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Finite-Time Synchronization of Clifford-Valued Neural Networks With Infinite Distributed Delays and Impulses

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ABSTRACT We study the issue of finite-time synchronization pertaining to a class of Clifford-valued neural networks with discrete and infinite distributed delays and impulse phenomena. Since multiplication of Clifford numbers is of non-commutativity, we decompose the original n-dimensional Clifford-valued drive and response systems into the equivalent 2^m-dimensional real-valued counterparts. We then derive the finite-time synchronization criteria concerning the decomposed real-valued drive and response models through new Lyapunov-Krasovskii functional and suitable controller as well as new computational techniques. We also demonstrate the usefulness of the results through a simulation example.

INDEX TERMS Clifford-valued neural networks, finite-time synchronization, infinite distributed delay, Lyapunov-Krasovskii fractional.

I. INTRODUCTION

Neural networks (NNs) have been successfully utilized to undertake complex problems in different domains, e.g. image processing, signal processing, optimization problems, and other tasks. Recently, researchers have conducted investigations on dynamic analysis of NN models, leading to many effective methods for analyzing the stability conditions of NNs [1]–[4]. While many useful results have been achieved, one challenge remains, i.e., the unavoidable phenomenon of time delay in NNs. In this regard, poor signal propagation, chaos, low functionality, divergence, and uncertainty are all examples of dynamics caused by time delays [5]–[9]. As such, it is worthwhile to analyze the dynamics of NNs subject to constant or time-varying delays. Recently, quaternion- and complex-valued NNs have been used to solve a variety of problems, including classification of polarized signals, investigation on $3D$ wind forecast, examination of radar images, analysis of night vision, and many other challenging tasks [10], [11], [15]–[17]. Many dynamic analysis results with respect to complex-valued and quaternion-valued NNs have been reported recently [12]-[14], [18]-[21].

As a strong foundation for tackling many geometry problems, Clifford algebra is useful for handling different tasks, including computing devices, control, robotic systems, and neural modeling [22]–[27]. Clifford-valued NNs generalize real-valued, complex-valued, and quaternion-valued NNs. In particular, Clifford-valued NNs are effective to solve spatial geometric transformation as well as high-dimensional data problems [24]-[30]. However, Clifford-valued NNs have complex dynamical characteristics than those of real-valued, complex-valued, and quaternion-valued NNs. Since Clifford numbers are subject to non-commutativity of multiplication, there have been few studies on the dynamical characteristics of Clifford-valued NNs [28]–[36]. As an example, by using appropriate Lyapunov-Krasovskii fractional (LKF) and linear matrix inequality (LMI) techniques, the study in [28] derived the global exponential stability conditions with respect to the recurrent type of Clifford-valued NNs with time delays. Leveraging the decomposition process, the key concern in

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Clifford-valued recurrent NNs from the perspective of global asymptotic stability was investigated in [29]. On the other hand, an effective feedback controller was designed in [31] for the recurrent type of Clifford-valued NNs to handle the presence of globally asymptotic almost automorphic synchronization. The Banach fixed point principle and a suitable LKF candidate was utilized to address the global asymptotic almost periodic synchronization issues of Clifford-valued NNs in [33]. Other similar results on the dynamical analysis of Clifford-valued NNs have also been reported, e.g. see [34]–[37].

In [38], the authors studied the fundamentals of chaotic systems, particularly on drive-response synchronization. The research on this phenomenon has now become popular, in view of its wide influence in different areas, e.g. communication, neural modeling, image processing, to name a few. As such, various synchronization methods for different NNs are available nowadays, especially exponential and asymptotic synchronization [39]–[31]. However, in some realistic applications, such as the limited life span of machines, synchronization problems are insufficient [42]–[43]. Thus, in order to resolve these problems, the concept of finite-time synchronization was developed. The finite-time synchronization due to discontinuous activation functions and time-varying delays in the recurrent type of complex-valued NNs was investigated in [12]. While finite-time synchronization with infinite-time distributed delays in complex-valued NNs has been examined [47]. Similar issues due to time delays in NNs was studied in [45], [46], [48].

However, studies on finite-time synchronization pertaining to impulse and infinite-time distributed delays in Clifford-valued NNs are limited. Therefore, our paper is focused on the derivation of sufficient conditions to finite-time synchronization of Clifford-valued NNs with impulse and infinite-time distributed delays. The key contributions of this work are listed below:

(1) The effects of impulsive and infinite distributed delays on finite-time synchronization of Clifford-valued NNs are studied for the first time.
(2) The sufficient conditions on finite-time synchronization of Clifford-valued NNs are formulated by decomposition into real-valued counterparts and by the use of appropriate LKF candidate, control scheme and computing techniques.
(3) The results in this paper are valid as special cases when we transform Clifford-valued NNs to its real-valued, complex-valued, as well as quaternion-valued counterparts.
(4) An illustrative example is given and the associated results are analyzed to show the effectiveness of our results.

The remaining part of this paper is structured as follows. The definition of Clifford-valued NNs is presented in Section 2. In section 3, the sufficient conditions subject to finite-time synchronization with respect to considered Clifford-valued NNs are given. Simulation results and concluding remarks are presented in Sections 4 and 5, respectively.

II. BACKGROUND

A. NOTATIONS

In the following sections, $\|\mathfrak{r}\| = \sum_{x=1}^{n} |\mathfrak{r}_x|$ denotes the norm of $\mathbb{R}^n$. Given $\mathfrak{A} = (a_{xy})_{n\times n} \in \mathbb{R}^{n\times n}$, one has $\|\mathfrak{A}\| = \max_{1\le x\le n}\left\{\sum_{y=1}^{n}|a_{xy}|\right\}$. Similarly, $\mathfrak{r} = \sum_A \mathfrak{r}^A e_A \in \mathbb{A}$, one has $|\mathfrak{r}|_{\mathbb{A}} = \sum_A |\mathfrak{r}^A|$, and given $\mathfrak{A} = (a_{xy})_{n\times n} \in \mathbb{A}^{n\times n}$, one has $\|\mathfrak{A}\|_{\mathbb{A}} = \max_{1\le x\le n}\left\{\sum_{y=1}^{n}|a_{xy}|_{\mathbb{A}}\right\}$. Besides, $\varphi \in \mathcal{C}((-\infty,0],\mathbb{A}^n)$. Matrix transposition and matrix involution transposition are indicated by superscripts of T and $*$. Clifford algebra with m generators operating on real number $\mathbb{R}$ is denoted by $\mathbb{A}$. The n-dimensional real vector space and Clifford vector are represented by $\mathbb{R}^n$ and $\mathbb{A}^n$, while the set of all $n \times n$ real and Clifford matrices are represented by $\mathbb{R}^{n\times n}$ and $\mathbb{A}^{n\times n}$, respectively. Moreover, $sign(\mathfrak{r}) = \begin{cases} 1, & \text{if } \mathfrak{r} > 0, \\ 0, & \text{if } \mathfrak{r} = 0, \\ -1, & \text{if } \mathfrak{r} < 0. \end{cases}$

B. CLIFFORD ALGEBRA

Define the Clifford real algebra over $\mathbb{R}^m$ as

$$\mathbb{A} = \left\{ \sum_{A\subseteq\{1,2,\ldots,m\}} a^A e_A,\ a^A \in \mathbb{R} \right\},$$

where $e_A = e_{\ell_1}e_{\ell_2}\ldots e_{\ell_\nu}$ with $A = \{\ell_1, \ell_2, \ldots, \ell_\nu\}$, $1 \le \ell_1 < \ell_2 < \ldots < \ell_\nu \le m$. For the Clifford generators that $e_x e_y + e_y e_x = 0,\ x \ne y,\ \ x, y = 1, 2, \ldots, m, e_x^2 = -1,\ x = 1, 2, \ldots, m$ are denoted by $e_\emptyset = e_0 = 1$ and $e_\ell = e_{\{\ell\}}$, $\ell = 1, 2, \ldots, m$.

Remark 1: When an element is the product of multiple Clifford generators, the subscripts are organized together: $e_4e_5e_6e_7 = e_{4567}$.

Given $\Lambda = \{\emptyset, 1, 2, \ldots, A, \ldots, 12\ldots m\}$, and one has

$$\mathbb{A} = \left\{ \sum_A a^A e_A,\ a^A \in \mathbb{R} \right\},$$

where $\sum_A$ indicates $\sum_{A\in\Lambda}$ and $dim\mathbb{A} = \sum_{k=0}^{m}\binom{m}{k} = 2^m$. Given a Clifford number $\mathfrak{r} = \sum_A \mathfrak{r}^A e_A$, one can define the involution of $\mathfrak{r}$ as

$$\bar{\mathfrak{r}} = \sum_A \mathfrak{r}^A \bar{e}_A,$$

where $\bar{e}_A = (-1)^{\frac{\sigma[A](\sigma[A]+1)}{2}} e_A$. If $A = \emptyset$, then $\sigma[A] = 0$, while if $A = \ell_1\ell_2\ldots\ell_\nu$, then $\sigma[A] = \nu$. As such, based on the definition, one has $e_A\bar{e}_A = \bar{e}_A e_A = 1$. Given $\mathfrak{r} = \sum_A \mathfrak{r}^A e_A : \mathbb{R} \to \mathbb{A}$, where $\mathfrak{r}^A : \mathbb{R} \to \mathbb{R}$, $A \in \Lambda$, one has $\dot{\mathfrak{r}}(t) = \sum_A \dot{\mathfrak{r}}^A(t) e_A$ as its derivative. Readers can refer to [28]–[30] for a detailed understanding of Clifford algebra.

Remark 2: Since $e_B\bar{e}_A = (-1)^{\frac{\sigma[A](\sigma[A]+1)}{2}} e_B e_A$, *one has* $e_B\bar{e}_A = e_C$ *or* $e_B\bar{e}_A = -e_C$, *where* e_C *is a basis of Clifford algebra* $\mathbb{A}$. *One can then identify a unique basis* e_C *pertaining to a given* $e_B\bar{e}_A$. *Let*

$$\sigma[B.\bar{A}] = \begin{cases} 0, & \text{if } e_B\bar{e}_A = e_C, \\ 1, & \text{if } e_B\bar{e}_A = -e_C, \end{cases}$$

then $e_B\bar{e}_A = (-1)^{\sigma[B.\bar{A}]} e_C$. *Given* $\mathscr{F} \in \mathbb{A}$, *one has a unique* $\mathscr{F}^C$ *that fulfills* $\mathscr{F}^{B.\bar{A}} = (-1)^{\sigma[B.\bar{A}]}\mathscr{F}^C$ *pertaining to* $e_B\bar{e}_A = (-1)^{\sigma[B.\bar{A}]}e_C$. *Hence,*

$$\begin{aligned} \mathscr{F}^{B.\bar{A}} e_B\bar{e}_A &= \mathscr{F}^{B.\bar{A}}(-1)^{\sigma[B.\bar{A}]}e_C \\ &= (-1)^{\sigma[B.\bar{A}]}\mathscr{F}^C(-1)^{\sigma[B.\bar{A}]}e_C \\ &= \mathscr{F}^C e_C, \end{aligned}$$

and $\mathscr{F} = \sum_C \mathscr{F}^C e_C \in \mathbb{A}$.

As an example, consider the second term in an NN (1), and for all $y = 1, 2, \ldots, n$

$$\begin{aligned} &\sum_{y=1}^{n} a_{xy}\mathfrak{h}_y(\mathfrak{r}_y(t)) \\ &= \sum_{y=1}^{n}\sum_C a_{xy}^C e_C \sum_B \mathfrak{h}_y^B(\mathfrak{r}_y(t))e_B \\ &= \sum_{y=1}^{n}\sum_A\sum_B (-1)^{\sigma[A.\bar{B}]}a_{xy}^{A.\bar{B}}(-1)^{\sigma[A.\bar{B}]}e_A\bar{e}_B\mathfrak{h}_y^B(\mathfrak{r}_y(t))e_B \\ &= \sum_{y=1}^{n}(-1)^{2\sigma[A.\bar{B}]}\sum_A\sum_B a_{xy}^{A.\bar{B}}\mathfrak{h}_y^B(\mathfrak{r}_y(t))e_A\bar{e}_B e_B \\ &= \sum_{y=1}^{n}\sum_A\sum_B a_{xy}^{A.\bar{B}}\mathfrak{h}_y^B(\mathfrak{r}_y(t))e_A. \end{aligned}$$

C. PROBLEM DEFINITION

We consider the Clifford-valued NNs subject to time-varying delays, infinite distributed delays and impulsive effects, as follows:

$$\begin{cases} \dot{\mathfrak{r}}_x(t) = -d_x\mathfrak{r}_x(t) + \sum_{y=1}^{n} a_{xy}\mathfrak{h}_y(\mathfrak{r}_y(t)) \\ \qquad + \sum_{y=1}^{n} b_{xy}\mathfrak{h}_y(\mathfrak{r}_y(t-\tau_y(t))) \\ \qquad + \sum_{y=1}^{n} c_{xy}\int_{-\infty}^{t}\theta_{xy}(t-s)\mathfrak{h}_y(\mathfrak{r}_y(s))ds \\ \qquad + \mathfrak{k}_x,\ t \geq 0,\ t \neq t_k, \\ \Delta\mathfrak{r}_x(t) = \mathfrak{r}_x(t_k) - \mathfrak{r}_x(t_k^-) = \alpha_{xk}\mathfrak{r}_x(t_k^-),\ t = t_k, \\ \mathfrak{r}_x(t) = \varphi_x(t) \in \mathcal{C}((-\infty, 0], \mathbb{A}), \end{cases} \tag{1}$$

where $x,\ y \in N$ $(N = 1, 2, \ldots, n)$; the state vector of the xth node is $\mathfrak{r}_x(t) \in \mathbb{A}$. In addition, the rate with which the xth node resets its potential to achieve a resting state in isolation after being disconnected from the network and external inputs is indicated by $d_x \in \mathbb{R}^+$, while the connection weights are $a_{xy}, b_{xy}, c_{xy} \in \mathbb{A}$, and the external input is $\mathfrak{k}_x \in \mathbb{A}$. Moreover, the activation function is $\mathfrak{h}_y(\cdot) : \mathbb{A}^n \rightarrow \mathbb{A}^n$, the time-varying delay is $\tau_y(t)$, and the bounded scalar function is $\theta_{xy} : [0, +\infty) \rightarrow [0, +\infty)$. On the other hand, $\Delta\mathfrak{r}_x(t_k) = \mathfrak{r}_x(t_k) - \mathfrak{r}_x(t_k^-)$ represents the jump of $\mathfrak{r}_x$ at impulsive moments $t_k,\ k = 1, 2, \ldots$ that fulfills $0 < t_1 < t_2 < \ldots < t_k < \ldots$, which is strictly increasing sequence such that $\lim_{k\rightarrow+\infty} t_k = +\infty$. The right and left limits at t_k are $\mathfrak{r}_x(t_k)$ and $\mathfrak{r}_x(t_k^-)$, while the strength of impulse is $\alpha_{xk} = diag\{\alpha_{1k}, \alpha_{2k}, \ldots, \alpha_{nk}\}$. With respect to the Banach space, $\varphi_x(t) \in \mathcal{C}((-\infty, 0], \mathbb{A})$ represents a continuous function mapping $(-\infty, 0]$ into $\mathbb{A}$.

(A1) Positive constants τ_y and $\mu_y < 1$ exist, in which $0 \leq \tau_y(t) \leq \tau_y$, $\dot{\tau}_y(t) \leq \mu_y$, $y \in N$.

Remark 3: The NN (1) covers real-valued, complex-valued, and quaternion-valued counterparts. As such, the formulated NN in (1) stands as a generic version as compared with those in the existing literature. As an example, the Clifford-valued NN representation in (1) covers real-valued NNs $(m = 0)$, *complex-valued NNs* $(m = 1)$, *and quaternion-valued NNs* $(m = 2)$ *as its special cases.*

The definition of its associated response model is as follows:

$$\begin{cases} \dot{\mathfrak{s}}_x(t) = -d_x\mathfrak{s}_x(t) + \sum_{y=1}^{n} a_{xy}\mathfrak{h}_y(\mathfrak{s}_y(t)) \\ \qquad + \sum_{y=1}^{n} b_{xy}\mathfrak{h}_y(\mathfrak{s}_y(t-\tau_y(t))) \\ \qquad + \sum_{y=1}^{n} c_{xy}\int_{-\infty}^{t}\theta_{xy}(t-s)\mathfrak{h}_y(\mathfrak{s}_y(s))ds \\ \qquad + \mathfrak{k}_x + \mathfrak{u}_x(t),\ t \geq 0,\ t \neq t_k, \\ \Delta\mathfrak{s}_x(t) = \mathfrak{s}_x(t_k) - \mathfrak{s}_x(t_k^-) = \alpha_{xk}\mathfrak{s}_x(t_k^-),\ t = t_k, \\ \mathfrak{s}_x(t) = \phi_x(t) \in \mathcal{C}((-\infty, 0], \mathbb{A}), \end{cases} \tag{2}$$

where $x,\ y \in N$ $(N = 1, 2, \ldots, n)$, $\mathfrak{s}_x(t) \in \mathbb{A}$ is the state vector of the xth node. In addition, $\phi_x(t) \in \mathcal{C}((-\infty, 0], \mathbb{A})$ represents the Banach space of all continuous functions mapping $(-\infty, 0]$ to $\mathbb{A}$. On the other hand, $\Delta\mathfrak{s}_x(t_k) = \mathfrak{s}_x(t_k) - \mathfrak{s}_x(t_k^-)$ represents the jump of $\mathfrak{s}_x$ at impulsive moment t_k. Besides, $\mathfrak{u}_x(t)$ is the control input, while all other notations in NN (2) are similar to those in NN (1).

Remark 4: When the delay kernels meet the following condition

$$\theta_{xy}(t) = \begin{cases} 0, & t > \varsigma_{xy}, \\ 1, & 0 \leq t \leq \varsigma_{xy}, \end{cases}$$

where $\varsigma_{xy} > 0$, $x,\ y \in N$ *are constants, the NN (1) becomes the following NN with finite-time distributed delays*:

$$\begin{cases} \dot{\mathfrak{r}}_x(t) = -d_x\mathfrak{r}_x(t) + \sum_{y=1}^{n} a_{xy}\mathfrak{h}_y(\mathfrak{r}_y(t)) \\ \qquad + \sum_{y=1}^{n} b_{xy}\mathfrak{h}_y(\mathfrak{r}_y(t-\tau_y(t))) \\ \qquad + \sum_{y=1}^{n} c_{xy}\int_{t-\varsigma_{xy}}^{t}\theta_{xy}(t-s)\mathfrak{h}_y(\mathfrak{r}_y(s))ds \\ \qquad + \mathfrak{k}_x,\ t \geq 0,\ t \neq t_k, \\ \Delta\mathfrak{r}_x(t) = \mathfrak{r}_x(t_k) - \mathfrak{r}_x(t_k^-) = \alpha_{xk}\mathfrak{r}_x(t_k^-),\ t = t_k, \\ \mathfrak{r}_x(t) = \varphi_x(t) \in \mathcal{C}((-\infty, 0], \mathbb{A}). \end{cases}$$

(A2) Function $\mathfrak{h}_y(\cdot)$ satisfies the Lipschitz continuity condition with respect to the n-dimensional Clifford vector. Consider each $y \in N$, there is a positive constant l_y such that for any $p, q \in \mathbb{A}$,

$$|\mathfrak{h}_y(p) - \mathfrak{h}_y(q)|_{\mathbb{A}} \leq l_y|p - q|_{\mathbb{A}}, \ y \in N, \tag{3}$$

where l_y ($y \in N$) is the Lipschitz constant and $\mathfrak{h}_y(0) = 0$. Furthermore, a positive constant l_y exists such that $|\mathfrak{h}(p)|_{\mathbb{A}} \leq l_y$ for any $p \in \mathbb{A}$.

(A3) There is a positive constant $\tilde{\theta}_{xy}$ such that

$$\int_0^{+\infty} \theta_{xy}(s)ds = \tilde{\theta}_{xy}, \ x, y \in N. \tag{4}$$

III. MAIN RESULTS

Based on $e_A\bar{e}_A = \bar{e}_A e_A = 1$ and $e_B\bar{e}_A e_A = e_B$, the following real-valued NN can be obtained through transformation of (1):

$$\begin{cases} \dot{\mathfrak{r}}_x^A(t) = -d_x\mathfrak{r}_x^A(t) + \sum_{y=1}^n \sum_B a_{xy}^{A.\bar{B}} \mathfrak{h}_y^B(\mathfrak{r}_y(t)) \\ \quad + \sum_{y=1}^n \sum_B b_{xy}^{A.\bar{B}} \mathfrak{h}_y^B(\mathfrak{r}_y(t - \tau_y(t))) \\ \quad + \sum_{y=1}^n \sum_B c_{xy}^{A.\bar{B}} \int_{-\infty}^t \theta_{xy}(t-s)\mathfrak{h}_y^B(\mathfrak{r}_y(s))ds \\ \quad + \mathfrak{k}_x^A, \ t \geq 0, \ t \neq t_k, \\ \Delta\mathfrak{r}_x^A(t) = \mathfrak{r}_x^A(t_k) - \mathfrak{r}_x^A(t_k^-) = \alpha_{xk}\mathfrak{r}_x^A(t_k^-), \ t = t_k, \\ \mathfrak{r}_x^A(t) = \varphi_x^A(t) \in \mathcal{C}((-\infty, 0], \mathbb{R}^{2^m n}), \end{cases} \tag{5}$$

where

$$\begin{aligned} \mathfrak{r}^A(t) &= (\mathfrak{r}_1^A(t), \mathfrak{r}_2^A(t), \ldots, \mathfrak{r}_n^A(t))^T, \\ \mathfrak{r}_x(t) &= \sum_A \mathfrak{r}_x^A(t)e_A, \\ \mathfrak{k}^A &= (\mathfrak{k}_1^A, \mathfrak{k}_2^A, \ldots, \mathfrak{k}_n^A)^T, \ \mathfrak{k}_x = \sum_A \mathfrak{k}_x^A e_A, \\ \mathfrak{h}_y(\mathfrak{r}_y(t)) &= \sum_B \mathfrak{h}_y^B\big(\mathfrak{r}_y^{C_1}(t), \mathfrak{r}_y^{C_2}(t), \ldots, \mathfrak{r}_y^{C_{2m}}(t)\big)e_B \\ &= \sum_B \mathfrak{h}_y^B(\mathfrak{r}_y(t))e_B, \\ \mathfrak{h}_y(\mathfrak{r}_y(t - \tau_y(t))) &= \sum_B \mathfrak{h}_y^B\big(\mathfrak{r}_y^{C_1}(t - \tau_y(t)), \\ &\quad \mathfrak{r}_y^{C_2}(t - \tau_y(t)), \ldots, \mathfrak{r}_y^{C_{2m}}(t - \tau_y(t))\big)e_B \\ &= \sum_B \mathfrak{h}_y^B(\mathfrak{r}_y(t - \tau_y(t)))e_B, \\ a_{xy} &= \sum_C a_{xy}^C e_C, \ a_{xy}^{A.\bar{B}} = (-1)^{\sigma[A.\bar{B}]}a_{xy}^C, \\ b_{xy} &= \sum_C b_{xy}^C e_C, \ b_{xy}^{A.\bar{B}} = (-1)^{\sigma[A.\bar{B}]}b_{xy}^C, \\ c_{xy} &= \sum_C c_{xy}^C e_C, \ c_{xy}^{A.\bar{B}} = (-1)^{\sigma[A.\bar{B}]}c_{xy}^C, \\ e_A\bar{e}_B &= (-1)^{\sigma[A.\bar{B}]}e_C. \end{aligned}$$

Using the same method, one can obtain the following real-valued NN through transformation of (2):

$$\begin{cases} \dot{\mathfrak{s}}_x^A(t) = -d_x\mathfrak{s}_x^A(t) + \sum_{y=1}^n \sum_B a_{xy}^{A.\bar{B}} \mathfrak{h}_y^B(\mathfrak{s}_y(t)) \\ \quad + \sum_{y=1}^n \sum_B b_{xy}^{A.\bar{B}} \mathfrak{h}_y^B(\mathfrak{s}_y(t - \tau_y(t))) \\ \quad + \sum_{y=1}^n \sum_B c_{xy}^{A.\bar{B}} \int_{-\infty}^t \theta_{xy}(t-s)\mathfrak{h}_y^B(\mathfrak{s}_y(s))ds \\ \quad + \mathfrak{k}_x^A + \mathfrak{u}_x^A(t), \ t \geq 0, \ t \neq t_k, \\ \Delta\mathfrak{s}_x^A(t) = \mathfrak{s}_x^A(t_k) - \mathfrak{s}_x^A(t_k^-) = \alpha_{xk}\mathfrak{s}_x^A(t_k^-), \ t = t_k, \\ \mathfrak{s}_x^A(t) = \phi_x^A(t) \in \mathcal{C}((-\infty, 0], \mathbb{R}^{2^m n}), \end{cases} \tag{6}$$

where

$$\begin{aligned} \mathfrak{s}^A(t) &= (\mathfrak{s}_1^A(t), \mathfrak{s}_2^A(t), \ldots, \mathfrak{s}_n^A(t))^T, \\ \mathfrak{s}_x(t) &= \sum_A \mathfrak{s}_x^A(t)e_A, \\ \mathfrak{u}^A(t) &= (\mathfrak{u}_1^A(t), \mathfrak{u}_2^A(t), \ldots, \mathfrak{u}_n^A(t))^T, \\ \mathfrak{u}_x(t) &= \sum_A \mathfrak{u}_x^A(t)e_A, \\ \mathfrak{h}_y(\mathfrak{s}_y(t)) &= \sum_B \mathfrak{h}_y^B\big(\mathfrak{s}_y^{C_1}(t), \mathfrak{s}_y^{C_2}(t), \ldots, \mathfrak{s}_y^{C_{2m}}(t)\big)e_B \\ &= \sum_B \mathfrak{h}_y^B(\mathfrak{s}_y(t))e_B, \\ \mathfrak{h}_y(\mathfrak{s}_y(t - \tau_y(t))) &= \sum_B \mathfrak{h}_y^B\big(\mathfrak{s}_y^{C_1}(t - \tau_y(t)), \\ &\quad \mathfrak{s}_y^{C_2}(t - \tau_y(t)), \ldots, \mathfrak{s}_y^{C_{2m}}(t - \tau_y(t))\big)e_B \\ &= \sum_B \mathfrak{h}_y^B(\mathfrak{s}_y(t - \tau_y(t)))e_B. \end{aligned}$$

Note that the notation in (5) is applicable in (6).

Remark 5: If $\mathfrak{r} = (\mathfrak{r}_1^0, \mathfrak{r}_1^1, \ldots, \mathfrak{r}_1^{1.2\ldots m}, \mathfrak{r}_2^0, \mathfrak{r}_2^1, \ldots, \mathfrak{r}_2^{1.2\ldots m}, \ldots, \mathfrak{r}_n^0, \mathfrak{r}_n^1, \ldots, \mathfrak{r}_n^{1.2\ldots m})^T = \{\mathfrak{r}_x^A\}$ *is a solution to (5), then* $\mathfrak{r} = (\mathfrak{r}_1, \ldots, \mathfrak{r}_n)^T$ *must be a solution to (1), where* $\mathfrak{r}_x = \sum_A \mathfrak{r}_x^A e_A, \ i = 1, 2, \ldots, n, \ A \in \Lambda.$

The error vectors between (5) and (6) are indicated by $\mathfrak{e}_x^A(t) = \mathfrak{s}_x^A(t) - \mathfrak{r}_x^A(t)$ and $\psi_x^A(t) = \phi_x^A(t) - \varphi_x^A(t)$, respectively. Based on (5)-(6), one has:

$$\begin{cases} \dot{\mathfrak{e}}_x^A(t) = -d_x\mathfrak{e}_x^A(t) + \sum_{y=1}^n \sum_B a_{xy}^{A.\bar{B}} \mathfrak{h}_y^B(\mathfrak{e}_y(t)) \\ \quad + \sum_{y=1}^n \sum_B b_{xy}^{A.\bar{B}} \mathfrak{h}_y^B(\mathfrak{e}_y(t - \tau_y(t))) \\ \quad + \sum_{y=1}^n \sum_B c_{xy}^{A.\bar{B}} \int_{-\infty}^t \theta_{xy}(t-s)\mathfrak{h}_y^B(\mathfrak{e}_y(s))ds \\ \quad + \mathfrak{u}_x^A(t), \ t \geq 0, \ t \neq t_k, \\ \Delta\mathfrak{e}_x^A(t) = \mathfrak{e}_x^A(t_k) - \mathfrak{e}_x^A(t_k^-) = \alpha_{xk}\mathfrak{e}_x^A(t_k^-), \ t = t_k, \\ \mathfrak{e}_x^A(t) = \psi_x^A(t) \in \mathcal{C}((-\infty, 0], \mathbb{R}^{2^m n}). \end{cases} \tag{7}$$

The following definition applies in this work.

Definition 1: [48] Given an appropriate controller, if constant $t_1 > 0$, such that $\|\mathfrak{e}^A(t_1)\|_1 = 0$ and $\|\mathfrak{e}^A(t)\|_1 \equiv 0$ for $t > t_1$, where $\|\mathfrak{e}^A(t)\|_1 = \sum_{x=1}^{n}\sum_A |\mathfrak{e}_x^A(t)|$, then the NN (5) and (6) are finite-time synchronized.

A. FINITE TIME SYNCHRONIZATION

Referring to Definition (1), finite-time synchronization of (5) and (6) is equivalent to finite-time stabilization pertaining to (7) at the origin. Hence, the condition $\mathfrak{u}_x(t) = 0$ when $\mathfrak{e}_x(t) = 0,\ x \in N$ must be met by controller $\mathfrak{u}_x(t)$. In this case, the following controller is formulated:

$$\begin{aligned}\mathfrak{u}_x(t) &= \sum_A \mathfrak{u}_x^A(t)e_A,\\ \mathfrak{u}_x^A(t) &= -\beta_x\mathfrak{e}_x^A(t) - \gamma_x\, sign(\mathfrak{e}_x^A(t)),\end{aligned} \tag{8}$$

where $x \in N, A \in \Lambda$, and the control gain $\beta_x > 0$, and the tunable constant is $\gamma_x > 0$. The following main results can be derived.

*Theorem 1: Assume that **(A1)-(A3)** are valid, given positive constants $\beta_x, \gamma_x,\ x \in N$, and the impulsive operator $\alpha_{xk}(\mathfrak{e}^A(t_k^-))$ satisfies $\alpha_{xk}(\mathfrak{e}^A(t_k^-)) = -\vartheta_{xk}(\mathfrak{e}^A(t_k^-)),\ x \in N$, $k = 1, 2, \ldots$. If $|1-\vartheta_{xk}| \le 1$, and the following inequality exists*

$$d_x + \beta_x - \Bigg(\sum_{y=1}^{n}\sum_A\sum_B l_x|a_{yx}^{A.\bar{B}}| + \sum_{y=1}^{n}\sum_A\sum_B \frac{1}{1-\mu_x} l_x|b_{yx}^{A.\bar{B}}| + \sum_{y=1}^{n}\sum_A\sum_B l_x|c_{yx}^{A.\bar{B}}|\tilde{\theta}_{yx}\Bigg) > 0, \quad x \in N, \tag{9}$$

then the model in (6) is synchronized with that in (5) in fixed-time under controller (8).

Proof: Consider the following LKF candidate:

$$\begin{aligned}\mathfrak{V}(t) &= \sum_{x=1}^{n}\sum_A |\mathfrak{e}_x^A(t)| + \sum_{x=1}^{n}\sum_{y=1}^{n}\sum_A\sum_B \frac{1}{1-\mu_y}|b_{xy}^{A.\bar{B}}|l_y\int_{t-\tau_y(t)}^{t}|\mathfrak{e}_y^A(s)|ds\\ &\quad + \sum_{x=1}^{n}\sum_{y=1}^{n}\sum_A\sum_B |c_{xy}^{A.\bar{B}}|l_y \times \int_{-\infty}^{0}\int_{t+s}^{t}\theta_{xy}(-s)|\mathfrak{e}_y^A(u)|duds.\end{aligned} \tag{10}$$

When $t = t_k,\ k = 1, 2, \ldots$, one can compute

$$\begin{aligned}\mathfrak{V}(t_k) &= \sum_{x=1}^{n}\sum_A |\mathfrak{e}_x^A(t_k)| + \sum_{x=1}^{n}\sum_{y=1}^{n}\sum_A\sum_B \times \frac{1}{1-\mu_y}|b_{xy}^{A.\bar{B}}|l_y\int_{t_k-\tau_y(t_k)}^{t_k}|\mathfrak{e}_y^A(s)|ds\\ &\quad + \sum_{x=1}^{n}\sum_{y=1}^{n}\sum_A\sum_B |c_{xy}^{A.\bar{B}}|l_y \times \int_{-\infty}^{0}\int_{t_k+s}^{t}\theta_{xy}(-s)|\mathfrak{e}_y^A(u)|duds\\ &= \sum_{x=1}^{n}\sum_A |\mathfrak{e}_x^A(t_k^-) + \alpha_{xk}(\mathfrak{e}_x^A(t_k^-))|\\ &\quad + \sum_{x=1}^{n}\sum_{y=1}^{n}\sum_A\sum_B \frac{1}{1-\mu_y}|b_{xy}^{A.\bar{B}}|l_y \times \int_{t_k-\tau_y(t_k)}^{t_k}|\mathfrak{e}_y^A(s)|ds + \sum_{x=1}^{n}\sum_{y=1}^{n}\sum_A\sum_B |c_{xy}^{A.\bar{B}}|l_y\\ &\quad \times \int_{-\infty}^{0}\int_{t_k+s}^{t}\theta_{xy}(-s)|\mathfrak{e}_y^A(u)|duds\\ &\le \sum_{x=1}^{n}\sum_A |\mathfrak{e}_x^A(t_k^-) - \vartheta_{xk}(\mathfrak{e}_x^A(t_k^-))|\\ &\quad + \sum_{x=1}^{n}\sum_{y=1}^{n}\sum_A\sum_B \frac{1}{1-\mu_y}|b_{xy}^{A.\bar{B}}|l_y \times \int_{t_k-\tau_y(t_k)}^{t_k}|\mathfrak{e}_y^A(s)|ds + \sum_{x=1}^{n}\sum_{y=1}^{n}\sum_A\sum_B |c_{xy}^{A.\bar{B}}|l_y\\ &\quad \times \int_{-\infty}^{0}\int_{t_k+s}^{t}\theta_{xy}(-s)|\mathfrak{e}_y^A(u)|duds\\ &= \sum_{x=1}^{n}\sum_A |1-\vartheta_{xk}|\mathfrak{e}_x^A(t_k^-)|\\ &\quad + \sum_{x=1}^{n}\sum_{y=1}^{n}\sum_A\sum_B \frac{1}{1-\mu_y}|b_{xy}^{A.\bar{B}}|l_y \times \int_{t_k-\tau_y(t_k)}^{t_k}|\mathfrak{e}_y^A(s)|ds + \sum_{x=1}^{n}\sum_{y=1}^{n}\sum_A\sum_B |c_{xy}^{A.\bar{B}}|l_y\\ &\quad \times \int_{-\infty}^{0}\int_{t_k+s}^{t}\theta_{xy}(-s)|\mathfrak{e}_y^A(u)|duds\\ &= |1-\vartheta_{xk}|\mathfrak{V}(t_k^-)\\ \mathfrak{V}(t_k) &\le \mathfrak{V}(t_k^-).\end{aligned} \tag{11}$$

When $t \neq t_k$, and by assumptions **(A1)-(A3)**, one can compute the upper right derivatives of $\mathfrak{V}(t)$ with respect to the solution of (7) to yield

$$\begin{aligned}\dot{\mathfrak{V}}(t) &= \sum_{x=1}^{n}\sum_A sign(\mathfrak{e}_x^A(t))\dot{\mathfrak{e}}_x^A(t)\\ &= \sum_{x=1}^{n}\sum_A sign(\mathfrak{e}_x^A(t))\Big[-d_x\mathfrak{e}_x^A(t)\\ &\quad + \sum_{y=1}^{n}\sum_B a_{xy}^{A.\bar{B}}\mathfrak{h}_y^B(\mathfrak{e}_y(t))\\ &\quad + \sum_{y=1}^{n}\sum_B b_{xy}^{A.\bar{B}}\mathfrak{h}_y^B(\mathfrak{e}_y(t-\tau_y(t)))\end{aligned}$$

$$+\sum_{y=1}^{n}\sum_{B} c_{xy}^{A.\bar{B}}\int_{-\infty}^{t}\theta_{xy}(t-s)\mathfrak{h}_y^B(\mathfrak{e}_y(s))ds$$

$$+\mathfrak{u}_x^A(t)\Big] + \sum_{x=1}^{n}\sum_{y=1}^{n}\sum_{A}\sum_{B}\frac{1}{1-\mu_y}|b_{xy}^{A.\bar{B}}|l_y|\mathfrak{e}_y^A(t)|$$

$$-\sum_{x=1}^{n}\sum_{y=1}^{n}\sum_{A}\sum_{B}\frac{1}{1-\mu_y}(1-\dot{\tau}_y(t))|b_{xy}^{A.\bar{B}}|l_y$$

$$\times|\mathfrak{e}_y^A(t-\tau_y(t))| + \sum_{x=1}^{n}\sum_{y=1}^{n}\sum_{A}\sum_{B}|c_{xy}^{A.\bar{B}}|l_y$$

$$\times\int_{-\infty}^{0}\theta_{xy}(-s)|\mathfrak{e}_y^A(t)|ds - \sum_{x=1}^{n}\sum_{y=1}^{n}\sum_{A}\sum_{B}|c_{xy}^{A.\bar{B}}|l_y$$

$$\times\int_{-\infty}^{0}\theta_{xy}(-s)|\mathfrak{e}_y^A(t+s)|ds$$

$$\leq \sum_{x=1}^{n}\sum_{A} sign(\mathfrak{e}_x^A(t))\Big[-d_x\mathfrak{e}_x^A(t)$$

$$+\sum_{y=1}^{n}\sum_{B} a_{xy}^{A.\bar{B}}\mathfrak{h}_y^B(\mathfrak{e}_y(t))$$

$$+\sum_{y=1}^{n}\sum_{B} b_{xy}^{A.\bar{B}}\mathfrak{h}_y^B(\mathfrak{e}_y(t-\tau_y(t)))$$

$$+\sum_{y=1}^{n}\sum_{B} c_{xy}^{A.\bar{B}}\int_{-\infty}^{t}\theta_{xy}(t-s)\mathfrak{h}_y^B(\mathfrak{e}_y(s))ds$$

$$-\beta_x\mathfrak{e}_x^A(t) - \gamma_x\, sign(\mathfrak{e}_x^A(t))\Big]$$

$$+\sum_{x=1}^{n}\sum_{y=1}^{n}\sum_{A}\sum_{B}\frac{1}{1-\mu_y}|b_{xy}^{A.\bar{B}}|l_y|\mathfrak{e}_y^A(t)|$$

$$-\sum_{x=1}^{n}\sum_{y=1}^{n}\sum_{A}\sum_{B}\frac{1}{1-\mu_y}(1-\mu_y)|b_{xy}^{A.\bar{B}}|l_y$$

$$\times|\mathfrak{e}_y^A(t-\tau_y(t))| + \sum_{x=1}^{n}\sum_{y=1}^{n}\sum_{A}\sum_{B}|c_{xy}^{A.\bar{B}}|l_y\tilde{\theta}_{xy}|\mathfrak{e}_y^A(t)|$$

$$-\sum_{x=1}^{n}\sum_{y=1}^{n}\sum_{A}\sum_{B}|c_{xy}^{A.\bar{B}}|l_y\int_{-\infty}^{t}\theta_{xy}(t-s)|\mathfrak{e}_y^A(s)|ds$$

$$= -\sum_{x=1}^{n}\sum_{A} d_x|\mathfrak{e}_x^A(t)| + \sum_{x=1}^{n}\sum_{A} sign(\mathfrak{e}_x^A(t))$$

$$\times\sum_{y=1}^{n}\sum_{B} a_{xy}^{A.\bar{B}}\mathfrak{h}_y^B(\mathfrak{e}_y(t)) + \sum_{x=1}^{n}\sum_{A} sign(\mathfrak{e}_x^A(t))$$

$$\times\sum_{y=1}^{n}\sum_{B} b_{xy}^{A.\bar{B}}\mathfrak{h}_y^B(\mathfrak{e}_y(t-\tau_y(t))) + \sum_{x=1}^{n}\sum_{A} sign(\mathfrak{e}_x^A(t))$$

$$\times\sum_{y=1}^{n}\sum_{B} c_{xy}^{A.\bar{B}}\int_{-\infty}^{t}\theta_{xy}(t-s)\mathfrak{h}_y^B(\mathfrak{e}_y(s))ds$$

$$-\sum_{x=1}^{n}\sum_{A}\beta_x|\mathfrak{e}_x^A(t)| - \sum_{x=1}^{n}\sum_{A}\gamma_x\,|sign(\mathfrak{e}_x^A(t))|$$

$$+\sum_{x=1}^{n}\sum_{y=1}^{n}\sum_{A}\sum_{B}\frac{1}{1-\mu_y}|b_{xy}^{A.\bar{B}}|l_y|\mathfrak{e}_y^A(t)|$$

$$-\sum_{x=1}^{n}\sum_{y=1}^{n}\sum_{A}\sum_{B}\frac{1}{1-\mu_y}(1-\mu_y)|b_{xy}^{A.\bar{B}}|l_y$$

$$\times|\mathfrak{e}_y^A(t-\tau_y(t))| + \sum_{x=1}^{n}\sum_{y=1}^{n}\sum_{A}\sum_{B}|c_{xy}^{A.\bar{B}}|l_y\tilde{\theta}_{xy}|\mathfrak{e}_y^A(t)|$$

$$-\sum_{x=1}^{n}\sum_{y=1}^{n}\sum_{A}\sum_{B}|c_{xy}^{A.\bar{B}}|l_y\int_{-\infty}^{t}\theta_{xy}(t-s)|\mathfrak{e}_y^A(s)|ds$$

$$= -\sum_{x=1}^{n}\sum_{A} d_x|\mathfrak{e}_x^A(t)| + \sum_{x=1}^{n}\sum_{y=1}^{n}\sum_{A}\sum_{B}|a_{xy}^{A.\bar{B}}|l_y|\mathfrak{e}_y^A(t)|$$

$$+\sum_{x=1}^{n}\sum_{y=1}^{n}\sum_{A}\sum_{B}|b_{xy}^{A.\bar{B}}|l_y|\mathfrak{e}_y^A(t-\tau_y(t))|$$

$$+\sum_{x=1}^{n}\sum_{y=1}^{n}\sum_{A}\sum_{B}|c_{xy}^{A.\bar{B}}|l_y\int_{-\infty}^{t}\theta_{xy}(t-s)|\mathfrak{e}_y(s)|ds$$

$$-\sum_{x=1}^{n}\sum_{A}\beta_x|\mathfrak{e}_x^A(t)| - \sum_{x=1}^{n}\gamma_x\lambda_x$$

$$+\sum_{x=1}^{n}\sum_{y=1}^{n}\sum_{A}\sum_{B}\frac{1}{1-\mu_y}|b_{xy}^{A.\bar{B}}|l_y|\mathfrak{e}_y^A(t)|$$

$$-\sum_{x=1}^{n}\sum_{y=1}^{n}\sum_{A}\sum_{B}|b_{xy}^{A.\bar{B}}|l_y|\mathfrak{e}_y^A(t-\tau_y(t))|$$

$$+\sum_{x=1}^{n}\sum_{y=1}^{n}\sum_{A}\sum_{B}|c_{xy}^{A.\bar{B}}|l_y\tilde{\theta}_{xy}|\mathfrak{e}_y^A(t)|$$

$$-\sum_{x=1}^{n}\sum_{y=1}^{n}\sum_{A}\sum_{B}|c_{xy}^{A.\bar{B}}|l_y\int_{-\infty}^{t}\theta_{xy}(t-s)|\mathfrak{e}_y^A(s)|ds$$

where $\lambda_x = |sign(\mathfrak{e}_x^A(t))|$. Then, one has

$$\dot{\mathfrak{V}}(t) \leq \sum_{x=1}^{n}\sum_{A} -(d_x+\beta_x)|\mathfrak{e}_x^A(t)|$$

$$+\sum_{x=1}^{n}\sum_{y=1}^{n}\sum_{A}\sum_{B}|a_{xy}^{A.\bar{B}}|l_y|\mathfrak{e}_y^A(t)|$$

$$+\sum_{x=1}^{n}\sum_{y=1}^{n}\sum_{A}\sum_{B}|b_{xy}^{A.\bar{B}}|l_y|\mathfrak{e}_y^A(t-\tau_y(t))|$$

$$+\sum_{x=1}^{n}\sum_{y=1}^{n}\sum_{A}\sum_{B}|c_{xy}^{A.\bar{B}}|l_y\int_{-\infty}^{t}\theta_{xy}(t-s)|\mathfrak{e}_y(s)|ds$$

$$-\sum_{x=1}^{n}\gamma_x\lambda_x + \sum_{x=1}^{n}\sum_{y=1}^{n}\sum_{A}\sum_{B}\frac{1}{1-\mu_y}|b_{xy}^{A.\bar{B}}|l_y|\mathfrak{e}_y^A(t)|$$

$$-\sum_{x=1}^{n}\sum_{y=1}^{n}\sum_{A}\sum_{B}|b_{xy}^{A.\bar{B}}|l_y|\mathfrak{e}_y^A(t-\tau_y(t))|$$

$$+\sum_{x=1}^{n}\sum_{y=1}^{n}\sum_{A}\sum_{B}|c_{xy}^{A.\bar{B}}|l_y\tilde{\theta}_{xy}|\mathfrak{e}_y^A(t)|$$

$$-\sum_{x=1}^{n}\sum_{y=1}^{n}\sum_{A}\sum_{B}|c_{xy}^{A.\bar{B}}|l_y\int_{-\infty}^{t}\theta_{xy}(t-s)|\mathfrak{e}_y^A(s)|ds$$
$$=\sum_{x=1}^{n}\sum_{A}-(d_x+\beta_x)|\mathfrak{e}_x^A(t)|$$
$$+\sum_{x=1}^{n}\sum_{y=1}^{n}\sum_{A}\sum_{B}|a_{xy}^{A.\bar{B}}|l_y|\mathfrak{e}_y^A(t)|$$
$$-\sum_{x=1}^{n}\gamma_x\lambda_x+\sum_{x=1}^{n}\sum_{y=1}^{n}\sum_{A}\sum_{B}\frac{1}{1-\mu_y}|b_{xy}^{A.\bar{B}}|l_y|\mathfrak{e}_y^A(t)|$$
$$+\sum_{x=1}^{n}\sum_{y=1}^{n}\sum_{A}\sum_{B}|c_{xy}^{A.\bar{B}}|l_y\tilde{\theta}_{xy}|\mathfrak{e}_y^A(t)|$$
$$=\sum_{x=1}^{n}\sum_{A}\Bigg[-(d_x+\beta_x)+\sum_{y=1}^{n}\sum_{B}l_x|a_{yx}^{A.\bar{B}}|+\frac{1}{1-\mu_x}\sum_{y=1}^{n}\sum_{B}l_x|b_{yx}^{A.\bar{B}}|+\sum_{y=1}^{n}\sum_{B}l_x|c_{yx}^{A.\bar{B}}|\tilde{\theta}_{yx}\Bigg]|\mathfrak{e}_x^A(t)|-\sum_{x=1}^{n}\gamma_x\lambda_x. \quad (12)$$

By applying (9) into (12), one can obtain:

$$\dot{\mathfrak{V}}(t)\leq-\sum_{x=1}^{n}\gamma_x\lambda_x. \quad (13)$$

When $\|\mathfrak{e}(t)\|_1=0$, at least one index $x\in N$ exists such that $\lambda_x=1$. As such, one has $\sum_{x=1}^{n}\lambda_x=1$, and

$$\dot{\mathfrak{V}}(t)\leq-\gamma<0. \quad (14)$$

where $\gamma=min\{\gamma_x\}$, $x\in N$. Integrating both sides of (14) from 0 to t gives

$$\mathfrak{V}(t)-\mathfrak{V}(0)\leq\ -\gamma t. \quad (15)$$

Consider $|\mathfrak{e}_x^A(t_1)|=0$ at instant $t_1\in(0,+\infty)$ for $x\in N$, one can analyze further with respect to (15). If $\|\mathfrak{e}_x^A(t_1)\|>0$ for all $t\in[0,+\infty)$, then $\sum_{x=1}^{n}\lambda_x=1$, $\gamma<0$ for all $t\in[0,+\infty)$. Under this scenario, one can interpret inequality (15) to have $\lim_{t\to t_1}\mathfrak{V}(t)=-\infty$, which contracts $\mathfrak{V}(t)\geq0$. As a result, a nonnegative constant $\mathfrak{V}^*$ exists and $t_1\in(0,+\infty)$ such as

$$\lim_{t\to t_1}=\mathfrak{V}^*\text{ and }\mathfrak{V}(t)\equiv\mathfrak{V}^*,\ \forall t\geq t_1. \quad (16)$$

One can subsequently indicate that

$$\|\mathfrak{e}^A(t_1)\|_1=0\text{ and }\|\mathfrak{e}^A(t_1)\|_1\equiv0,\quad\forall t\geq t_1. \quad (17)$$

Firstly, one can prove that $\|\mathfrak{e}^A(t_1)\|_1=0$; otherwise $\|\mathfrak{e}^A(t_1)\|_1>0$. In this respect, there exists a small constant $\epsilon>0$ where $\|\mathfrak{e}^A(t_1)\|_1>0$ for all $t\in[t_1,t_1+\epsilon]$. Hence, at least one $q_0\in N$ exists such that $|\mathfrak{e}_{q_0}^A(t)|>0$ for $t\in[t_1,t_1+\epsilon]$, resulting in $\dot{\mathfrak{V}}(t)\leq-\gamma_{q_0}<0$ which holds for all $t\in[t_1,t_1+\epsilon]$. This contradicts (16).

Secondly, one can prove $\|\mathfrak{e}^A(t)\|_1\equiv0$ for $t\geq t_1$. In contradiction and without loss of generality, suppose that there exists at least one $q_0\in N$ and $t_2>t_1$ such that $|\mathfrak{e}_{q_0}^A(t_2)|_1>0$. Let $t_s=sup\{t\in[t_1,t_2]:\|\mathfrak{e}^A(t)\|_1=0\}$, one can have $t_s<t_2$, $\|\mathfrak{e}^A(t_s)\|_1=0$ and $|\mathfrak{e}_{q_0}^A(t)|_1>0$ for all $t\in(t_s,t_2]$. In addition, there exists $t_3\in(t_s,t_2]$ in which $|\mathfrak{e}_{q_0}^A(t)|$ is increasing in monotonous manner subject to the interval $[t_s,t_3]$. Hence, $\mathfrak{V}(t)$ also increases in a monotonous manner subject to $[t_s,t_3]$, i.e., $\dot{\mathfrak{V}}(t)>0$ for $t\in(t_s,t_3]$. In addition, based on the explanation in the first part, one can observe that $\dot{\mathfrak{V}}(t)\leq-\gamma<0$ holds for all $t\in[t_s,t_3]$, resulting in a contradiction outcome. Hence, $\|\mathfrak{e}^A(t)\|_1\equiv0$ for $t\geq t_1$.

In short, one can see that the conditions in (17) hold. In view of Definition (1), NN (6) is synchronized with NN (5) in finite time under controller (8). The proof is completed.

Remark 6: When the impulsive effect is absent, (7) reduces to:

$$\begin{cases}\dot{\mathfrak{e}}_x^A(t)=-d_x\mathfrak{e}_x^A(t)+\sum_{y=1}^{n}\sum_{B}a_{xy}^{A.\bar{B}}\mathfrak{h}_y^B(\mathfrak{e}_y(t))\\ \quad+\sum_{y=1}^{n}\sum_{B}b_{xy}^{A.\bar{B}}\mathfrak{h}_y^B(\mathfrak{e}_y(t-\tau_y(t)))\\ \quad+\sum_{y=1}^{n}\sum_{B}c_{xy}^{A.\bar{B}}\int_{-\infty}^{t}\theta_{xy}(t-s)\mathfrak{h}_y^B(\mathfrak{e}_y(s))ds\\ \quad+\mathfrak{u}_x^A(t),\quad t\geq0,\\ \mathfrak{e}_x^A(t)=\psi_x^A(t)\in\mathcal{C}((-\infty,0],\mathbb{R}^{2^m n}).\end{cases} \quad (18)$$

By applying the similar procedure in Theorem (1), the finite-time synchronization criteria pertaining to (18) can be obtained, as follows.

Corollary 1: Assume that ***(A1)-(A3)*** *are valid, given positive constants β_x,γ_x, $x\in N$, and the impulsive operator $\alpha_{xk}(\mathfrak{e}^A(t_k^-))$ satisfies $\alpha_{xk}(\mathfrak{e}^A(t_k^-))=0$, and the following inequality exists*

$$d_x+\beta_x-\Bigg(\sum_{y=1}^{n}\sum_{A}\sum_{B}l_x|a_{yx}^{A.\bar{B}}|+\sum_{y=1}^{n}\sum_{A}\sum_{B}\frac{1}{1-\mu_x}l_x|b_{yx}^{A.\bar{B}}|+\sum_{y=1}^{n}\sum_{A}\sum_{B}l_x|c_{yx}^{A.\bar{B}}|\tilde{\theta}_{yx}\Bigg)>0,\quad x\in N, \quad (19)$$

then the model in (18) is fixed-time synchronization under controller (8).

Remark 7: When $c_{xy}^{A.\bar{B}}=0$, model (7) reduces to:

$$\begin{cases}\dot{\mathfrak{e}}_x^A(t)=-d_x\mathfrak{e}_x^A(t)+\sum_{y=1}^{n}\sum_{B}a_{xy}^{A.\bar{B}}\mathfrak{h}_y^B(\mathfrak{e}_y(t))\\ \quad+\sum_{y=1}^{n}\sum_{B}b_{xy}^{A.\bar{B}}\mathfrak{h}_y^B(\mathfrak{e}_y(t-\tau_y(t)))\\ \quad+\mathfrak{u}_x^A(t),\ t\geq0,\ t\neq t_k,\\ \Delta\mathfrak{e}_x^A(t)=\mathfrak{e}_x^A(t_k)-\mathfrak{e}_x^A(t_k^-)=\alpha_{xk}\mathfrak{e}_x^A(t_k^-),\ t=t_k,\\ \mathfrak{e}_x^A(t)=\psi_x^A(t)\in\mathcal{C}([-\tau,0],\mathbb{R}^{2^m n}).\end{cases} \quad (20)$$

By applying the similar procedure in Theorem (1), the finite-time synchronization criteria pertaining to (20) can be obtained, as follows.

*Corollary 2: Assume that **(A1)-(A2)** are valid, given positive constants $\beta_x, \gamma_x,\ x \in N$, and the impulsive operator $\alpha_{xk}(\mathfrak{e}^A(t_k^-))$ satisfies $\alpha_{xk}(\mathfrak{e}^A(t_k^-)) = -\vartheta_{xk}(\mathfrak{e}^A(t_k^-)),\ x \in N$, $k = 1, 2, \ldots$. If $|1-\vartheta_{xk}| \leq 1$, and the following inequality exists*

$$d_x + \beta_x - \Bigg(\sum_{y=1}^{n}\sum_{A}\sum_{B} l_x|a_{yx}^{A.\bar{B}}| + \sum_{y=1}^{n}\sum_{A}\sum_{B}\frac{1}{1-\mu_x} l_x|b_{yx}^{A.\bar{B}}|\Bigg) > 0,\ x \in N, \quad (21)$$

then the model in (20) is finite-time synchronization under controller (8).

Remark 8: Note that the term sign(·) in controller (8) is discontinuity and it may not be applicable in some practical cases. Moreover, it can induce chattering phenomenon. As such, one can modify controller (8) to overcome the chattering issue, as follows

$$\mathfrak{u}_x^A(t) = -\beta_x \mathfrak{e}_x^A(t) - \frac{\gamma_x \mathfrak{e}_x^A(t)}{|\mathfrak{e}_x^A(t)| + \delta_x}$$

where β_x and γ_x for $x \in N$ are the control gains, while $\delta_x,\ x \in N$ are sufficiently small positive constants.

Remark 9: The dynamical analysis of Clifford-valued NNs is challenging because of the non-communicative principle of multiplication in Clifford numbers. To overcome this issue, the decomposition method is useful. This is evident in many existing studies that exploit the decomposition method to transform Clifford-valued NNs to real-valued NNs [29], [30], [33], [36], [37].

Remark 10: Clifford-valued NNs offer new capabilities to undertake problems that the real-valued, complex-valued, quaternion-valued counterparts fail to solve. Indeed, the results in [44] and [48] on fixed-time synchronization of quaternion-valued NNs and finite-time synchronization of complex-valued NNs, respectively, can be considered as special cases of the results in this paper.

Remark 11: Recently, various dynamics of Clifford-valued NNs can be found in the literature, e.g. [28]–[37] that present global exponential stability of pseudo almost periodic solution, globally asymptotic almost automorphic synchronization S^p-Almost periodic solutions, and other stability analysis. Nonetheless, our review reveals limited studies on finite synchronization of Clifford-valued NNs subject to both impulse effects and infinite distributed delays, and this work aims to fill this gap. Comparing with the existing studies, we contribute new results in this work.

IV. NUMERICAL EXAMPLE

The following example indicates the usefulness of our results.

Example 1: The two-neuron drive model in (1) with $m = 2$ and $n = 2$ is considered, i.e.,

$$\begin{cases} \dot{\mathfrak{r}}_x(t) = -d_x\mathfrak{r}_x(t) + \sum_{y=1}^{2} a_{xy}\mathfrak{h}_y(\mathfrak{r}_y(t)) \\ \qquad + \sum_{y=1}^{2} b_{xy}\mathfrak{h}_y(\mathfrak{r}_y(t-\tau_y(t))) \\ \qquad + \sum_{y=1}^{2} c_{xy}\int_{-\infty}^{t}\theta_{xy}(t-s)\mathfrak{h}_y(\mathfrak{r}_y(s))ds \\ \qquad + \mathfrak{k}_x,\ t \geq 0,\ t \neq t_k, \\ \Delta\mathfrak{r}_x(t) = \mathfrak{r}_x(t_k) - \mathfrak{r}_x(t_k^-) = \alpha_{xk}\mathfrak{r}_x(t_k),\ t = t_k, \\ \mathfrak{r}_x(t) = \varphi_x(t) \in \mathcal{C}((-\infty, 0], \mathbb{A}),\ x = 1, 2. \end{cases}$$

As such, one has the corresponding response model in (2)

$$\begin{cases} \dot{\mathfrak{s}}_x(t) = -d_x\mathfrak{s}_x(t) + \sum_{y=1}^{2} a_{xy}\mathfrak{h}_y(\mathfrak{s}_y(t)) \\ \qquad + \sum_{y=1}^{2} b_{xy}\mathfrak{h}_y(\mathfrak{s}_y(t-\tau_y(t))) \\ \qquad + \sum_{y=1}^{2} c_{xy}\int_{-\infty}^{t}\theta_{xy}(t-s)\mathfrak{h}_y(\mathfrak{s}_y(s))ds \\ \qquad + \mathfrak{k}_x + \mathfrak{u}_x(t),\ t \geq 0,\ t \neq t_k, \\ \Delta\mathfrak{s}_x(t) = \mathfrak{s}_x(t_k) - \mathfrak{s}_x(t_k^-) = \alpha_{xk}\mathfrak{s}_x(t_k),\ t = t_k, \\ \mathfrak{s}_x(t) = \phi_x(t) \in \mathcal{C}((-\infty, 0], \mathbb{A}),\ x = 1, 2. \end{cases}$$

The multiplication generators are: $e_1^2 = e_2^2 = e_{12}^2 = e_1e_2e_{12} = -1$, $e_1e_2 = -e_2e_1 = e_{12}$, $e_1e_{12} = -e_{12}e_1 = -e_2$, $e_2e_{12} = -e_{12}e_2 = e_1$, $\mathfrak{r}_1 = \mathfrak{r}_1^0e_0 + \mathfrak{r}_1^1e_1 + \mathfrak{r}_1^2e_2 + \mathfrak{r}_1^{12}e_{12}$, $\mathfrak{r}_2 = \mathfrak{r}_2^0e_0 + \mathfrak{r}_2^1e_1 + \mathfrak{r}_2^2e_2 + \mathfrak{r}_2^{12}e_{12}$, $\mathfrak{s}_1 = \mathfrak{s}_1^0e_0 + \mathfrak{s}_1^1e_1 + \mathfrak{s}_1^2e_2 + \mathfrak{s}_1^{12}e_{12}$, $\mathfrak{s}_2 = \mathfrak{s}_2^0e_0 + \mathfrak{s}_2^1e_1 + \mathfrak{s}_2^2e_2 + \mathfrak{s}_2^{12}e_{12}$.

Furthermore, one can take

$$\mathfrak{D} = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix},$$

$$\mathfrak{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 0.5e_0 + 0.1e_1 & 0.1e_0 + 0.2e_2 + 0.6e_{12} \\ 0.5e_0 - 0.1e_1 + 0.3e_2 & 0.3e_0 + 0.1e_1 + 0.5e_{12} \end{bmatrix},$$

$$\mathfrak{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} 0.1e_0 + 0.2e_1 + 0.5e_2 & 0.3e_0 + 0.1e_2 + 0.4e_{12} \\ 0.6e_0 - 0.2e_1 + 0.3e_2 & 0.4e_0 + 0.1e_{12} \end{bmatrix},$$

$$\mathfrak{C} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} 0.1e_0 + 0.2e_2 + 0.6e_{12} & 0.3e_0 + 0.1e_1 + 0.5e_{12} \\ 0.1e_0 + 0.2e_1 + 0.5e_2 & 0.6e_0 - 0.2e_1 + 0.3e_2 \end{bmatrix},$$

$$\mathfrak{K} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0.1e_0 - 0.2e_1 + 0.2e_2 + 0.1e_{12} \\ -0.2e_0 + 0.2e_1 + 0.1e_2 + 0.1e_{12} \end{bmatrix},$$

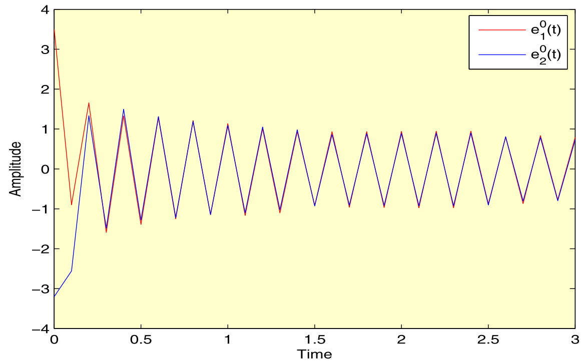

FIGURE 1. Synchronization curves of state variables $\mathfrak{e}_1^0(t)$, $\mathfrak{e}_2^0(t)$ of (7) under controller (8) without impulsive effects.

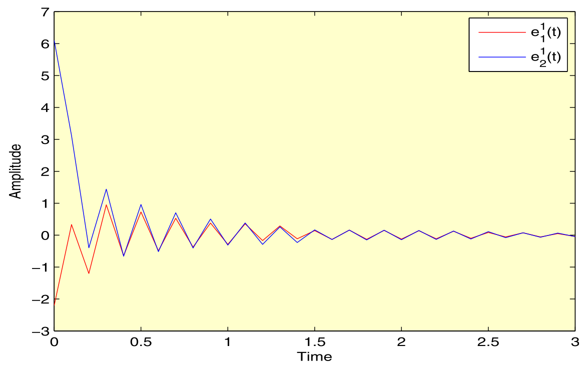

FIGURE 2. Synchronization curves of state variables $\mathfrak{e}_1^1(t)$, $\mathfrak{e}_2^1(t)$ of (7) under controller (8) without impulsive effects.

$$\mathfrak{h}_1(\mathfrak{e}_1) = \frac{1-e^{-\mathfrak{e}_1^0}}{1+e^{-\mathfrak{e}_1^0}}e_0 + \frac{1}{1+e^{-\mathfrak{e}_1^1}}e_1 + \frac{1-e^{-\mathfrak{e}_1^2}}{1+e^{-\mathfrak{e}_1^2}}e_2 + \frac{1}{1+e^{-\mathfrak{e}_1^{12}}}e_{12},$$

$$\mathfrak{h}_2(\mathfrak{e}_2) = \frac{1-e^{-\mathfrak{e}_2^0}}{1+e^{-\mathfrak{e}_2^0}}e_0 + \frac{1}{1+e^{-\mathfrak{e}_2^1}}e_1 + \frac{1-e^{-\mathfrak{e}_2^2}}{1+e^{-\mathfrak{e}_2^2}}e_2 + \frac{1}{1+e^{-\mathfrak{e}_2^{12}}}e_{12},$$

in which $\mathfrak{e}_x(t) = \mathfrak{s}_x(t) - \mathfrak{r}_x(t)$ and $\mathfrak{e}_x^A(t) = \mathfrak{s}_x^A(t) - \mathfrak{r}_x^A(t)$, $x = 1, 2$. For further analysis, the following time-varying delays are considered: $\tau_1(t) = \tau_2(t) = 0.6|cos(t)| + 0.4$, $\alpha_{1k} = \alpha_{2k} = -0.8$, $\theta_{xy}(t) = e^{-0.5t}$, $x, y = 1, 2$. As such, one can easily obtain $\tau_1 = \tau_2 = 1$, $\mu_1 = \mu_2 = 0.6 < 1$ and $\tilde{\theta}_{xy} = 2$, $x, y = 1, 2$. Assumption (**A2**) holds for the activation function with $l_1 = l_2 = 0.5$. Select $\beta_1 = 2.5$, $\beta_2 = 2.6$, $\gamma_1 = 3.5$ and $\gamma_2 = 3.8$.

Besides, it is easy to obtain $d_1 = 5$, $d_2 = 5$, $a_{11}^{A.\bar{B}} = 0.6$, $a_{12}^{A.\bar{B}} = 0.9$, $a_{21}^{A.\bar{B}} = 0.7$, $a_{22}^{A.\bar{B}} = 0.9$, $b_{11}^{A.\bar{B}} = 0.8$, $b_{12}^{A.\bar{B}} = 0.8$, $b_{21}^{A.\bar{B}} = 0.7$, $b_{22}^{A.\bar{B}} = 0.5$, $c_{11}^{A.\bar{B}} = 0.9$, $c_{12}^{A.\bar{B}} = 0.9$, $c_{21}^{A.\bar{B}} = 0.8$, $c_{22}^{A.\bar{B}} = 0.7$. The initial conditions pertaining to the drive-response representations in (1) and (2) are: $\varphi_1(t) = -1.5e_0 + 1.2e_1 + 0.9e_2 + 0.5e_{12}$ for $t \in [-1, 0]$, $\varphi_2(t) = 1.6e_0 - 3.5e_1 + 2.2e_2 - 0.9e_{12}$ for $t \in [-1, 0]$, $\phi_1(t) = 2.5e_0 - e_1 + 1.5e_2 - 1.2e_{12}$ for $t \in [-1, 0]$, and $\phi_2(t) = -2.6e_0 - 2.6e_1 - 2.2e_2 + 0.8e_{12}$ for $t \in [-1, 0]$.

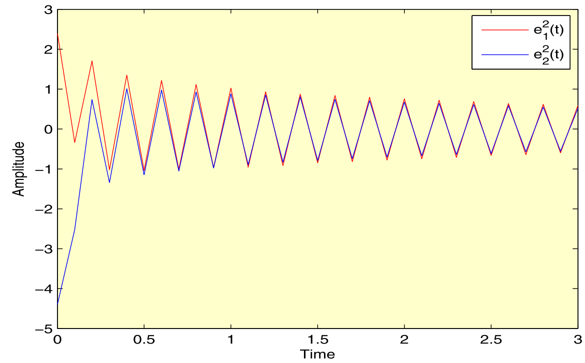

FIGURE 3. Synchronization curves of state variables $\mathfrak{e}_1^2(t)$, $\mathfrak{e}_2^2(t)$ of (7) under controller (8) without impulsive effects.

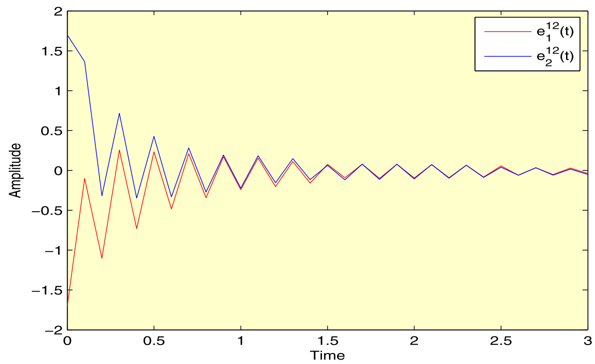

FIGURE 4. Synchronization curves of state variables $\mathfrak{e}_1^{12}(t)$, $\mathfrak{e}_2^{12}(t)$ of (7) under controller (8) without impulsive effects.

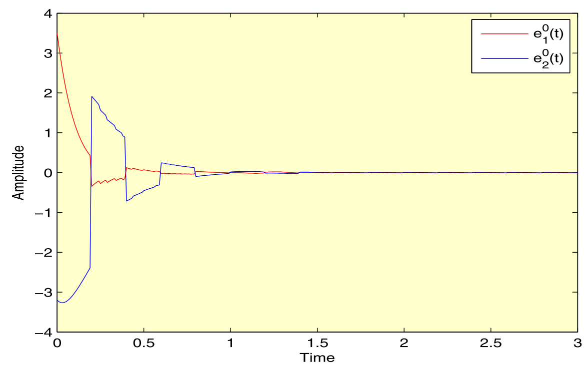

FIGURE 5. Synchronization curves of state variables $\mathfrak{e}_1^0(t)$, $\mathfrak{e}_2^0(t)$ of (7) under controller (8) with impulsive effects.

Through some straightforward computation, one can obtain

$$d_1 + \beta_1 - \Bigg(\sum_{y=1}^{2}\sum_A\sum_B l_1|a_{y1}^{A.\bar{B}}| + \sum_{y=1}^{2}\sum_A\sum_B \frac{1}{1-\mu_1}l_1 \times|b_{y1}^{A.\bar{B}}| + \sum_{y=1}^{2}\sum_A\sum_B l_1|c_{y1}^{A.\bar{B}}|\tilde{\theta}_{y1}\Bigg) = 3.275 > 0,$$

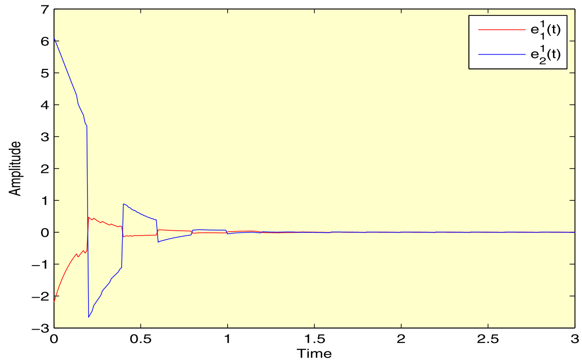

FIGURE 6. Synchronization curves of state variables $e_1^1(t)$, $e_2^1(t)$ of (7) under controller (8) with impulsive effects.

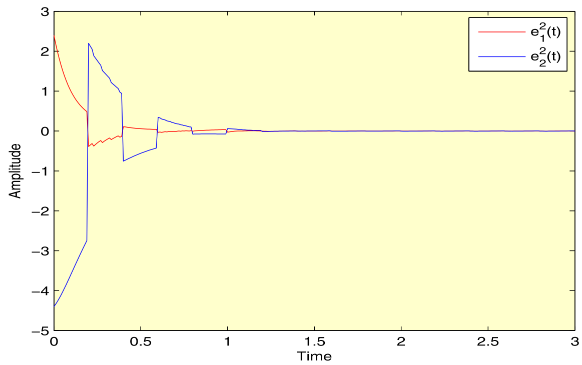

FIGURE 7. Synchronization curves of state variables $e_1^2(t)$, $e_2^2(t)$ of (7) under controller (8) with impulsive effects.

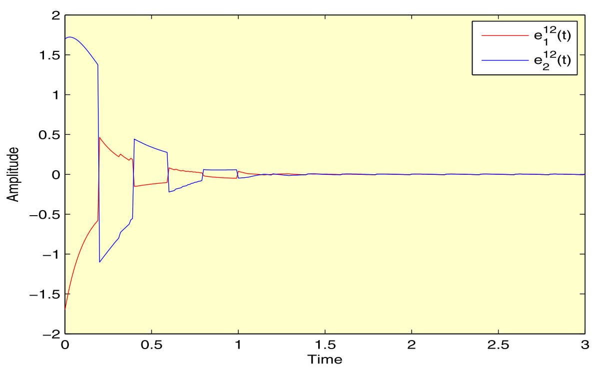

FIGURE 8. Synchronization curves of state variables $e_1^{12}(t)$, $e_2^{12}(t)$ of (7) under controller (8) with impulsive effects.

$$d_2 + \beta_2 - \Bigg(\sum_{y=1}^{2} \sum_{A} \sum_{B} l_2 |a_{y2}^{A.\bar{B}}| + \sum_{y=1}^{2} \sum_{A} \sum_{B} \frac{1}{1-\mu_2} l_2 \times |b_{y2}^{A.\bar{B}}| + \sum_{y=1}^{2} \sum_{A} \sum_{B} l_2 |c_{y2}^{A.\bar{B}}| \tilde{\theta}_{y2} \Bigg) = 3.475 > 0.$$

All conditions pertaining to Theorem (1) hold. As such, under controller (8), finite-time synchronization can be achieved for the drive-response representations in (1) and (2). The synchronization curves of state variables of (7) without the impulsive effects are depicted in Figures (1), (2), (3) and (4) indicates. Meanwhile, synchronization curves of state variables of (7) with impulsive effects are displayed in Figures (5), (6), (7) and (8). From the results in Figures (1)-(8), NN representations in (1) and (2) are able to synchronize in finite-time subject to controller (8) based on the given initial conditions.

V. CONCLUSION

A dynamical analysis on finite-time synchronization of Clifford-valued NNs subject to impulse effects, infinite distributed delays and discrete-time-varying delays has been presented in this paper. The decomposition method has been adopted to transform Clifford-valued drive and response model into the corresponding real-valued counterparts. This is performed to overcome the non-commutativity of multiplication of Clifford numbers. To analyze the synchronization issue of the error model, a suitable controller has been designed. The finite-time synchronization criteria of the resulting real-valued counterparts have been analyzed by formulating a new LKF candidate and utilizing new computational techniques. To demonstrate the usefulness of the obtained results, a numerical example has been presented, along with the simulation results. For further work, the proposed method will be extended to analyze other types of Clifford-valued NNs subject to different types of time delays and controllers.

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