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Research Article

Comparative Study on Numerical Methods for Singularly Perturbed Advanced-Delay Differential Equations

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In this paper, we consider a class of singularly perturbed advanced-delay differential equations of convection-diffusion type. We use finite and hybrid difference schemes to solve the problem on piecewise Shishkin mesh. We have established almost first- and second-order convergence with respect to finite difference and hybrid difference methods. An error estimate is derived with the discrete norm. In the end, numerical examples are given to show the advantages of the proposed results (Mathematics Subject Classification: 65L11, 65L12, and 65L20).

1. Introduction

Differential equations depend both on past and future values (mixed delay) called functional differential equations. It attains many application problems such as optimal control problems [1], nerve conduction theory [2], economic dynamics [3], traveling waves in a spatial lattice [4] and has discussed both linear and nonlinear functional differential equations.

The functional differential equation has been multiplied by small parameter $(0 < \varepsilon < 1)$ in the highest order derivative term called the singularly perturbed mixed delay differential equations. The main determination for such a problem is the study of biological science, epidemics, and population [5–10].

The authors in [11] have considered functional differential equation in singularly perturbed problems, such as

$$\left(\frac{\sigma^2}{2}\right)y''(x) + (\mu - x)y'(x) + \lambda_E y(x + a_E) + \lambda_I y(x - a_I) - (\lambda_E + \lambda_I)y(x) = -1, \tag{1}$$

and considered the problem of determining the expected time for the generation of action potentials in nerve cells by random synaptic inputs in the dendrites. The general linear second-order functional differential equation with the boundary-value problem arises in the modeling of neuron activation, where σ and μ are the variance and drift parameters and y is the expected first-exit time. The first-order derivative term $-xy'(x)$ corresponds to exponential decay between synaptic and inputs. The undifferentiated terms correspond to excitatory and inhibitory synaptic inputs modeled as a Poisson process with mean rates λ_E and λ_I; they produce jumps in the membrane potential of amounts a_E

and a_I, which are small quantities and could depend on the voltage. The boundary condition is

$$y(x) = 0, \quad \forall x \notin (x_1, x_2), \tag{2}$$

where the values $x = x_1$ and $x = x_2$ correspond to inhibitory reversal potential and the threshold value of membrane potential for action potential generation. This biological problem motivates the investigation of boundary-value problems for differential-difference equations with mixed shifts. In this biological model, using the Taylor series for the small delay term, provided the delay is of order ε, the small delay problem has oscillatory solution that has been discussed in [12]. The same authors discussed the signal transmission problem in [13].

The authors in [14, 15] have considered the singularly perturbed problem with derivative depending on small delay term such as

$$\begin{aligned}&\varepsilon y''(t) + a(t)y'(t) + b(t)y(t)\\ &\quad + c(t)y(t-\tau) = f(t), \quad \text{where } 0 < \tau \ll 1,\end{aligned} \tag{3}$$

to solve the boundary-value problem using the following numerical method such as the finite difference scheme [14, 16], fitted mesh B-spline collocation method [17], and hybrid difference scheme [15].

The authors in [18, 19] investigated various concepts of singularly perturbed differential equation with derivative depending on both past and future small variables,

$$\begin{aligned}&\varepsilon y''(t) + a(t)y'(t) + b(t)y(t) + c(t)y(t-\tau)\\ &\quad + \mathrm{d}(t)y(t+\tau) = f(t), \quad \text{where } 0 < \tau \ll 1,\end{aligned} \tag{4}$$

also proposed a finite difference scheme to solve singular perturbation problems in [18, 20, 21].

The authors in [19] have been proposed to solve the singular perturbation problem with mixed small shifts using the fitted operator method. In recent years, the authors in [22–25] considered singular perturbation problem with derivative depending on large delay $(\tau = 1)$ variable, such as

$$\varepsilon y''(t) + a(t)y'(t) + b(t)y(t) + c(t)y(t-1) = f(t), \tag{5}$$

has been developed various numerical schemes are finite and hybrid difference method [22], iterative scheme [26], finite element method [27, 28]. The study in [23] proposed solving singularly perturbed delay differential equation with integral boundary condition using finite difference method.

Throughout the literature, the researcher concentrates on solving the singular perturbation problem with a small delay or mixed small delay or large delay using finite or hybrid or finite element methods on uniform meshes or nonuniform mesh. To the best of the author's knowledge, up to now, no theoretical results are given for comparative study on numerical methods for singularly perturbed advanced-delay differential equations. Moreover, we proposed two numerical methods such as the finite and hybrid difference scheme on nonuniform meshes, to solve the singular perturbation problem with mixed large delay using the finite difference scheme and hybrid difference scheme on Shishkin mesh.

This paper is structured as follows: Section 2 describes the problem statement. Section 3 proves the maximum principle and stability result. Moreover, it introduces the terminology for Shishkin decomposition and proves many inequalities. In Section 4, we introduce the numerical methods to discretize the continuous problem. Error analysis for finite and hybrid difference scheme approximate solution is given in Sections 5 and 6. Finally, Section 7 presents numerical results.

Throughout our analysis, we use the following notations: $\overline{\Gamma} = [0,3]$, $\Gamma = (0,3)$, $\Gamma_1 = (0,1)$, $\Gamma_2 = (1,2)$, $\Gamma_3 = (2,3)$, $\Gamma^* = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. $\overline{\Gamma}^{3N} = \{0,1,2,\ldots,3N\}$, $\Gamma_1^{3N} = \{1,2,\ldots,N-1\}$, $\Gamma_2^{3N} = \{N+1,\ldots,2N-1\}$, $\Gamma_3^{3N} = \{2N+1,\ldots,3N-1\}$. The parameter ε and mesh points $3N$ are independent of C and C_1 are positive constants. The norm is $\|y\|_\Gamma = \sup_{r\in\Gamma}|y(r)|$.

2. Statement of the Problem

Consider the following singularly perturbed mixed delay differential equation:

$$\begin{cases}\mathcal{K}y(r) = -\varepsilon y''(r) + a(r)y'(r) + b(r)y(r) + c(r)y(r-1) + d(r)y(r+1) = f(r), & r \in \Gamma,\\ y(r) = \phi(r), & r \in [-1,0],\\ y(r) = \varphi(r), & r \in [3,4],\end{cases} \tag{6}$$

where $\phi(r)$ and $\varphi(r)$ are history function on $[-1,0]$ and $[3,4]$. Assume that $a(r) \geq \alpha_1 > \alpha > 0$, $b(r) \geq \beta \geq 0$, $\gamma \leq c(r) \leq 0$, $d(r) \geq \eta \geq 0$, $\beta + \gamma \geq \beta_1 > 0$, and the coefficients are smooth function on $r \in \overline{\Gamma}$. The above problem solution satisfies $y(r) \in G = C^0(\overline{\Gamma}) \cap C^1(\Gamma) \cap C^2(\Gamma^*)$. Problem (1) is rewritten as $\mathcal{K}y(r) = g(r)$, where

$$\mathscr{K}y(r)=\begin{cases}\mathscr{K}_1y(r)=-\varepsilon y''(r)+a(r)y'(r)+b(r)y(r)+d(r)y(r+1), & r\in\Gamma_1,\\ \mathscr{K}_2y(r)=-\varepsilon y''(r)+a(r)y'(r)+b(r)y(r)+c(r)y(r-1)+d(r)y(r+1), & r\in\Gamma_2,\\ \mathscr{K}_3y(r)=-\varepsilon y''(r)+a(r)y'(r)+b(r)y(r)+c(r)y(r-1), & r\in\Gamma_3,\end{cases}\tag{7}$$

$$g(r)=\begin{cases}f(r)-c(r)\phi(r-1), & r\in\Gamma_1,\\ f(r), & r\in\Gamma_2,\\ f(r)-d(r)\varphi(r+1), & r\in\Gamma_3,\end{cases}\tag{8}$$

with boundary conditions

$$\begin{cases}y(r)=\phi(r), & r\in[-1,0],\\ y(1^-)=y(1^+), & y'(1^-)=y'(1^+),\\ y(2^-)=y(2^+), & y'(2^-)=y'(2^+),\\ y(r)=\varphi(r), & r\in[3,4].\end{cases}\tag{9}$$

3. Analytical Results

Lemma 1 (maximum principle). *If $y(r)\in G$ such that $y(0)\geq 0$, $y(3)\geq 0$, $\mathscr{K}_1y(r)\geq 0$, $\forall r\in\Gamma_1$, $\mathscr{K}_2y(r)\geq 0$, $\forall r\in\Gamma_2$, $\mathscr{K}_3y(r)\geq 0$, $\forall r\in\Gamma_3$, $[y'](1)\leq 0$, and $[y'](2)\leq 0$, then $y(r)\geq 0$, $\forall r\in\overline{\Gamma}$.*

Proof. Let

$$s(r)=\begin{cases}\dfrac{1}{12}+\dfrac{r}{4}, & r\in[0,1],\\ \dfrac{2}{12}+\dfrac{r}{6}, & r\in[1,2],\\ \dfrac{4}{12}+\dfrac{r}{12}, & r\in[2,3].\end{cases}\tag{10}$$

Clearly, $s(r)>0,\forall x\in\overline{\Gamma}$, $\mathscr{K}s(r)>0,\forall r\in\Gamma^*$, $s(0)>0$, $s(3)>0$, $[s'](1)<0$, and $[s'](2)<0$. Consider that $\mu=\max\{((-\psi(r))/s(r))\colon r\in\overline{\Gamma}\}$; then, there exists $r_0\in\overline{\Gamma}$ such that $\psi(r_0)+\mu s(r_0)=0$ and $\psi(r)+\mu s(r)\geq 0,\forall r\in\overline{\Gamma}$ implies that $(\psi+\mu s)$ obtain minimum at $t=r_0$. If $\mu<0$, then $\psi(r)\geq 0$.

If $\mu>0$, then the function $\psi(r)$ nonnegative is not possible. The following cases are easy to prove the contradiction if $\mu>0$.

Case (i): $r_0=0$:

$$0<(\psi+\mu s),\quad (0)=\psi(0)+\mu s(0)=0.\tag{11}$$

Case (ii): $r_0\in\Gamma_1$:

$$0<\mathscr{K}_1(\psi+\mu s)(r_0)\leq 0.\tag{12}$$

Case (iii): $r_0=1$:

$$0\leq[(\psi+\mu s)'](1)=[\psi'](1)+\mu[s'](1)<0.\tag{13}$$

Case (iv): $r_0\in\Gamma_2$:

$$0<\mathscr{K}_2(\psi+\mu s)(r_0)\leq 0.\tag{14}$$

Case (v): $r_0=2$:

$$0\leq[(\psi+\mu s)'](2)=[\psi'](2)+\mu[s'](2)<0.\tag{15}$$

Case (vi): $r_0\in\Gamma_3$:

$$0<\mathscr{K}_3(\psi+\mu s)(r_0)\leq 0.\tag{16}$$

Case (vii): $r_0=3$:

$$0\leq(\psi+\mu s)(3)=(\psi)(3)+\mu(s)(3)=0.\tag{17}$$

All the cases are contradiction.

Lemma 2 (stability result). *If $y(r)$ is a solution of problems (7)–(9), then*

$$|y(r)|\leq C\max\left\{|y(0)|,|y(3)|,\sup_{r\in\Gamma^*}|\mathscr{K}y(r)|\right\},\quad r\in\overline{\Gamma}.\tag{18}$$

Lemma 3. *If $y(r)$ is a solution of problems (7)–(9), then*

$$\left|y^{(k)}(r)\right|_{\Gamma^*}\leq C\varepsilon^{-k},\quad \text{where } k=1,2,3,4.\tag{19}$$

Proof. First, to prove $y'(r)$ is bound on Γ_1,

$$\begin{aligned}\mathscr{K}_1y(r)&=-\varepsilon y''(r)+a(r)y'(r)+b(r)y(r)\\&\quad+d(r)y(r+1)=f(r)-c(r)\phi(r-1).\end{aligned}\tag{20}$$

Integrating the above equation on both sides, we have

$$\begin{aligned}-\varepsilon(y'(r)-y'(0))=&-[a(r)y(r)-a(0)u(0)]\\&+\int_0^r a'(t)y(t)\,dt-\int_0^r b(t)y(t)\,dt\\&-\int_0^r d(t)y(t+1)\,dt\\&+\int_0^r[f(t)-c(t)\phi(t-1)]\,dt.\end{aligned}\tag{21}$$

Therefore,

$$\begin{aligned}\varepsilon y'(0)=&\,\varepsilon y'(r)-[a(r)y(r)-a(0)y(0)]\\&+\int_0^r a'(t)y(t)\,dt-\int_0^r b(t)y(t)\,dt\\&-\int_0^r d(t)y(t+1)\,dt+\int_0^r[f(t)-c(t)\phi(t-1)]\,dt.\end{aligned}\tag{22}$$

Using Mean Value Theorem, then $|\varepsilon y'(x)| \le C(\|y(r)\|, \|f(r)\|, \|\phi\|_{[-1,0]})$, for some $x \in (0,\varepsilon)$ and $|\varepsilon y'(0)| \le C(\|y(r)\| + \|f(r)\| + \|\phi(r)\|)$. Then, we have $|\varepsilon y'(r)| \le C\max(\|y(r)\|, \|f(r)\|, \|\phi\|), r \in \Gamma_1$.

To prove $y'(r)$ is bound on Γ_2,

$$\begin{aligned}\mathscr{K}_2 y(r) &= -\varepsilon y''(r) + a(r)y'(r) + b(r)y(r)\\ &\quad + c(r)y(r-1) + d(r)y(r+1) = f(r).\end{aligned} \tag{23}$$

Integrating the above equation on both sides, we have

$$\begin{aligned}-\varepsilon(y'(r) - y'(0)) &= -[a(r)y(r) - a(0)u(0)] + \int_0^r a'(t)y(t)\,dt\\ &\quad - \int_0^r b(t)y(t)\,dt - \int_0^r c(t)y(t-1)\,dt\\ &\quad - \int_0^r \mathrm{d}(t)y(t+1)\,dt + \int_0^r f(t)\,dt.\end{aligned} \tag{24}$$

Therefore,

$$\begin{aligned}\varepsilon y'(0) &= \varepsilon y'(r) - [a(r)y(r) - a(0)u(0)] + \int_0^r a'(t)y(t)\,dt\\ &\quad - \int_0^r b(t)y(t)\,dt - \int_0^r c(t)y(t-1)\,dt\\ &\quad - \int_0^r d(t)y(t+1)\,dt + \int_0^r f(t)\,dt.\end{aligned} \tag{25}$$

Using Mean Value Theorem, then $|\varepsilon y'(x)| \le C(\|y(r)\|, \|f(r)\|, \|\phi\|_{[-1,0]})$, for some $x \in (0,\varepsilon)$ and $|\varepsilon y'(0)| \le C(\|y(r)\| + \|f(r)\| + \|\phi(r)\|)$. Then, we have $|\varepsilon y'(r)| \le C\max(\|y(r)\|, \|f(r)\|, \|\phi\|), r \in \Gamma_2$.

Next, to prove $y'(r)$ is bound on Γ_3,

$$\begin{aligned}\mathscr{K}_3 y(r) &= -\varepsilon y''(r) + a(r)y'(r) + b(r)y(r)\\ &\quad + c(r)y(r-1) = f(r) - d(r)\varphi(r+1).\end{aligned} \tag{26}$$

Integrating the above equation on both sides, we have

$$\begin{aligned}-\varepsilon(y'(r) - y'(0)) &= -[a(r)y(r) - a(0)u(0)]\\ &\quad + \int_0^r a'(t)y(t)\,dt - \int_0^r b(t)y(t)\,dt\\ &\quad - \int_0^r c(t)y(t-1)\,dt\\ &\quad - \int_0^r d(t)y(t+1)\,dt + \int_0^r f(t)\,dt.\end{aligned} \tag{27}$$

Therefore,

$$\begin{aligned}\varepsilon y'(0) &= \varepsilon y'(r) - [a(r)y(r) - a(0)u(0)] + \int_0^r a'(t)y(t)\,dt\\ &\quad - \int_0^r b(t)y(t)\,dt - \int_0^r c(t)y(t-1)\,dt\\ &\quad - \int_0^r d(t)y(t+1)\,dt + \int_0^r f(t)\,dt.\end{aligned} \tag{28}$$

Using Mean Value Theorem, then $|\varepsilon y'(x)| \le C(\|y(r)\|, \|f(r)\|, \|\phi\|_{[-1,0]})$, for some $x \in (0,\varepsilon)$ and $|\varepsilon y'(0)| \le C(\|y(r)\| + \|f(r)\| + \|\phi(r)\|)$. Then, we have, $|\varepsilon y'(r)| \le C\max(\|y(r)\|, \|f(r)\|, \|\phi\|), r \in \Gamma_3$.

Hence, $|y^{(k)}(r)|_{\Gamma^*} \le C\varepsilon^{-k}$, where $k = 2, 3, 4$. □

3.1. Shishkin Decomposition. The solution $y(r)$ is decomposed into $v(r)$ smooth component and $w(r)$-layer component. Furthermore, $v(r) = v_0(r) + \varepsilon v_1(r) + \varepsilon^2 v_2(r)$, where $v_0(r)$, $v_1(r)$, and $v_2(r)$ are solutions of the following differential equations.

Obtain reduced problem solution $v_0(r) \in X$ such that

$$\begin{cases} a(r)v_0'(r) + b(r)v_0(r) + c(r)v_0(r-1) = f(r), & r \in \Gamma \cap (\Gamma_1 \cup \Gamma_2),\\ v_0(r) = \phi(r), & r \in [-1,0].\end{cases} \tag{29a}$$

$$\begin{cases} a(r)v_0'(r) + b(r)v_0(r) + \mathrm{d}(r)v_0(r+1) = f(r), & r \in \Gamma \cap (\Gamma_2 \cup \Gamma_3) \cup \{3\},\\ v_0(r) = \varphi(r), & r \in [3,4].\end{cases} \tag{29b}$$

If $v_1(r) \in C^0(\overline{\Gamma}) \cap C^1(\Gamma^* \cup \{3\})$,

$$\begin{cases} a(r)v_1'(r) + b(r)v_1(r) + c(r)v_1(r-1) = v_0''(r), & r \in \Gamma \cap (\Gamma_1 \cup \Gamma_2),\\ v_1(r) = 0, & r \in [-1,0].\end{cases} \tag{30a}$$

$$\begin{cases} a(r)v_1'(r) + b(r)v_1(r) + d(r)v_1(r+1) = v_0''(r), & r \in \Gamma \cap (\Gamma_2 \cup \Gamma_3) \cup \{3\},\\ v_1(r) = 0, & r \in [3,4].\end{cases} \tag{30b}$$

If $v_2(r) \in X$,

$$\begin{cases} -\varepsilon v_2''(r) + a(r)v_2''(r) + b(r)v_2(r) + c(r)v_2(r-1) + d(r)v_2(r+1) = v_1''(r), & r \in \Gamma^* \\ v_2(r) = 0, & r \in [-1,0], \\ v_2(r) = 0, & r \in [3,4]. \end{cases} \tag{31}$$

If $v(r) \in C^0(\overline{\Gamma}) \cap C^2(\Gamma^*)$,

$$\begin{cases} \mathcal{K}v(r) = -\varepsilon v''(r) + a(r)v'(r) + b(r)v(r) + c(r)v(r-1) + \mathrm{d}(r)v(r+1) = f(r), & r \in \Gamma^*, \\ v(r) = \phi(r), & r \in [-1,0], \\ v(1) = v_0(1) + \varepsilon v_1(1) + \varepsilon^2 v_2(1), & \\ v(2) = v_0(2) + \varepsilon v_1(2) + \varepsilon^2 v_2(2), & \\ v(r) = 0, & r \in [3,4]. \end{cases} \tag{32}$$

Also, $w(r)$ satisfies the following problem: if the singular component $w(r) \in C^0(\overline{\Gamma}) \cap C^2(\Gamma^*)$,

$$\begin{cases} \mathcal{K}w(r) = -\varepsilon w''(r) + a(r)w'(r) + b(r)w(r) + c(r)w(r-1) + d(r)w(r+1) = 0, & r \in \Gamma^*, \\ w(r) = 0, & r \in [-1,0], \\ [w'](1) = -[v'](1), & \\ [w'](2) = -[v'](2), & \\ w(r) = \varphi(r), & r \in [3,4]. \end{cases} \tag{33}$$

Furthermore, we decompose $w(r)$ as $w(r) = w_B(r) + w_{I_1}(r) + w_{I_2}(r)$, where the function $w_B(r)$ is boundary layer component and $w_{I_1}(r)$, $w_{I_2}(r)$ are interior layer components.

If the boundary layer $w_B(r) \in X$,

$$\begin{cases} \mathcal{K}w_B(r) = -\varepsilon w_B''(r) + a(r)w_B'(r) + b(r)w_B(r) + c(r)w_B(r-1) + d(r)w_B(r+1) = 0, & \\ w_B(r) = 0, & r \in [-1,0], \\ w_B(r) = \varphi(r), & r \in [3,4]. \end{cases} \tag{34}$$

If the first interior layer $w_{I_1}(r) \in C^0(\overline{\Gamma}) \cap C^2(\Gamma^*)$,

$$\begin{cases} \mathcal{K}w_{I_1}(r) = -\varepsilon w_{I_1}''(r) + a(r)w_{I_1}'(r) + b(r)w_{I_1}(r) + c(r)w_{I_1}(r-1) + \mathrm{d}(r)w_{I_1}(r+1) = 0, & \\ w_{I_1}(r) = 0, & r \in [-1,0], \\ [w_{I_1}'](1) = -[v'](1), & \\ w_{I_1}(r) = 0, & r \in [3,4]. \end{cases} \tag{35}$$

If the second interior layer $w_{I_2}(r) \in C^0(\overline{\Gamma}) \cap C^2(\Gamma^*)$,

$$\begin{cases} \mathcal{K}w_{I_2}(r) = -\varepsilon w_{I_2}''(r) + a(r)w_{I_2}'(r) + b(r)w_{I_2}(r) + c(r)w_{I_2}(r-1) + d(r)w_{I_2}(r+1) = 0, & \\ w_{I_2}(r) = 0, & r \in [-1,0], \\ \left[w_{I_2}'\right](2) = -[v'](2), & \\ w_{I_2}(r) = 0, & r \in [3,4]. \end{cases} \quad (36)$$

Theorem 1. *If $y(r)$ and $v_0(r)$ are solutions of problems (7)–(9) and (29a)-(29b), then*

$$\left|y(r) - v_0(r)\right| \le C_1\left(\varepsilon + \exp\left(\frac{-\alpha(3-r)}{\varepsilon}\right)\right), \quad r \in \overline{\Gamma}. \quad (37)$$

Proof. Consider

$$\Theta^{\pm}(r) = C_1\left(\varepsilon s(r) + \exp\left(\frac{-\alpha(3-r)}{\varepsilon}\right)\right) \pm \left(y(r) - v_0(r)\right), \quad r \in \overline{\Gamma}. \quad (38)$$

Clearly, $\Theta^{\pm}(r) \in C^0(\overline{\Gamma}) \cap C^2(\Gamma^*)$. Note that $\Theta^{\pm}(0) \ge 0$, $\Theta^{\pm}(3) \ge 0$, for a suitable choice of $C_1 > 0$.

If $r \in \Gamma_1$,

$$\begin{aligned} \mathcal{K}_1\Theta^{\pm}(r) &= C_1\left[\left(\frac{\alpha}{\varepsilon}(a(r)-\alpha) + b(r) + d(r)\exp\left(\frac{\alpha}{\varepsilon}\right)\exp\left(\frac{-\alpha(3-r)}{\varepsilon}\right) + \varepsilon\left(a(r)s'(r) + b(r)s(r) + d(r)s(r+1)\right)\right)\right] \pm \varepsilon v_0''(r), \\ &\ge C_1\left[\left(\frac{\alpha}{\varepsilon}(\alpha_1-\alpha) + \beta + \eta\exp\left(\frac{\alpha}{\varepsilon}\right)\right)\exp\left(\frac{-\alpha(3-r)}{\varepsilon}\right) + \varepsilon\left(\frac{\alpha}{4} + \beta s(r) + \eta s(r+1)\right)\right] \pm C\varepsilon \ge 0. \end{aligned} \quad (39)$$

If $r \in \Gamma_2$,

$$\begin{aligned} \mathcal{K}_2\Theta^{\pm}(r) &= C_1\left[\left(\frac{\alpha}{\varepsilon}(a(r)-\alpha) + b(r) + c(r)\exp\left(-\frac{\alpha}{\varepsilon}\right) + d(r)\exp\left(\frac{\alpha}{\varepsilon}\right)\right)\exp\left(\frac{-\alpha(3-r)}{\varepsilon}\right)\right. \\ &\quad \left. + \varepsilon\left(a(r)s'(r) + b(r)s(r) + c(r)s(r-1) + d(r)s(r+1)\right)\right] \pm \varepsilon v_0''(r), \\ &\ge C_1\left[\left(\frac{\alpha}{\varepsilon}(\alpha_1-\alpha) + \beta + \gamma + \eta\exp\left(\frac{\alpha}{\varepsilon}\right)\right)\exp\left(\frac{-\alpha(3-r)}{\varepsilon}\right) + \varepsilon\left(\frac{\alpha}{6} + \beta s(r) + \gamma s(r-1) + \eta s(r+1)\right)\right] \mp C\varepsilon \ge 0. \end{aligned} \quad (40)$$

Following the same process, we have $\mathcal{K}_3\Theta^{\pm}(r) \ge 0$.

Using Lemma 1, then $\Theta^{\pm}(r) \ge 0, r \in \overline{\Gamma}$. Therefore, $|y(r) - v_0(r)| \le C_1(\varepsilon + \exp((-\alpha(3-r))/\varepsilon))$. □

Lemma 4. *If $v(r)$ and $w(r)$ are the solution of regular and singular component problems (32) and (33), then*

$$\left|v^k(r)\right|_{\Gamma^*} \le C\left(1 + \varepsilon^{2-k}\right), \quad \text{for } k = 0,1,2,3,4, \quad (41)$$

$$\left|w_B^k(r)\right| \le C\varepsilon^{-k}\exp\left(\frac{-\alpha(3-r)}{\varepsilon}\right), \quad r \in \Gamma^*, \quad (42)$$

$$\left|w_{I_1}^k(r)\right| \le C\begin{cases} \varepsilon^{1-k}\exp\left(\dfrac{-\alpha(1-r)}{\varepsilon}\right), & r \in \Gamma_1, \\ \varepsilon^{1-k}, & r \in \Gamma_2, r \in \Gamma_3, \end{cases} \quad (43)$$

$$\left|w_{I_2}^{k}(r)\right| \leq C \begin{cases} \varepsilon^{1-k} \exp\left(\dfrac{-\alpha(2-r)}{\varepsilon}\right), & r \in \Gamma_2, \\ \varepsilon^{1-k}, & r \in \Gamma_1, r \in \Gamma_3, \end{cases} \tag{44}$$

where $k = 0, 1, 2, 3, 4$.

Proof. The smooth component derivative bound is easy to prove by using stability result and integrating (30a), (30b), and (31). Next, to prove (42), consider that

$$\Phi^{\pm}(r) = C\left(\exp\left(\frac{-\alpha(3-r)}{\varepsilon}\right)\right) \pm w_B(r), \quad r \in \bar{\Gamma}. \tag{45}$$

Note that $\Phi^{\pm}(0) \geq 0$, $\Phi^{\pm}(3) \geq 0$, and

$$\mathscr{K}\Phi^{\pm}(r) \geq C\left[\frac{\alpha}{\varepsilon}(\alpha_1 - \alpha) + \beta + \gamma + \eta \exp\left(\frac{\alpha}{\varepsilon}\right)\right]\exp\left(\frac{-\alpha(3-r)}{\varepsilon}\right) \pm \mathscr{K}w_B(r) \geq 0. \tag{46}$$

By Lemma 1,

$$\left|w_B(r)\right| \leq C \exp\left(\frac{-\alpha(3-r)}{\varepsilon}\right). \tag{47}$$

Integration of (34) yields the estimates of $|w_B'(r)|$. From the differential equations (33), one can derive the rest of the derivative estimates (42).

Inequalities (43) and (44) can be proved, using Theorem 1 and maximum principle for the barrier functions:

$$\Phi^{\pm}(r) = C \begin{cases} \varepsilon\left(\exp\left(\dfrac{-\alpha(1-r)}{\varepsilon}\right)\right) \pm w_{I_1}(r), & r \in \Gamma_1, \\ r\varepsilon \pm w_{I_1}(r), & r \in \Gamma_2, \\ r\varepsilon \pm w_{I_1}(r), & r \in \Gamma_3, \end{cases}$$
$$\Phi^{\pm}(r) = C \begin{cases} \varepsilon\left(\exp\left(\dfrac{-\alpha(2-r)}{\varepsilon}\right)\right) \pm w_{I_2}(r), & r \in \Gamma_1, \\ r\varepsilon \pm w_{I_2}(r), & r \in \Gamma_2, \\ r\varepsilon \pm w_{I_2}(r), & r \in \Gamma_3. \end{cases} \tag{48}$$

Hence, it is proved. □

Remark. The following inequalities are easy to prove, using Theorem 1 and Lemma 4:

$$|y(r) - v(r)| \leq C \begin{cases} \varepsilon + \varepsilon \exp\left(\dfrac{-\alpha(1-r)}{\varepsilon}\right) + \exp\left(\dfrac{-\alpha(3-r)}{\varepsilon}\right), & r \in \Gamma_1, \\ \varepsilon + \varepsilon \exp\left(\dfrac{-\alpha(2-r)}{\varepsilon}\right) + \exp\left(\dfrac{-\alpha(3-r)}{\varepsilon}\right), & r \in \Gamma_2, \\ \varepsilon + \exp\left(\dfrac{-\alpha(3-r)}{\varepsilon}\right), & r \in \Gamma_3. \end{cases} \tag{49}$$

4. The Discrete Problem

4.1. Shishkin Mesh. Problems (7)–(9) are convection-diffusion type containing delay term. Then, the layers occur in boundary at $t = 3$ and interior at $t = 1$ and $t = 2$.

The intervals $[0, 1]$, $[1, 2]$, and $[2, 3]$ are partitioned into $[0, 1-\sigma]$, $[1-\sigma, 1]$, $[1, 2-\sigma]$, $[2-\sigma, 2]$, $[2, 3-\sigma]$, and $[3-\sigma, 3]$ for each interval $(N/2)$ mesh points and $\sigma = \min\{(1/2), 2(\varepsilon/\alpha)\ln N\}$ is transition parameter.

The interior of points is denoted by $\overline{\Gamma}^{3N} = \{r_0, r_1, \ldots, r_{3N}\}$. Then, the mesh widths are

$$h(r_i) = \begin{cases} H = \dfrac{2(1-\sigma)}{N}, & \text{for } i = 1 \text{ to } \dfrac{N}{2}, i = N+1 \text{ to } \dfrac{3N}{2}, i = 2N+1 \text{ to } \dfrac{5N}{2}, \\ h = \dfrac{2\sigma}{N}, & \text{for } i = \dfrac{N}{2}+1 \text{ to } N, i = \dfrac{3N}{2}+1 \text{ to } 2N, i = \dfrac{5N}{2}+1 \text{ to } 3N. \end{cases} \tag{50}$$

4.2. Finite Difference Method. The discrete scheme corresponding to the original problems (7)–(9) is as follows:

$$\mathscr{K}^N Y(r_i) = \begin{cases} \mathscr{K}_1^N Y(r_i) = -\varepsilon\delta^2 Y(r_i) + a(r_i)D^- Y(r_i) + b(r_i)Y(r_i) + d(r_i)Y(r_{i+N}) = f_i - c_i\phi_{i-N}, & r_i \in \Gamma_1^{3N}, \\ \mathscr{K}_2^N Y(r_i) = -\varepsilon\delta^2 Y(r_i) + a(r_i)D^- Y(r_i) + b(r_i)Y(r_i) + c(r_i)Y(r_{i-N}) + d(r_i)Y(r_{i+N}) = f_i, & r_i \in \Gamma_2^{3N}, \\ \mathscr{K}_3^N Y(r_i) = -\varepsilon\delta^2 Y(r_i) + a(r_i)D^- Y(r_i) + b(r_i)Y(r_i) + c(r_i)Y(r_{i-N}) = f_i - d_i\varphi_{i+N}, & r_i \in \Gamma_3^{3N}, \end{cases} \tag{51}$$

with

$$\begin{cases} Y(r(0)) = \phi_0, \\ D^- U_N = D^+ U_N, \\ D^- U_{2N} = D^+ U_{2N}, \\ Y(r(3)) = \varphi_{3N}. \end{cases} \tag{52}$$

4.3. Hybrid Difference Scheme. The hybrid scheme corresponding to the original problems (7)–(9) is as follows:

$$\mathscr{K}_1^N Y(r_i) = \begin{cases} -\varepsilon\delta^2 Y(r_i) + a_{i-(1/2)}D^- Y(r_i) + b(r_i)\widehat{U}(r_i) + d(r_i)\widehat{U}(r_{i+N}) = f_{i-(1/2)} - c_i\widehat{\phi}_{i-N}, & i = 1 \text{ to } \dfrac{N}{2}, \\ -\varepsilon\delta^2 Y(r_i) + a(r_i)D^0 Y(r_i) + b(r_i)Y(r_i) + d(r_i)Y(r_{i+N}) = f_i - c_i\phi_{i-N}, & i = \dfrac{N}{2}+1 \text{ to } N-1, \end{cases} \tag{53}$$

$$\mathscr{K}_2^N Y(r_i) = \begin{cases} -\varepsilon\delta^2 Y(r_i) + a_{i-(1/2)}D^- Y(r_i) + b(r_i)\widehat{U}(r_i) + c(r_i)\widehat{U}(r_{i-N}) + d(r_i)\widehat{U}(r_{i+N}) = f_{i-(1/2)}, & i = N+1 \text{ to } \dfrac{3N}{2}, \\ -\varepsilon\delta^2 Y(r_i) + a(r_i)D^0 Y(r_i) + b(r_i)Y(r_i) + c(r_i)Y(r_{i-N}) + d(r_i)Y(r_{i+N}) = f_i, & i = \dfrac{3N}{2}+1 \text{ to } 2N-1, \end{cases} \tag{54}$$

$$\mathscr{K}_3^N Y(r_i) = \begin{cases} -\varepsilon\delta^2 Y(r_i) + a_{i-(1/2)}D^- Y(r_i) + b(r_i)\widehat{U}(r_i) + c(r_i)\widehat{U}(r_{i-N}) = f_{i-(1/2)} - d_i\widehat{\varphi}_{i+N}, & i = 2N+1 \text{ to } \dfrac{5N}{2}, \\ -\varepsilon\delta^2 Y(r_i) + a(r_i)D^0 Y(r_i) + b(r_i)Y(r_i) + c(r_i)Y(r_{i-N}) = f_i - d_i\varphi_{i+N}, & i = \dfrac{5N}{2}+1 \text{ to } 3N-1, \end{cases} \tag{55}$$

$$\mathscr{K}^N Y(r_i) = \frac{Y_{i-2} - 4Y_{i-1} + 3Y_i}{2h} - \frac{-Y_{i+2} + 4Y_{i+1} - 3Y_i}{2H}, \quad i = N, 2N, \tag{56}$$

where

$$\delta^2 Y(r_i) = \frac{2}{h_i + h_{i+1}}\left(\frac{Y(r_{i+1}) - Y(r_i)}{h_{i+1}} - \frac{Y(r_i) - Y(r_{i-1})}{h_i}\right),$$
$$\widehat{U}(r_i) = \frac{Y(r_i) + Y(r_{i-1})}{2},$$
$$D^0 Y(r_i) = \frac{Y(r_{i+1}) - Y(r_{i-1})}{h_i + h_{i+1}}, \tag{57}$$
$$D^- Y(r_i) = \frac{Y(r_i) - Y(r_{i-1})}{h_i},$$
$$a_{i-(1/2)} = a\left(\frac{(r_{i-1} + r_i)}{2}\right).$$

5. Numerical Estimates for the Finite Difference Method

Lemma 5 (discrete maximum principle). *If $U(r_i)$ satisfies $U(r_0) \geq 0$, $U(r_{3N}) \geq 0$, $\mathscr{K}_1^N U(r_i) \geq 0$, $\mathscr{K}_2^N U(r_i) \geq 0$, $\mathscr{K}_3^N U(r_i) \geq 0$, $D^+(U(r_N)) - D^-(U(r_N)) \leq 0$, and $D^+(U(r_{2N})) - D^-(U(r_{2N})) \leq 0$, then $U(r_i) \geq 0$, $\forall r_i \in \overline{\Gamma}^{3N}$.*

Proof.

$$\text{Define } S(r_i) = \begin{cases} \dfrac{1}{12} + \dfrac{r_i}{4}, & r_i \in [0,1] \cap \overline{\Gamma}^{3N}, \\ \dfrac{2}{12} + \dfrac{r_i}{6}, & r_i \in [1,2] \cap \overline{\Gamma}^{3N}, \\ \dfrac{4}{12} + \dfrac{r_i}{12}, & r_i \in [2,3] \cap \overline{\Gamma}^{3N}. \end{cases} \tag{58}$$

It is easy to see that $s(r_i) > 0, \forall r_i \in \overline{\Gamma}^{3N}$, $\mathscr{K}s(r_i) > 0, \forall r_i \in \Gamma_1^{3N} \cup \Gamma_2^{3N} \cup \Gamma_3^{3N}$, $s(r_0) > 0$, $s(r_{3N}) > 0$, $D^+(s(r_N)) - D^-(s(r_N)) < 0$, and $D^+(s(r_{2N})) - D^-(s(r_{2N})) < 0$. Let $\mu = \min\left\{((-\psi(r_i))/s(r_i)) \colon r_i \in \overline{\Gamma}^{3N}\right\}$.

Then, there exists $r_k \in \overline{\Gamma}^{3N}$ such that $\psi(r_k) + \mu s(r_k) = 0$ and $\psi(r_i) + \mu s(r_i) \geq 0, \forall r_i \in \overline{\Gamma}^{3N}$. Then, $(\psi + \mu s)$ attains its maximum at $r_i = r_k$. If $\mu < 0$, then $\psi \geq 0$. Suppose $\mu > 0$.

Case (i): $r_k = r_0$:

$$0 < (\psi + \mu s)(r_0) = 0. \tag{59}$$

Case (ii): $r_k \in \Gamma_1^{3N}$:

$$\begin{aligned} 0 < \mathscr{K}_1(\psi + \mu s)(r_k) &= -\varepsilon\delta^2(\psi + \mu s)(r_k) \\ &+ a(r_k)D^-(\psi + \mu s)(r_k) + b(r_k)(\psi + \mu s)(r_k) \\ &+ d(r_k)(\psi + \mu s)(r_{k+N}) \leq 0. \end{aligned} \tag{60}$$

Case (iii): $r_k = r_N$:

$$0 \leq [D](\psi + \mu s)(r_N) < 0. \tag{61}$$

Case (iv): $r_k \in \Gamma_2^{3N}$:

$$\begin{aligned} 0 < \mathscr{K}_2(\psi + \mu s)(r_k) &= -\varepsilon\delta^2(\psi + \mu s)(r_k) \\ &+ a(r_k)D^-(\psi + \mu s)(r_k) + b(r_k)(\psi + \mu s)(r_k) \\ &+ c(r_k)(\psi + \mu s)(r_{k-N}) + \mathrm{d}(r_k)(\psi + \mu s)(r_{k+N}) \leq 0. \end{aligned} \tag{62}$$

Case (v): $r_k = r_{2N}$:

$$0 \leq [D](\psi + \mu s)r_{2N} < 0. \tag{63}$$

Case (vi): $r_k \in \Gamma_3^{3N}$:

$$\begin{aligned} 0 < \mathscr{K}_3(\psi + \mu s)(r_k) &= -\varepsilon\delta^2(\psi + \mu s)(r_k) \\ &+ a(r_k)D^-(\psi + \mu s)(r_k) + b(r_k)(\psi + \mu s)(r_k) \\ &+ c(r_k)(\psi + \mu s)(r_{k-N}) \leq 0. \end{aligned} \tag{64}$$

Case (vii): $r_k = r_{3N}$:

$$0 < (\psi + \mu s)r_{3N} = 0. \tag{65}$$

All the cases are a contradiction. □

Lemma 6. *The discrete solution of (51) and (52) is bounded:*

$$|U(r_i)| \leq C \max\left\{|U(r_0)|, |U(r_{3N})|, \max_{i \in \Gamma_1^{3N} \cup \Gamma_2^{3N} \cup \Gamma_3^{3N}} \left|\mathscr{K}^N U(r_i)\right|\right\}. \tag{66}$$

Proof. Consider $\psi^{\pm}(r_i) = \mathrm{CM}s(r_i) \pm U(r_i)$, $0 \leq i \leq 3N$, where $M = \max\left\{|U(r_0)|, |U(r_{3N})|, \max_{i \in \Gamma_1^{3N} \cup \Gamma_2^{3N} \cup \Gamma_3^{3N}} |\mathscr{K}^N U(r_i)|\right\}$.

Observe $\psi^{\pm}(r_0) \geq 0$ and $\psi^{\pm}(r_{3N}) \geq 0$:

$$\begin{aligned} &\mathscr{K}_1^N \psi^{\pm}(r_i) \geq 0, \forall r_i \in \Gamma_1^N,\\ &\mathscr{K}_2^N \psi^{\pm}(r_i) \geq 0, \forall r_i \in \Gamma_2^N,\\ &\mathscr{K}_3^N \psi^{\pm}(r_i) \geq 0, \forall r_i \in \Gamma_3^N,\\ &D^+\psi^{\pm}(r_N) - D^-\psi^{\pm}(r_N) \leq 0.\\ &D^+\psi^{\pm}(r_{2N}) - D^-\psi^{\pm}(r_{2N}) \leq 0. \end{aligned} \tag{67}$$

Using Lemma 5, $\psi^{\pm}(r_i) \geq 0$, $\forall r_i \in \overline{\Gamma}^{3N}$.

To decompose numerical solution $Y(r_i)$ into $V(r_i)$ and $W(r_i)$ satisfy the following equations, respectively:

$$\begin{cases} \mathscr{K}^N V(r_i) = -\varepsilon\delta^2 V(r_i) + a(r_i)D^- V(r_i) + b(r_i)V(r_i) + c(r_i)V(r_{i-N}) + d(r_i)V(r_{i+N}) = f_i, \quad i \in \overline{\Gamma}^{3N}\backslash\{0, N, 2N, 3N\},\\ V(r_0) = v(0),\\ [D]V(r_N) = [v'](1),\\ [D]V(r_{2N}) = [v'](2),\\ V(r_{3N}) = v(3), \end{cases} \tag{68}$$

$$\begin{cases} \mathscr{K}^N W(r_i) = -\varepsilon\delta^2 W(r_i) + a(r_i)D^- W(r_i) + b(r_i)W(r_i) + c(r_i)W(r_{i-N}) + d(r_i)W(r_{i+N}) = f_i, \quad i \in \overline{\Gamma}^{3N}\backslash\{0, N, 2N, 3N\},\\ W(r_0) = w(0),\\ [D]W(r_N) = -[D]V(r_N),\\ [D]W(r_{2N}) = -[D]V(r_{2N}),\\ W(r_{3N}) = w(3). \end{cases} \tag{69}$$

□

Theorem 2. *If $Y(r_i)$ and $V(r_i)$ are a solution of discretization problem (51), (52), and (68), then* $|Y(r_i) - V(r_i)| \leq CN^{-1}$.

Proof. Consider

$$\theta^{\pm}(r_i) = CN^{-1}s(r_i) \pm (Y(r_i) - V(r_i)), \quad \forall r_i \in \overline{\Gamma}^{3N}. \tag{70}$$

Note that $\theta^{\pm}(r_0) \geq 0$ and $\theta^{\pm}(r_{3N}) \geq 0$:

$$\begin{aligned} &\mathscr{K}_1^N \theta^{\pm}(r_i) \geq 0, \quad \text{for all } i \in \{1, 2, \ldots, N-1\},\\ &\mathscr{K}_2^N \theta^{\pm}(r_i) \geq 0, \quad \text{for all } i \in \{N+1, \ldots, 2N-1\},\\ &\mathscr{K}_3^N \theta^{\pm}(r_i) \geq 0, \quad \text{for all } i \in \{2N+1, \ldots, 3N-1\}, \end{aligned} \tag{71}$$

$[D]\theta^{\pm}(r_N) = -C(N^{-1}/6) \pm [v'](1) < 0$, $[D]\theta^{\pm}(r_{2N}) < 0$, using Lemma 5; then, the theorem has been proved.

Theorem 3. *The error estimates for smooth components bounded by CN^{-1}:*

$$|v(r_i) - V(r_i)| \leq CN^{-1}, \quad r_i \in \overline{\Gamma}^{3N}. \tag{72}$$

Proof. The proof of Theorem 3 has the same idea in [29]:

$$\left|\mathscr{K}^N (v(r_i) - V(r_i))\right| \leq CN^{-1}, \quad i \in \Gamma_1^{3N} \cup \Gamma_2^{3N} \cup \Gamma_3^{3N}. \tag{73}$$

Using Lemma 6, then

$$|v(r_i) - V(r_i)| \leq CN^{-1}, \quad i \in \Gamma_1^{3N} \cup \Gamma_2^{3N} \cup \Gamma_3^{3N}. \tag{74}$$

Therefore, we get $|v(r_i) - V(r_i)| \leq CN^{-1}$, $i \in \overline{\Gamma}^{3N}$. □

Theorem 4. *Derive the error estimates for singular components bounded by $CN^{-1}\log^2 N$:*

$$|w(r_i) - W(r_i)| \leq CN^{-1}\log^2 N, \quad r_i \in \overline{\Gamma}^{3N}. \tag{75}$$

Proof. Note that

$$|w(r_i) - W(r_i)| \leq |y(r_i) - Y(r_i)| + |v(r_i) - V(r_i)|. \tag{76}$$

Then, by (49) and from Theorems 1, 3, we have

$$\begin{aligned} |y(r_i) - Y(r_i)| &= |Y(r_i) - V(r_i)| + |v(r_i) - V(r_i)| + |y(r_i) - v(r_i)|,\\ |y(r_i) - Y(r_i)| &\leq |Y(r_i) - V(r_i)| + |v(r_i) - V(r_i)| + |y(r_i) - v(r_i)|. \end{aligned} \tag{77}$$

Now,

$$
\begin{aligned}
|w(r_i) - W(r_i)| &\leq |Y(r_i) - V(r_i)| + 2|v(r_i) - V(r_i)| + |y(r_i) - v(r_i)| \\
&\leq C_1 N^{-1} + C_1 \exp\left(\frac{-\alpha(3-r)}{\varepsilon}\right) + \varepsilon \\
&\leq C_1 N^{-1} + C_1 \exp\left(\frac{-\alpha(3-r)}{\varepsilon}\right) + C_1 N^{-1} \\
&\leq C_1 \exp\left(\frac{-\alpha\sigma}{\varepsilon}\right) + C_1 N^{-1} \\
&\leq C N^{-1}, \quad i = 0 \text{ to } \frac{5N}{2}.
\end{aligned} \tag{78}
$$

Consider the mesh functions:

$$
\begin{aligned}
\Phi^{\pm}(r_i) &= C_1 N^{-1} s(r_i) + C_1 N^{-1} \frac{\sigma}{\varepsilon^2}(r_i - (3-\sigma)) \\
&\pm (w(r_i) - W(r_i)), \quad r_i \in [3-\sigma, 3] \cap \overline{\Gamma}^{3N}.
\end{aligned} \tag{79}
$$

Observe that $\Phi^{\pm}(r_{(5N/2)}) \geq 0$ and $\Phi^{\pm}(r_{3N}) \geq 0$, and $\mathscr{K}^N \Phi^{\pm}(r_i) \geq 0$.

Then, by the Lemma 5, we have $\Phi^{\pm}(r_i) \geq 0, r_i \in \overline{\Gamma}^{3N}$. Therefore,

$$
|w(r_i) - W(r_i)| \leq C N^{-1} \log^2 N, \quad r_i \in \overline{\Gamma}^{3N}. \tag{80}
$$

□

Theorem 5. *If $y(r_i)$ and $Y(r_i)$ are a solution of (7)–(9) and (51), (52),*

$$
|y(r_i) - Y(r_i)| \leq C N^{-1} \log^2 N, \quad r_i \in \overline{\Gamma}^{3N}. \tag{81}
$$

That is, the order of convergence is almost one.

Proof. The proof of Theorem 5 follows from $y_k = v_k + w_k$, $Y_k = V_k + W_k$, and Theorems 3 and 4. □

6. Numerical Estimates for the Hybrid Difference Method

Assume the following inequality:

$$
\frac{N}{\ln N} \geq 2 \frac{\alpha_1}{\alpha}. \tag{82}
$$

Lemma 7. *Assume (78) holds true. Let $\Psi(r_i)$ satisfy $\Psi(r_0) \geq 0$, $\Psi(r_{3N}) \geq 0$; the operator $\mathscr{K}^N$ defined by (53)–(55) satisfies $\mathscr{K}_1^N \Psi(r_i) \geq 0$, $\mathscr{K}_2^N \Psi(r_i) \geq 0$, $\mathscr{K}_3^N \Psi(r_i) \geq 0$; and then $\Psi(r_i) \geq 0$, $\forall r_i \in \overline{\Gamma}^{3N}$.*

Lemma 8. *If $\Psi(r_i)$ is discrete solution of problems (53)–(55), then*

$$
|\Psi(r_i)| \leq C \max\left\{ |\Psi(r_0)|, |\Psi(r_{3N})|, \max_{i \in \Gamma_1^{3N} \cup \Gamma_2^{3N} \cup \Gamma_3^{3N}} |\mathscr{K}^N \Psi(r_i)| \right\}. \tag{83}
$$

6.1. Error Estimate. To decompose the numerical solution $Y(r_i)$ into $V(r_i)$ and $W(r_i)$, satisfy the following equations, respectively:

$$
\mathscr{K}^N V(r_i) = \begin{cases} -\varepsilon\delta^2 V(r_i) + a_{i-(1/2)} D^- V(r_i) + b(r_i)\widehat{V}(r_i) + c(r_i)\widehat{V}(r_{i-N}) + \mathrm{d}(r_i)\widehat{U}(r_{i+N}) = f_{i-(1/2)}, \\ -\varepsilon\delta^2 V(r_i) + a(r_i) D^0 V(r_i) + b(r_i) V(r_i) + c(r_i) V(r_{i-N}) + \mathrm{d}(r_i) V(r_{i+N}) = f_i. \end{cases} \tag{84}
$$

$$
\mathscr{K}^N W(r_i) = \begin{cases} -\varepsilon\delta^2 W(r_i) + a_{i-(1/2)} D^- W(r_i) + b(r_i)\widehat{W}(r_i) + c(r_i)\widehat{W}(r_{i-N}) + \mathrm{d}(r_i)\widehat{W}(r_{i+N}) = 0, \\ -\varepsilon\delta^2 W(r_i) + a(r_i) D^0 W(r_i) + b(r_i) W(r_i) + c(r_i) W(r_{i-N}) + \mathrm{d}(r_i) W(r_{i+N}) = 0. \end{cases} \tag{85}
$$

Lemma 9. *Derive the error estimation of discretization original problems (53)–(56) and regular problem (84) solutions:*

$$
|Y(r_i) - V(r_i)| \leq C N^{-2}. \tag{86}
$$

Proof. The proof of Lemma 9 has the same idea in Lemma 7:

$$
\theta^{\pm}(r_i) = C N^{-2} s(r_i) \pm (Y(r_i) - V(r_i)), \quad \forall r_i \in \overline{\Gamma}^{3N}. \tag{87}
$$

□

Lemma 10. *The error estimates for smooth components are bounded by* CN^{-2}*:*

$$|V(r_i) - v(r_i)| \le CN^{-2}, \quad r_i \in \overline{\Gamma}^{3N}. \tag{88}$$

Proof. Utilizing the method adopted in [30],

$$|\mathcal{K}(Y-y)(r_i)| \le C \begin{cases} \varepsilon H\|y^{(3)}\| + H^2\left(\|y^{(3)}\| + \|y^{(2)}\|\right), & i = 1 \text{ to } \dfrac{N}{2}, \\ \varepsilon h^2\|y^{(4)}\| + h^2\|a_i\|\|y^{(3)}\|, & i = \dfrac{N}{2} + 1 \text{ to } N-1. \end{cases} \tag{89}$$

Using $\varepsilon \le CN^{-1}$ and the above equation, the bounds on the derivatives of v can be written as

$$|\mathcal{K}_1(V-v)(r_i)| \le C \begin{cases} \varepsilon H\|v^{(3)}\| + H^2\left(\|v^{(3)}\| + \|v^{(2)}\|\right), & i = 1 \text{ to } \dfrac{N}{2}, \\ \varepsilon h^2\|v^{(4)}\| + h^2\|a_i\|\|v^{(3)}\| + h^2\|c_i\|\|v^{(3)}\|, & i = \dfrac{N}{2} + 1 \text{ to } N-1, \end{cases}$$

$$|\mathcal{K}_1(V-v)(r_i)| \le C \begin{cases} N^{-1}\left(\varepsilon + N^{-1}\right), & i = 1 \text{ to } \dfrac{N}{2} \\ N^{-2}, & i = \dfrac{N}{2} + 1 \text{ to } N-1. \end{cases} \tag{90}$$

Then, we have $|\mathcal{K}_1(V-v)(r_i)| \le CN^{-2}$. Similarly, $|\mathcal{K}_j(V-v)(r_i)| \le CN^{-2}$, $j = 2, 3$, $|\mathcal{K}_j(V-v)(r_i)| \le CN^{-2}$, $j = 1, 2, 3$, $i \in \overline{\Gamma}^{3N}\backslash\{0, N, 2N, 3N\}$, and, by Lemma 8, we have

$$|V(r_i) - v(r_i)| \le CN^{-2}, \quad r_i \in \overline{\Gamma}^{3N}. \tag{91}$$

□

Lemma 11. *Derive the error estimates for singular components bounded by* $CN^{-2}\log^2 N$*:*

$$|w(r_i) - W(r_i)| \le CN^{-2}\log^2 N, \quad r_i \in \overline{\Gamma}^{3N}. \tag{92}$$

Proof. Note that

$$\begin{aligned} |w(r_i) - W(r_i)| &\le |y(r_i) - Y(r_i)| + |v(r_i) - V(r_i)|, \\ |y(r_i) - Y(r_i)| &\le |Y(r_i) - V(r_i)| + |v(r_i) - V(r_i)| + |y(r_i) - v(r_i)|. \end{aligned} \tag{93}$$

Now,

$$\begin{aligned} |w(r_i) - W(r_i)| &\le |Y(r_i) - V(r_i)| + 2|v(r_i) - V(r_i)| + |y(r_i) - v(r_i)|, \\ &\le C_1\exp\left(\frac{-\alpha\sigma}{\varepsilon}\right) + C_1N^{-2} \le CN^{-2}, \quad i = 0 \text{ to } \frac{5N}{2}. \end{aligned} \tag{94}$$

Consider the mesh functions

$$\begin{aligned} \Phi^{\pm}(r_i) &= C_1N^{-2}s(r_i) + C_1N^{-2}\frac{\sigma}{\varepsilon^2}(r_i - (3-\sigma)) \\ &\pm (w(r_i) - W(r_i)), \quad r_i \in [3-\sigma, 3] \cap \overline{\Gamma}^{3N}. \end{aligned} \tag{95}$$

Clearly, $\Phi^{\pm}(r_{(5N/2)}) \ge 0$ and $\Phi^{\pm}(r_{3N}) \ge 0$, for a suitable choice of $C_1 > 0$.

$$\mathcal{K}^N\Phi^{\pm}(r_i) \ge 0. \tag{96}$$

Then, by Lemma 7, we have $\Phi^{\pm}(r_i) \ge 0, r_i \in \overline{\Gamma}^{3N}$. Therefore,

$$|w(r_i) - W(r_i)| \le CN^{-2}\log^2 N, \quad r_i \in \overline{\Gamma}^{3N}. \tag{97}$$

□

Theorem 6. *If* $y(r_i)$ *and* $Y(r_i)$ *are the solution of (7)–(9) and (53)–(56), then*

$$|y(r_i) - Y(r_i)| \le CN^{-2}\log^2 N, \quad r_i \in \overline{\Gamma}^{3N}. \tag{98}$$

Table 1: Computed P^N rate of convergence and D^N maximum errors for Example 1.

ε	Number of mesh points 3N: 16	32	64	128	256	512	1024
Finite difference method							
10^{-3}	2.8530e – 03	1.6423e – 03	8.6257e – 04	4.3424e – 04	2.1521e – 04	1.0694e – 04	5.4208e – 05
10^{-4}	3.0817e – 03	1.8315e – 03	9.9232e – 04	5.1418e – 04	2.6100e – 04	1.3150e – 04	6.6324e – 05
10^{-5}	3.1526e – 03	1.8900e – 03	1.0321e – 03	5.3860e – 04	2.7489e – 04	1.3887e – 04	6.9901e – 05
10^{-6}	3.1750e – 03	1.9083e – 03	1.0446e – 03	5.4624e – 04	2.7922e – 04	1.4116e – 04	7.1008e – 05
10^{-7}	3.1820e – 03	1.9141e – 03	1.0486e – 03	5.4864e – 04	2.8058e – 04	1.4188e – 04	7.1355e – 05
10^{-8}	3.1842e – 03	1.9159e – 03	1.0498e – 03	5.4940e – 04	2.8101e – 04	1.4211e – 04	7.1465e – 05
10^{-9}	3.1849e–03	1.9165e–03	1.0502e – 03	5.4965e – 04	2.8115e– 04	1.4218e – 04	7.1499e – 05
10^{-10}	3.1851e – 03	1.9167e – 03	1.0503e – 03	5.4972e – 04	2.8119e – 04	1.4220e – 04	7.1510e – 05
D^N	3.1851e – 03	1.9167e – 03	1.0503e – 03	5.4972e – 04	2.8119e – 04	1.4220e – 04	7.1510e – 05
P^N	7.327520e – 01	8.6771e – 01	9.3412e – 01	9.6712e – 01	9.8357e – 01	9.9178e – 01	—
Hybrid difference method							
10^{-3}	1.6679e – 02	6.7615e – 03	2.4201e – 03	8.5271e – 04	2.6786e – 04	7.8677e – 05	2.6259e – 05
10^{-4}	1.6684e – 02	6.7758e – 03	2.4331e – 03	8.6109e – 04	2.7258e – 04	8.3394e – 05	2.5535e – 05
10^{-5}	1.6684e – 02	6.7772e – 03	2.4344e–03	8.6194e – 04	2.7307e – 04	8.4126e – 05	2.6113e – 05
10^{-6}	1.6684e – 02	6.7774e – 03	2.4345e–03	8.6202e – 04	2.7312e – 04	8.4200e – 05	2.6192e – 05
10^{-7}	1.6684e – 02	6.7774e – 03	2.4345e – 03	8.6203e – 04	2.7312e – 04	8.4207e – 05	2.6200e – 05
10^{-8}	1.6684e – 02	6.7774e – 03	2.4345e – 03	8.6203e – 04	2.7312e – 04	8.4208e – 05	2.6201e – 05
10^{-9}	1.6684e – 02	6.7774e – 03	2.4345e – 03	8.6204e – 04	2.7312e – 04	8.4209e – 05	2.6200e – 05
10^{-10}	1.6685e – 02	6.7774e – 03	2.4346e – 03	8.6201e – 04	2.7308e – 04	8.4224e – 05	2.6206e – 05
D^N	1.6685e – 02	6.7774e – 03	2.4346e – 03	8.6204e – 04	2.7312e – 04	8.4224e – 05	2.6259e – 05
P^N	1.2997e + 00	1.4770e + 00	1.4979e + 00	1.6581e + 00	1.6972e + 00	1.6814e + 00	—

Table 2: Computed P^N rate of convergence and D^N maximum errors for Example 2.

ε	Number of mesh points 3N: 16	32	64	128	256	512	1024
Finite difference method							
10^{-3}	5.5384e – 03	2.5272e – 03	1.1887e – 03	5.6672e – 04	2.7241e – 04	1.3205e – 04	6.4844e – 05
10^{-4}	5.8841e – 03	2.7585e – 03	1.3350e – 03	6.5461e – 04	3.2299e – 04	1.6003e – 04	7.9627e – 05
10^{-5}	5.9911e – 03	2.8296e – 03	1.3798e – 03	6.8149e – 04	3.3841e – 04	1.6851e – 04	8.4085e – 05
10^{-6}	6.0248e – 03	2.8519e – 03	1.3939e – 03	6.8989e – 04	3.4321e – 04	1.7116e – 04	8.5469e – 05
10^{-7}	6.0354e – 03	2.8590e – 03	1.3983e – 03	6.9253e – 04	3.4473e – 04	1.7199e – 04	8.5904e – 05
10^{-8}	6.0387e – 03	2.8612e – 03	1.3997e – 03	6.9337e – 04	3.4521e – 04	1.7225e – 04	8.6041e – 05
10^{-9}	6.0398e – 03	2.8619e – 03	1.4001e – 03	6.9363e – 04	3.4536e – 04	1.7233e – 04	8.6085e – 05
10^{-10}	6.0401e – 03	2.8621e – 03	1.4003e – 03	6.9371e – 04	3.4541e – 04	1.7236e – 04	8.6098e – 05
D^N	6.0401e – 03	2.8621e – 03	1.4003e – 03	6.9371e – 04	3.4541e – 04	1.7236e – 04	8.6098e – 05
P^N	1.0774e + 00	1.0313e + 00	1.0133e + 00	1.0060e + 00	1.0028e + 00	1.0013e + 00	—
Hybrid difference method							
10^{-3}	2.8953e – 02	1.1470e – 02	3.9972e – 03	1.4195e – 03	4.4816e – 04	1.3262e – 04	3.963906e – 05
10^{-4}	2.8954e – 02	1.1483e – 02	4.0105e – 03	1.4282e – 03	4.5308e – 04	1.3799e – 04	4.255447e – 05
10^{-5}	2.8954e – 02	1.1484e – 02	4.0119e – 03	1.4291e – 03	4.5358e – 04	1.3871e – 04	4.303554e – 05
10^{-6}	2.8954e – 02	1.1485e – 02	4.0120e – 03	1.4292e – 03	4.5363e – 04	1.3879e – 04	4.308411e – 05
10^{-7}	2.8954e – 02	1.1485e – 02	4.0120e – 03	1.4292e – 03	4.5364e – 04	1.3879e – 04	4.308899e – 05
10^{-8}	2.8954e – 02	1.1485e – 02	4.0120e – 03	1.4292e – 03	4.5364e – 04	1.3879e – 04	4.308959e – 05
10^{-9}	2.8954e – 02	1.1485e – 02	4.0120e – 03	1.4292e – 03	4.5363e – 04	1.3880e – 04	4.309005e – 05
10^{-10}	2.8954e – 02	1.1485e – 02	4.0121e – 03	1.4292e – 03	4.5358e – 04	1.3882e – 04	4.309514e – 05
D^N	2.8954e – 02	1.1485e – 02	4.0121e – 03	1.4292e – 03	4.5364e – 04	1.3882e – 04	4.309514e – 05
P^N	1.3340e + 00	1.5173e + 00	1.4891e + 00	1.6556e + 00	1.7082e + 00	1.6876e + 00	—

Proof. The proof of Theorem 6 follows from $y_k = v_k + w_k$ and $Y_k = V_k + W_k$ and using Theorems 3 and 4 □

7. Numerical Experiments

In this section, consider two examples for constant and variable coefficient problems and apply both of the numerical methods to find error and rate of convergence. The exact solution is not easy to find in these problems. Therefore, we use the double mesh principle:

$$D_\varepsilon^N = \max_{0\le i\le 3N}\left|U_i^N - U_{2i}^{2N}\right|. \quad (99)$$

We compute the uniform error and the rate of convergence as

$$D^N = \max_{\varepsilon} D_\varepsilon^N, \qquad P^N = \log_2\left(\frac{D^N}{D^{2N}}\right). \quad (100)$$

To solve the following numerical examples, we use two computational methods such as finite and hybrid difference scheme on the nonuniform mesh.

Example 1

$$\begin{aligned} &-\varepsilon y''(r) + 5y'(r) + 2y(r) - y(r-1) + y(r+1) = 1, \quad \text{for } r \in \Gamma^*,\\ &y(r) = 1, \quad \text{for } r \in [-1,0],\\ &y(r) = 2, \quad \text{for } r \in [3,4]. \end{aligned} \quad (101)$$

Example 2

$$\begin{aligned} &-\varepsilon y''(r) + (r+5)y'(r) + 2y(r) - y(r-1)\\ &\quad + x^2 y(r+1) = e^t, \quad \text{for } r \in \Gamma^*,\\ &y(r) = 1, \quad \text{for } r \in [-1,0],\\ &y(r) = 2, \quad \text{for } r \in [3,4]. \end{aligned} \quad (102)$$

We proved that the error is of order $O(N^{-1}\ln N)$ and $O(N^{-2}\ln^2 N)$. The theory has been validated with two examples; referring to these numerical results, it can be observed that the proposed method has been effective and applicable.

8. Discussion

In the literature, many authors have considered singular perturbation problem mixed delay $(\tau \ll 1)$ differential equation. In this paper, we consider a singular perturbation problem with mixed delay $(\tau = 1)$ differential equation. We suggested two computational methods such as finite and hybrid difference scheme. We proved that the error is of order $O(N^{-1}\ln N)$ and $O(N^{-2}\ln^2 N)$. Finally, two numerical examples are also presented to validate the theoretical results of this study. Maximum pointwise errors and order of convergence of Examples 1 and 2 are given in Tables 1 and 2, respectively.

Data Availability

No data were used to support this study

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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