e^{-\varphi(\xi_i-s)}\Bigg(\int_0^{s}\frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}(g_{2n})(a)da\Bigg)ds\Bigg\}\\
&+\Bigg\{\mu\int_0^{\eta}\frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\Bigg(\int_0^{s}e^{-\varphi(s-a)}\Bigg(\int_0^{a}\frac{(a-m)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}(g_{1n})(m)dm\Bigg)da\Bigg)ds\\
&-\mu\int_0^{\eta}\frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\Bigg(\int_0^{s}e^{-\varphi(s-a)}\Bigg(\int_0^{a}\frac{(a-m)^{(\varpi-2)}}{\Gamma(\varpi-1)}(g_{2n})(m)dm\Bigg)da\Bigg)ds\Bigg\}\Bigg)\Bigg\}\\
&-\frac{1}{\Delta_1}\Bigg\{\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-s)}\Bigg(\int_0^{s}\frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}(g_{1n})(a)da\Bigg)ds\\
&+\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-s)}\Bigg(\int_0^{s}\frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}(g_{2n})(a)da\Bigg)ds\Bigg\}\Bigg]\\
&+\int_0^{t}e^{-\varphi(t-s)}\Bigg(\int_0^{s}\frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}(g_{2n})(a)da\Bigg)ds.
\end{aligned}$$

Consider the $\Phi_1, \Phi_2 : L^1([0,\mathcal{T}],\mathcal{E}\times\mathcal{E}) \to C([0,\mathcal{T}],\mathcal{E}\times\mathcal{E})$ continuous linear operator defined by

$$\begin{aligned}\Phi_1(u,v)(t) = \frac{e^{-\varphi(t)}}{2}\Bigg[\frac{1}{\Delta_2}\Bigg\{\mathcal{A} - \Bigg(\sum_{i=1}^{m} x_i\Bigg\{\int_0^{\xi_i} e^{-\varphi(\xi_i - s)}\Bigg(\int_0^s \frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)} g_1(a)da\Bigg)ds\Bigg\} \\ -\sum_{i=1}^{m} x_i\Bigg\{\int_0^{\xi_i} e^{-\varphi(\xi_i - s)}\Bigg(\int_0^s \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)} g_2(a)da\Bigg)ds\Bigg\} \\ +\Bigg\{\mu\int_0^{\eta}\frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\Bigg(\int_0^s e^{-\varphi(s-a)}\Bigg(\int_0^a \frac{(a-m)^{(\vartheta-2)}}{\Gamma(\vartheta-1)} g_1(m)dm\Bigg)da\Bigg)ds \\ -\mu\int_0^{\eta}\frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\Bigg(\int_0^s e^{-\varphi(s-a)}\Bigg(\int_0^a \frac{(a-m)^{(\varpi-2)}}{\Gamma(\varpi-1)} g_2(m)dm\Bigg)da\Bigg)ds\Bigg\}\Bigg)\Bigg\} \\ -\frac{1}{\Delta_1}\Bigg\{\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\Bigg(\int_0^s \frac{(s-a)^{\vartheta-2}}{\Gamma(\vartheta-1)} g_1(a)da\Bigg)ds \\ +\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\Bigg(\int_0^s \frac{(s-a)^{\varpi-2}}{\Gamma(\varpi-1)} g_2(a)da\Bigg)ds\Bigg\}\Bigg] \\ +\int_0^t e^{-\varphi(t-s)}\Bigg(\int_0^s \frac{(s-a)^{\vartheta-2}}{\Gamma(\vartheta-1)} g_1(a)da\Bigg)ds,\end{aligned}$$

and

$$\begin{aligned}\Phi_2(u,v)(t) = \frac{e^{-\varphi(t)}}{2}\Bigg[-\frac{1}{\Delta_2}\Bigg\{\mathcal{A} - \Bigg(\sum_{i=1}^{m} x_i\Bigg\{\int_0^{\xi_i} e^{-\varphi(\xi_i - s)}\Bigg(\int_0^s \frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)} g_1(a)da\Bigg)ds\Bigg\} \\ -\sum_{i=1}^{m} x_i\Bigg\{\int_0^{\xi_i} e^{-\varphi(\xi_i - s)}\Bigg(\int_0^s \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)} g_2(a)da\Bigg)ds\Bigg\} \\ +\Bigg\{\mu\int_0^{\eta}\frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\Bigg(\int_0^s e^{-\varphi(s-a)}\Bigg(\int_0^a \frac{(a-m)^{(\vartheta-2)}}{\Gamma(\vartheta-1)} g_1(m)dm\Bigg)da\Bigg)ds \\ -\mu\int_0^{\eta}\frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\Bigg(\int_0^s e^{-\varphi(s-a)}\Bigg(\int_0^a \frac{(a-m)^{(\varpi-2)}}{\Gamma(\varpi-1)} g_2(m)dm\Bigg)da\Bigg)ds\Bigg\}\Bigg)\Bigg\} \\ -\frac{1}{\Delta_1}\Bigg\{\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\Bigg(\int_0^s \frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)} g_1(a)da\Bigg)ds \\ +\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\Bigg(\int_0^s \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)} g_2(a)da\Bigg)ds\Bigg\}\Bigg] \\ +\int_0^t e^{-\varphi(t-s)}\Bigg(\int_0^s \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)} g_2(a)da\Bigg)ds.\end{aligned}$$

We can deduce from [38] that $(\Phi_1, \Phi_2) \circ (\mathcal{W}_{\mathcal{G}_1}, \mathcal{W}_{\mathcal{G}_2})$ is closed graph operator. Further, we have $(h_n, \bar{h}_n) \in (\Phi_1, \Phi_2) \circ (\mathcal{W}_{\mathcal{G}_1(u_n,v_n)}, \mathcal{W}_{\mathcal{G}_2(u_n,v_n)})$ for all n. Since $(u_n, v_n) \to (u_*, v_*)$, $(h_n, \bar{h}_n) \to (h_*, \bar{h}_*)$ it follows that $g_{1n} \in \mathcal{W}_{\mathcal{G}_1(u,v)}$ and $g_{2n} \in \mathcal{W}_{\mathcal{G}_2(u,v)}$ such that

$$\begin{aligned}& h_*(u_*, v_*)(t) \\ &= \frac{e^{-\varphi(t)}}{2}\Bigg[\frac{1}{\Delta_2}\Bigg\{\mathcal{A} - \Bigg(\sum_{i=1}^{m} x_i\Bigg\{\int_0^{\xi_i} e^{-\varphi(\xi_i - s)}\Bigg(\int_0^s \frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)} (g_{1*})(a)da\Bigg)ds\Bigg\}\end{aligned}$$

$$-\sum_{i=1}^{m} x_i \left\{\int_0^{\xi_i} e^{-\varphi(\xi_i-s)}\left(\int_0^s \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}(g_{2*})(a)da\right)ds\right\}$$
$$+\left\{\mu\int_0^{\eta}\frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\left(\int_0^s e^{-\varphi(s-a)}\left(\int_0^a \frac{(a-m)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}(g_{1*})(m)dm\right)da\right)ds\right.$$
$$\left.-\mu\int_0^{\eta}\frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\left(\int_0^s e^{-\varphi(s-a)}\left(\int_0^a \frac{(a-m)^{(\varpi-2)}}{\Gamma(\varpi-1)}(g_{2*})(m)dm\right)da\right)ds\right\}\Bigg)\Bigg\}$$
$$-\frac{1}{\Delta_1}\left\{\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\left(\int_0^s \frac{(s-a)^{\vartheta-2}}{\Gamma(\vartheta-1)}(g_{1*})(a)da\right)ds\right.$$
$$\left.+\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\left(\int_0^s \frac{(s-a)^{\varpi-2}}{\Gamma(\varpi-1)}(g_{2*})(a)da\right)ds\right\}\Bigg]$$
$$+\int_0^t e^{-\varphi(t-s)}\left(\int_0^s \frac{(s-a)^{\vartheta-2}}{\Gamma(\vartheta-1)}(g_{1*})(a)da\right)ds,$$

$$\bar{h}_*(u_*, v_*)(t)$$
$$=\frac{e^{-\varphi(t)}}{2}\Bigg[-\frac{1}{\Delta_2}\Bigg\{\mathcal{A}-\Bigg(\sum_{i=1}^{m} x_i\left\{\int_0^{\xi_i} e^{-\varphi(\xi_i-s)}\left(\int_0^s \frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}(g_{1*})(a)da\right)ds\right\}$$
$$-\sum_{i=1}^{m} x_i \left\{\int_0^{\xi_i} e^{-\varphi(\xi_i-s)}\left(\int_0^s \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}(g_{2*})(a)da\right)ds\right\}$$
$$+\left\{\mu\int_0^{\eta}\frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\left(\int_0^s e^{-\varphi(s-a)}\left(\int_0^a \frac{(a-m)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}(g_{1*})(m)dm\right)da\right)ds\right.$$
$$\left.-\mu\int_0^{\eta}\frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\left(\int_0^s e^{-\varphi(s-a)}\left(\int_0^a \frac{(a-m)^{(\varpi-2)}}{\Gamma(\varpi-1)}(g_{2*})(m)dm\right)da\right)ds\right\}\Bigg)\Bigg\}$$
$$-\frac{1}{\Delta_1}\left\{\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\left(\int_0^s \frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}(g_{1*})(a)da\right)ds\right.$$
$$\left.+\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\left(\int_0^s \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}(g_{2*})(a)da\right)ds\right\}\Bigg]$$
$$+\int_0^t e^{-\varphi(t-s)}\left(\int_0^s \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}(g_{2*})(a)da\right)ds.$$

i.e., $(h_n, \bar{h}_n) \in K(u_*, v_*)$. Let $(u, v) \in \nu K(u, v)$. Then there exist $g_1 \in \mathcal{W}_{\mathcal{G}_1(u,v)}$ and $g_2 \in \mathcal{W}_{\mathcal{G}_2(u,v)}$ such that

$$\|u\| \leq \Upsilon_1 \left(\|p_1\|(\psi_1(r)+\phi_1(r)) + \Upsilon_2\|p_2\|(\psi_2(r)+\phi_2(r))\right),$$

$$\|v\| \leq \Upsilon_1 \left(\|p_1\|(\psi_1(r)+\phi_1(r)) + \Upsilon_2\|p_2\|(\psi_2(r)+\phi_2(r))\right),$$

and, for each $t \in \mathcal{J}$, we obtain

$$\|(u,v)\| = \|u\| + \|v\| \leq 2\Upsilon_1 \left(\|p_1\|(\psi_1(r)+\phi_1(r)) + 2\Upsilon_2\|p_2\|(\psi_2(r)+\phi_2(r))\right),$$

which implies that

$$\frac{\|(u,v)\|}{(2\Upsilon_1)\|p_1\|(\psi_1(\mathcal{N})+\phi_1(\mathcal{N}))+(2\Upsilon_2)\|p_2\|(\psi_2(\mathcal{N})+\phi_2(\mathcal{N}))}\leq 1.$$

According to B_3 there exists $\mathcal{N}$ such that $\|(u,v)\|\neq\mathcal{N}$. Let us set

$$\mathcal{E}=\{(u,v)\in\mathcal{S}\times\mathcal{S}:\|(u,v)\|<\mathcal{N}\}.$$

It should be noted that operator $K:\bar{\mathcal{E}}\to\mathcal{U}_{cp,cv}(\mathcal{S})\times\mathcal{U}_{cp,cv}(\mathcal{S})$ is C.C and U.S.C. There is no $(u,v)\in vK(u,v)$ for some $v\in(0,1)$ by choice of $\mathcal{U}$. As a consequence, we can deduce from the nonlinear alternative of Leray-Schauder [36] that K has a fixed point $(u,v)\in\bar{\mathcal{E}}$, which is a solution of system (1.3). □

Let $(\mathcal{S},d)$ be a metric space induced from the normed space $(\mathcal{S};\|.\|)$, and let $\mathcal{H}_d:\mathcal{U}(\mathcal{S})\times\mathcal{U}(\mathcal{S})\to\mathbb{R}\infty$ be defined by $\mathcal{H}_d(\mathcal{E},\mathcal{V})=\max\{\sup_{e\in\mathcal{E}}d(e,\mathcal{V}),\sup_{v\in\mathcal{V}}d(\mathcal{E},v)\}$, where $d(\mathcal{E},v)=\inf_{e\in\mathcal{E}}d(e,v)$ and $d(v,\mathcal{V})=\inf_{v\in\mathcal{V}}d(e,v)$. Then $(\mathcal{U}_{b,cl}(\mathcal{S}),\mathcal{H}_d)$ is a metric space and $(\mathcal{U}_{cl}(\mathcal{S}),\mathcal{H}_d)$ is a generalized metric space (see [39]).

Covitz and Nadler's theorem for multi-valued maps are used in the following result.

Theorem 3.5. *Suppose that holds* (B_4) *and* (B_5) *holds. Then system (1.3) has at least one solution on* $\mathcal{J}$ *provided that*

$$(2\Upsilon_1)\|m_1\|+(2\Upsilon_2)\|m_2\|<1. \tag{3.14}$$

Proof. Assuming (B_4) that the sets $\mathcal{W}_{\mathcal{G}_1(u,v)}$ and $\mathcal{W}_{\mathcal{G}_2(u,v)}$ are non-empty for each $(u,v)\in\mathcal{S}\times\mathcal{S}$, $\mathcal{G}_1$ and $\mathcal{G}_2$ have measurable selections (see Theorem III.6 in [40]). Next, we show that the operator K satisfies the assumptions of Covitz and Nadler's theorem [35]. Further we show that $K(u,v)\in\mathcal{U}_{cl}(\mathcal{S})\times\mathcal{U}_{cl}(\mathcal{S})$ for each $(u,v)\in\mathcal{S}\times\mathcal{S}$. Let $(h_n,\bar{h}_n)\in K(u_n,v_n)$ such that $(h_n,\bar{h}_n)\to(h,\bar{h})$ in $\mathcal{S}\times\mathcal{S}$. Then $(h,\bar{h})\in\mathcal{S}\times\mathcal{S}$ and there exists $g_{1n}\in\mathcal{W}_{\mathcal{G}_1(u_n,v_n)}$ and $g_{2n}\in\mathcal{W}_{\mathcal{G}_2(u_n,v_n)}$ such that

$$\begin{aligned}
&h_n(u_n,v_n)(t)\\
&=\frac{e^{-\varphi(t)}}{2}\left[\frac{1}{\Delta_2}\left\{\mathcal{A}-\left(\sum_{i=1}^{m}x_i\left\{\int_0^{\xi_i}e^{-\varphi(\xi_i-s)}\left(\int_0^s\frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}(g_{1n})(a)da\right)ds\right\}\right.\right.\right.\\
&-\sum_{i=1}^{m}x_i\left\{\int_0^{\xi_i}e^{-\varphi(\xi_i-s)}\left(\int_0^s\frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}(g_{2n})(a)da\right)ds\right\}\\
&+\left\{\mu\int_0^{\eta}\frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\left(\int_0^s e^{-\varphi(s-a)}\left(\int_0^a\frac{(a-m)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}(g_{1n})(m)dm\right)da\right)ds\right.\\
&\left.\left.\left.-\mu\int_0^{\eta}\frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\left(\int_0^s e^{-\varphi(s-a)}\left(\int_0^a\frac{(a-m)^{(\varpi-2)}}{\Gamma(\varpi-1)}(g_{2n})(m)dm\right)da\right)ds\right\}\right)\right\}\\
&-\frac{1}{\Delta_1}\left\{\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-s)}\left(\int_0^s\frac{(s-a)^{\vartheta-2}}{\Gamma(\vartheta-1)}(g_{1n})(a)da\right)ds\right.\\
&\left.\left.+\int_0^{\mathcal{T}}e^{-\varphi(\mathcal{T}-s)}\left(\int_0^s\frac{(s-a)^{\varpi-2}}{\Gamma(\varpi-1)}(g_{2n})(a)da\right)ds\right\}\right]
\end{aligned}$$

$$+\int_0^t e^{-\varphi(t-s)}\left(\int_0^s \frac{(s-a)^{\vartheta-2}}{\Gamma(\vartheta-1)}(g_{1n})(a)da\right)ds,$$

and

$$\begin{aligned}
&\bar{h}_n(u_n,v_n)(t)\\
&=\frac{e^{-\varphi(t)}}{2}\left[-\frac{1}{\Delta_2}\left\{\mathcal{A}-\left(\sum_{i=1}^m x_i\left\{\int_0^{\xi_i} e^{-\varphi(\xi_i-s)}\left(\int_0^s \frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}(g_{1n})(a)da\right)ds\right\}\right.\right.\right.\\
&-\sum_{i=1}^m x_i\left\{\int_0^{\xi_i} e^{-\varphi(\xi_i-s)}\left(\int_0^s \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}(g_{2n})(a)da\right)ds\right\}\\
&+\left\{\mu\int_0^\eta \frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\left(\int_0^s e^{-\varphi(s-a)}\left(\int_0^a \frac{(a-m)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}(g_{1n})(m)dm\right)da\right)ds\right.\\
&\left.\left.\left.-\mu\int_0^\eta \frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\left(\int_0^s e^{-\varphi(s-a)}\left(\int_0^a \frac{(a-m)^{(\varpi-2)}}{\Gamma(\varpi-1)}(g_{2n})(m)dm\right)da\right)ds\right\}\right)\right\}\\
&-\frac{1}{\Delta_1}\left\{\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\left(\int_0^s \frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}(g_{1n})(a)da\right)ds\right.\\
&\left.\left.+\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\left(\int_0^s \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}(g_{2n})(a)da\right)ds\right\}\right]\\
&+\int_0^t e^{-\varphi(t-s)}\left(\int_0^s \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}(g_{2n})(a)da\right)ds.
\end{aligned}$$

Due to the fact that $\mathcal{G}_1$ and $\mathcal{G}_2$ have compact values, we pass onto subsequences to ensure that g_{1n} and g_{2n} converge to g_1 and g_2 in $L^1(\mathcal{J},\mathbb{R})$. Thus $g_1 \in \mathcal{W}_{\mathcal{G}_1(u,v)}$ and $g_2 \in \mathcal{W}_{G_2(u,v)}$ for each $t \in \mathcal{J}$ and that

$$\begin{aligned}
&h_n(u_n,v_n)(t)\to h(u,v)(t)\\
&=\frac{e^{-\varphi(t)}}{2}\left[\frac{1}{\Delta_2}\left\{\mathcal{A}-\left(\sum_{i=1}^m x_i\left\{\int_0^{\xi_i} e^{-\varphi(\xi_i-s)}\left(\int_0^s \frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}(g_1)(a)da\right)ds\right\}\right.\right.\right.\\
&-\sum_{i=1}^m x_i\left\{\int_0^{\xi_i} e^{-\varphi(\xi_i-s)}\left(\int_0^s \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}(g_2)(a)da\right)ds\right\}\\
&+\left\{\mu\int_0^\eta \frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\left(\int_0^s e^{-\varphi(s-a)}\left(\int_0^a \frac{(a-m)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}(g_1)(m)dm\right)da\right)ds\right.\\
&\left.\left.\left.-\mu\int_0^\eta \frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\left(\int_0^s e^{-\varphi(s-a)}\left(\int_0^a \frac{(a-m)^{(\varpi-2)}}{\Gamma(\varpi-1)}(g_2)(m)dm\right)da\right)ds\right\}\right)\right\}\\
&-\frac{1}{\Delta_1}\left\{\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\left(\int_0^s \frac{(s-a)^{\vartheta-2}}{\Gamma(\vartheta-1)}(g_1)(a)da\right)ds\right.\\
&\left.\left.+\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\left(\int_0^s \frac{(s-a)^{\varpi-2}}{\Gamma(\varpi-1)}(g_2)(a)da\right)ds\right\}\right]\\
&+\int_0^t e^{-\varphi(t-s)}\left(\int_0^s \frac{(s-a)^{\vartheta-2}}{\Gamma(\vartheta-1)}(g_1)(a)da\right)ds,
\end{aligned}$$

$$
\begin{aligned}
&\bar{h}_n(u_n, v_n)(t) \to \bar{h}_n(u, v)(t)\\
&= \frac{e^{-\varphi(t)}}{2}\Bigg[-\frac{1}{\Delta_2}\Bigg\{\mathcal{A} - \Bigg(\sum_{i=1}^{m} x_i \Bigg\{\int_0^{\xi_i} e^{-\varphi(\xi_i - s)}\Bigg(\int_0^{s} \frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}(g_1)(a)da\Bigg)ds\Bigg\}\\
&-\sum_{i=1}^{m} x_i \Bigg\{\int_0^{\xi_i} e^{-\varphi(\xi_i - s)}\Bigg(\int_0^{s} \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}(g_2)(a)da\Bigg)ds\Bigg\}\\
&+\Bigg\{\mu\int_0^{\eta} \frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\Bigg(\int_0^{s} e^{-\varphi(s-a)}\Bigg(\int_0^{a} \frac{(a-m)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}(g_1)(m)dm\Bigg)da\Bigg)ds\\
&-\mu\int_0^{\eta} \frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\Bigg(\int_0^{s} e^{-\varphi(s-a)}\Bigg(\int_0^{a} \frac{(a-m)^{(\varpi-2)}}{\Gamma(\varpi-1)}(g_2)(m)dm\Bigg)da\Bigg)ds\Bigg\}\Bigg)\Bigg\}\\
&-\frac{1}{\Delta_1}\Bigg\{\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\Bigg(\int_0^{s} \frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}(g_1)(a)da\Bigg)ds\\
&+\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\Bigg(\int_0^{s} \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}(g_2)(a)da\Bigg)ds\Bigg\}\Bigg]\\
&+\int_0^{t} e^{-\varphi(t-s)}\Bigg(\int_0^{s} \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}(g_2)(a)da\Bigg)ds.
\end{aligned}
$$

Hence $(h, \bar{h}) \in K$, which implies that K is closed. Next we show that there exists (defined by (3.14)) such that

$$
\mathcal{H}_d\left(K(u, v), K(\bar{u}, \bar{v})\right) \le \bar{\theta}(\|u - \bar{u}\| + \|v - \bar{v}\|) \ \text{ for each } \ u, \bar{u}, v, \bar{v} \in \mathcal{E}.
$$

Let $(\|u, \bar{u}\|), (u, \bar{v}) \in \mathcal{S} \times \mathcal{S}$ and $(h_1, \bar{h}_1) \in K(u, v)$. Then there exist $g_{11} \in \mathcal{W}_{\mathcal{G}_1(u,v)}$ and $g_{21} \in \mathcal{W}_{\mathcal{G}_2(u,v)}$ such that, for each $t \in \mathcal{J}$, we have

$$
\begin{aligned}
&h_1(u_n, v_n)(t)\\
&= \frac{e^{-\varphi(t)}}{2}\Bigg[\frac{1}{\Delta_2}\Bigg\{\mathcal{A} - \Bigg(\sum_{i=1}^{m} x_i \Bigg\{\int_0^{\xi_i} e^{-\varphi(\xi_i - s)}\Bigg(\int_0^{s} \frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}(g_{11})(a)da\Bigg)ds\Bigg\}\\
&-\sum_{i=1}^{m} x_i \Bigg\{\int_0^{\xi_i} e^{-\varphi(\xi_i - s)}\Bigg(\int_0^{s} \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}(g_{21})(a)da\Bigg)ds\Bigg\}\\
&+\Bigg\{\mu\int_0^{\eta} \frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\Bigg(\int_0^{s} e^{-\varphi(s-a)}\Bigg(\int_0^{a} \frac{(a-m)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}(g_{11})(m)dm\Bigg)da\Bigg)ds\\
&-\mu\int_0^{\eta} \frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\Bigg(\int_0^{s} e^{-\varphi(s-a)}\Bigg(\int_0^{a} \frac{(a-m)^{(\varpi-2)}}{\Gamma(\varpi-1)}(g_{21})(m)dm\Bigg)da\Bigg)ds\Bigg\}\Bigg)\Bigg\}\\
&-\frac{1}{\Delta_1}\Bigg\{\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\Bigg(\int_0^{s} \frac{(s-a)^{\vartheta-2}}{\Gamma(\vartheta-1)}(g_{11})(a)da\Bigg)ds\\
&+\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\Bigg(\int_0^{s} \frac{(s-a)^{\varpi-2}}{\Gamma(\varpi-1)}(g_{21})(a)da\Bigg)ds\Bigg\}\Bigg]\\
&+\int_0^{t} e^{-\varphi(t-s)}\Bigg(\int_0^{s} \frac{(s-a)^{\vartheta-2}}{\Gamma(\vartheta-1)}(g_{11})(a)da\Bigg)ds,
\end{aligned}
$$

and

$$
\begin{aligned}
&\bar{h}_1(u_n, v_n)(t) \\
&= \frac{e^{-\varphi(t)}}{2}\Bigg[-\frac{1}{\Delta_2}\Bigg\{\mathcal{A} - \Bigg(\sum_{i=1}^{m} x_i \Bigg\{\int_0^{\xi_i} e^{-\varphi(\xi_i - s)}\Bigg(\int_0^s \frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}(g_{11})(a)da\Bigg)ds\Bigg\} \\
&- \sum_{i=1}^{m} x_i \Bigg\{\int_0^{\xi_i} e^{-\varphi(\xi_i - s)}\Bigg(\int_0^s \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}(g_{21})(a)da\Bigg)ds\Bigg\} \\
&+ \Bigg\{\mu \int_0^{\eta} \frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\Bigg(\int_0^s e^{-\varphi(s-a)}\Bigg(\int_0^a \frac{(a-m)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}(g_{11})(m)dm\Bigg)da\Bigg)ds \\
&- \mu \int_0^{\eta} \frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\Bigg(\int_0^s e^{-\varphi(s-a)}\Bigg(\int_0^a \frac{(a-m)^{(\varpi-2)}}{\Gamma(\varpi-1)}(g_{21})(m)dm\Bigg)da\Bigg)ds\Bigg\}\Bigg)\Bigg\} \\
&- \frac{1}{\Delta_1}\Bigg\{\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\Bigg(\int_0^s \frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}(g_{11})(a)da\Bigg)ds \\
&+ \int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\Bigg(\int_0^s \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}(g_{21})(a)da\Bigg)ds\Bigg\}\Bigg] \\
&+ \int_0^{t} e^{-\varphi(t-s)}\Bigg(\int_0^s \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}(g_{21})(a)da\Bigg)ds.
\end{aligned}
$$

Using (B_5), we have

$$\mathcal{H}_d(\mathcal{G}_1(t,u,v), \mathcal{G}_1(t,\bar{u},\bar{v})) \le m_1(t)(|u(t)-\bar{u}(t)| + |v(t)-\bar{v}(t)|)$$

and

$$\mathcal{H}_d(\mathcal{G}_2(t,u,v), \mathcal{G}_2(t,\bar{u},\bar{v})) \le m_2(t)(|u(t)-\bar{u}(t)| + |v(t)-\bar{v}(t)|).$$

So, there exists $g_1 \in \mathcal{G}_1(t,u,v)$ and $g_2 \in \mathcal{G}_2(t,u,v)$ such that

$$|g_{11}(t) - w| \le m_1(t)(|u(t)-\bar{u}(t)| + |v(t)-\bar{v}(t)|)$$

and

$$|g_{22}(t) - z| \le m_2(t)(|u(t)-\bar{u}(t)| + |v(t)-\bar{v}(t)|).$$

Define $\mathcal{V}_1, \mathcal{V}_2 : \mathcal{J} \to \mathcal{U}(\mathbb{R})$ by

$$\mathcal{V}_1(t) = \{g_1 \in L^1(\mathcal{J},\mathbb{R}) : |g_1(t) - w| \le m_1(t)(|u(t)-\bar{u}(t)| + |v(t)-\bar{v}(t)|)\}$$

and

$$\mathcal{V}_2(t) = \{g_2 \in L^1(\mathcal{J},\mathbb{R}) : |g_2(t) - z| \le m_2(t)(|u(t)-\bar{u}(t)| + |v(t)-\bar{v}(t)|)\}.$$

There are functions $g_{12}(t), g_{22}(t)$ that are an observable selection for $\mathcal{V}_1$, $\mathcal{V}_2$ because the multi-valued operators $\mathcal{V}_1 \cap f(t,u(t),v(t))$ and $\mathcal{V}_2 \cap g(t,u(t),v(t))$ are measurable (Proposition III.4 in [17]). And $g_{12}(t) \in \mathcal{G}_1(t,u(t),v(t))$, $g_{22}(t) \in \mathcal{G}_2(t,u(t),v(t))$ such that, for a.e. $t \in \mathcal{J}$, we have

$$|g_{11}(t) - g_{12}(t)| \le m_1(t)(|u(t)-\bar{u}(t)| + |v(t)-\bar{v}(t)|)$$

and

$$|g_{21}(t) - g_{22}(t)| \leq m_2(t)(|u(t) - \bar{u}(t)| + |v(t) - \bar{v}(t)|).$$

Let

$$\begin{aligned}
&h_2(u_n, v_n)(t)\\
&= \frac{e^{-\varphi(t)}}{2}\left[\frac{1}{\Delta_2}\left\{\mathcal{A} - \left(\sum_{i=1}^{m} x_i\left\{\int_0^{\xi_i} e^{-\varphi(\xi_i - s)}\left(\int_0^s \frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}(g_{12})(a)da\right)ds\right\}\right.\right.\right.\\
&-\sum_{i=1}^{m} x_i\left\{\int_0^{\xi_i} e^{-\varphi(\xi_i - s)}\left(\int_0^s \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}(g_{22})(a)da\right)ds\right\}\\
&+\left\{\mu\int_0^{\eta}\frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\left(\int_0^s e^{-\varphi(s-a)}\left(\int_0^a \frac{(a-m)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}(g_{12})(m)dm\right)da\right)ds\right.\\
&\left.\left.\left.-\mu\int_0^{\eta}\frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\left(\int_0^s e^{-\varphi(s-a)}\left(\int_0^a \frac{(a-m)^{(\varpi-2)}}{\Gamma(\varpi-1)}(g_{22})(m)dm\right)da\right)ds\right\}\right)\right\}\\
&-\frac{1}{\Delta_1}\left\{\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\left(\int_0^s \frac{(s-a)^{\vartheta-2}}{\Gamma(\vartheta-1)}(g_{12})(a)da\right)ds\right.\\
&\left.\left.+\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\left(\int_0^s \frac{(s-a)^{\varpi-2}}{\Gamma(\varpi-1)}(g_{22})(a)da\right)ds\right\}\right]\\
&+\int_0^t e^{-\varphi(t-s)}\left(\int_0^s \frac{(s-a)^{\vartheta-2}}{\Gamma(\vartheta-1)}(g_{12})(a)da\right)ds,
\end{aligned}$$

and

$$\begin{aligned}
&\bar{h}_2(u_n, v_n)(t)\\
&= \frac{e^{-\varphi(t)}}{2}\left[-\frac{1}{\Delta_2}\left\{\mathcal{A} - \left(\sum_{i=1}^{m} x_i\left\{\int_0^{\xi_i} e^{-\varphi(\xi_i - s)}\left(\int_0^s \frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}(g_{12})(a)da\right)ds\right\}\right.\right.\right.\\
&-\sum_{i=1}^{m} x_i\left\{\int_0^{\xi_i} e^{-\varphi(\xi_i - s)}\left(\int_0^s \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}(g_{22})(a)da\right)ds\right\}\\
&+\left\{\mu\int_0^{\eta}\frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\left(\int_0^s e^{-\varphi(s-a)}\left(\int_0^a \frac{(a-m)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}(g_{12})(m)dm\right)da\right)ds\right.\\
&\left.\left.\left.-\mu\int_0^{\eta}\frac{(\eta-s)^{(\delta-1)}}{\Gamma(\delta)}\left(\int_0^s e^{-\varphi(s-a)}\left(\int_0^a \frac{(a-m)^{(\varpi-2)}}{\Gamma(\varpi-1)}(g_{22})(m)dm\right)da\right)ds\right\}\right)\right\}\\
&-\frac{1}{\Delta_1}\left\{\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\left(\int_0^s \frac{(s-a)^{(\vartheta-2)}}{\Gamma(\vartheta-1)}(g_{12})(a)da\right)ds\right.\\
&\left.\left.+\int_0^{\mathcal{T}} e^{-\varphi(\mathcal{T}-s)}\left(\int_0^s \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}(g_{22})(a)da\right)ds\right\}\right]\\
&+\int_0^t e^{-\varphi(t-s)}\left(\int_0^s \frac{(s-a)^{(\varpi-2)}}{\Gamma(\varpi-1)}(g_{22})(a)da\right)ds.
\end{aligned}$$

Hence

$$\|h_1(u,v)-h_2(u,v)\| \le \Upsilon_1(\|m_1\|+\|m_2\|)(\|u(t)-\bar{u}(t)\|+\|v(t)-\bar{v}(t)\|).$$

Similarly, we can define that

$$\|\bar{h}_1(u,v)-\bar{h}_2(u,v)\| \le \Upsilon_2(\|m_1\|+\|m_2\|)(\|u(t)-\bar{u}(t)\|+\|v(t)-\bar{v}(t)\|).$$

Thus

$$\|(h_1,\bar{h}_1),(h_2,\bar{h}_2)\| \le (2\Upsilon_1\|m_1\|+2\Upsilon_2\|m_2\|)(\|u(t)-\bar{u}(t)\|+\|v(t)-\bar{v}(t)\|).$$

Similarly, by swapping the positions of (u,v) and $(\bar{u},\bar{v})$, we can obtain

$$\mathcal{H}_d[\mathcal{F}(h_1,h_2),\mathcal{F}(\bar{h}_1,\bar{h}_2)] \le (2\Upsilon_1\|m_1\|+2\Upsilon_2\|m_2\|)(\|u(t)-\bar{u}(t)\|+\|v(t)-\bar{v}(t)\|).$$

In light of the assumption, K is a contraction (3.14). As a consequence of theorem Covitz and Nadler's [35], K has a fixed point (u,v) that is a solution to the system (1.3). □

4. Numerical examples

In this section, we present several instances following the systems (1.2) and (1.3) and the main theorems.

Example 4.1. *Consider the following system*

$$\begin{aligned}\left({}^C\mathcal{D}^{\frac{8}{3}}+\frac{2}{3}{}^C\mathcal{D}^{\frac{8}{3}-1}\right)u(t)&=\mathcal{G}_1(t,u(t),v(t)),\quad t\in\mathcal{J}=[0,1],\\ \left({}^C\mathcal{D}^{\frac{5}{4}}+\frac{2}{3}{}^C\mathcal{D}^{\frac{5}{4}-1}\right)v(t)&=\mathcal{G}_2(t,u(t),v(t)),\quad t\in\mathcal{J}=[0,1],\\ (u+v)(0)&=-(u+v)(\mathcal{T}),\\ \sum_{i=0}^{m}x_i(u-v)(\xi_i)+\mu\int_0^{\eta}\frac{(\eta-s)^{\delta-1}}{\Gamma(\delta)}(u-v)(s)ds&=\mathcal{A},\end{aligned}\tag{4.1}$$

where $\vartheta=8/3,\varpi=5/4,\eta=1/10,\delta=3/2,\mu=1,\xi_1=1/5,\xi_2=2/5,\xi_3=3/5,\mathcal{A}=3,\mathcal{T}=1$, $x_1=1,x_2=3/2,x_3=5/2$, and $\mathcal{G}_1(t,u(t),v(t))$ and $\mathcal{G}_2(t,u(t),v(t))$ will be fixed later. Using the above data, we get $\Upsilon_1=0.7579116416$ and $\Upsilon_2=1.313797297$, where Υ_1 and Υ_2 are respectively given by 3.4 and 3.5. From Theorem3.1, we will use

$$\begin{aligned}\mathcal{G}_1(t,u(t),v(t))&=\frac{e^{-t}}{2\sqrt{900+t^2}}(ut+\sin v+\cos t),\\ \mathcal{G}_2(t,u(t),v(t))&=\frac{1}{(3+t)^2}\left(\sin u+\frac{v}{2}+e^{-t}\right).\end{aligned}$$

Next, $\mathcal{G}_1$ and $\mathcal{G}_2$ are continuous and fulfil the hypothesis $(\mathcal{W}_1)$ with

$$\gamma_1(t) = \frac{e^{-t}\cos t}{2\sqrt{400+t^2}}, \gamma_2(t) = \frac{te^{-t}}{2\sqrt{400+t^2}}, \gamma_3(t) = \frac{e^{-t}}{2\sqrt{400+t^2}},$$
$$k_2 = \frac{e^{-t}}{(3+t)^2}, k_1 = \frac{1}{(3+t)^2} \ \text{and} \ k_3 = \frac{1}{2(3+t)^2}.$$

Also

$$\|\gamma_2\|\left(2\Upsilon_1 + \frac{T^{\vartheta-1}}{\varphi\Gamma(\vartheta)}(1-e^{-\varphi T})\right) + \|k_2\|\left(2\Upsilon_2 + \frac{T^{\varpi-1}}{\varphi\Gamma(\varpi)}(1-e^{-\varphi T})\right) \approx 0.0631747638 \ \text{and}$$
$$\|\gamma_3\|\left(2\Upsilon_1 + \frac{T^{\vartheta-1}}{\varphi\Gamma(\vartheta)}(1-e^{-\varphi T})\right) + \|k_3\|\left(2\Upsilon_2 + \frac{T^{\varpi-1}}{\varphi\Gamma(\varpi)}(1-e^{-\varphi T})\right) \approx 0.0324305431.$$

Thus, by Theorem 3.1, ∃ a solution to the system (4.1) on $[0, 1]$.

Example 4.2. *Consider system (4.1) again. For the application of Theorem 3.2, we choose*

$$\begin{aligned}\mathcal{G}_1(t, u(t), v(t)) &= \frac{1}{10(1+t^2)}\left(\frac{|u||}{1+|u} + \tan^{-1} v\right),\\ \mathcal{G}_2(t, u(t), v(t)) &= \frac{1}{\sqrt{400+t^2}}\left(\sin v + 2\tan^{-1} u\right),\end{aligned} \tag{4.2}$$

where $\mathcal{G}_1$ and $\mathcal{G}_2$ are continuous and fulfil the hypothesis $(\mathcal{W}_2)$ with $C_1 = C_2 = 1/10 = C$ and $\mathcal{K}_1 = 1/10, \mathcal{K}_2 = 1/20$ and $\mathcal{K} = 1/10$. In addition, we get

$$\left(C\left(2\Upsilon_1 + \frac{T^{\vartheta-1}}{\varphi\Gamma(\vartheta)}(1-e^{-\varphi T})\right) + \mathcal{K}\left(2\Upsilon_2 + \frac{T^{\varpi-1}}{\varphi\Gamma(\varpi)}(1-e^{-\varphi T})\right)\right) \approx 0.3520736817 < 1.$$

It is clear that all of Theorem 3.2's assumptions are satisfied. As a result, ∃ a unique solution to system (4.2).

Example 4.3. *Consider the following system*

$$\begin{aligned}\left({}^{C}\mathcal{D}^{\frac{8}{3}} + \frac{2}{3}{}^{C}\mathcal{D}^{\frac{8}{3}-1}\right)u(t) &\in \mathcal{G}_1(t, u(t), v(t)), \quad t \in \mathcal{J} = [0,1],\\ \left({}^{C}\mathcal{D}^{\frac{5}{4}} + \frac{2}{3}{}^{C}\mathcal{D}^{\frac{5}{4}-1}\right)v(t) &\in \mathcal{G}_2(t, u(t), v(t)), \quad t \in \mathcal{J} = [0,1],\\ (u+v)(0) &= -(u+v)(\mathcal{T}),\\ \sum_{i=0}^{m} x_i(u-v)(\xi_i) + \mu\int_0^{\eta}\frac{(\eta-s)^{\delta-1}}{\Gamma(\delta)}(u-v)(s)ds &= \mathcal{A},\end{aligned} \tag{4.3}$$

where $\vartheta = 8/3, \varpi = 5/4, \eta = 1/10, \delta = 3/2, \mu = 1, \xi_1 = 1/5, \xi_2 = 2/5, \xi_3 = 3/5, \mathcal{A} = 3, \mathcal{T} = 1,$ $x_1 = 1, x_2 = 3/2, x_3 = 5/2$, $\mathcal{G}_1(t,u,v) = \left[\frac{-1}{10}\frac{|u|}{1+|u|}, 0\right] \cup \left[0, \frac{1}{10}\frac{|\sin(v)|}{1+|\sin(v)|}\right]$ *and* $\mathcal{G}_2(t,u,v) = \left[\frac{-1}{10}\frac{|v|}{1+|v|}, 0\right] \cup \left[0, \frac{1}{10}\frac{|\cos(u)|}{1+|\cos(u)|}\right]$, *and on the other hand,*

$$\mathcal{H}_d(\mathcal{G}_1(t,u,v), \mathcal{G}_1(t,\widehat{v},\widehat{w})) \leq \frac{1}{10}|u - \widehat{v}| + \frac{1}{10}|v - \widehat{w}|, \ \forall\ u, \widehat{v}, v, \widehat{w} \in \mathbb{R},$$
$$\mathcal{G}_d(\mathcal{G}_2(t,u,v), \mathcal{G}_2(t,\widehat{v},\widehat{w})) \leq \frac{1}{10}|u - \widehat{v}| + \frac{1}{10}|v - \widehat{w}|, \ \forall\ u, \widehat{v}, v, \widehat{w} \in \mathbb{R}.$$

Using the above data, we get $\Upsilon_1 = 0.7579116416$ *and* $\Upsilon_2 = 1.313797297$ *and* $(2\Upsilon_1)m_1 + (2\Upsilon_2)m_2 \approx 0.4143417683 < 1$. *All of Theorem 3.5's assumptions are satisfied. As a consequence,* $\exists$ *a solution to the system (4.3).*

5. Discussion

We have introduced a new type of coupled boundary condition that deals with the sum of unknown functions at the boundary points and along an arbitrary segment of the given domain. We solved a nonlinear coupled system of Caputo SFDEs and inclusions under these conditions. The existence and uniqueness results for the given problem are new, and they add to our understanding of fully coupled fractional-order BVPs. Furthermore, this study can be expanded to include fractional differential and integral operators of the Riemann-Liouville and Hadamard types. Our findings are not only novel in the context of the problem, but they also lead to some novel special cases involving specific parameter choices. For instance, our results correspond to those for new coupled discrete boundary conditions of the form: $(u+v)(0) = -(u+v)(\mathcal{T})$ $\sum_{i=0}^{m} x_i(u-v)(\xi_i) = \mathcal{A}$, if we set $\mu = 0$ in (1.2) and (1.3). Letting $x_i = 0$, $(i = 0, 1, ..., m)$, our results correspond to the Riemann-Liouville integral boundary conditions: $(u+v)(0) = -(u+v)(\mathcal{T})$, $\mu \int_0^{\eta} \frac{(\eta-s)^{\delta-1}}{\Gamma(\delta)}(u-v)(s)ds = \mathcal{A}$, and also we can obtained new existence results through special cases as stated in Remark 3.1 and Remark 3.2. Future research may concentrate on various concepts of stability with control techniques [41, 42] and existence as they relate to a neutral time-delay system/inclusion and a time-delay system/inclusion with finite delay.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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