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Finite difference scheme for singularly perturbed reaction diffusion problem of partial delay differential equation with nonlocal boundary condition

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Abstract

This paper investigates singularly perturbed parabolic partial differential equations with delay in space, and the right end plane is an integral boundary condition on a rectangular domain. A small parameter is multiplied in the higher order derivative, which gives boundary layers, and due to the delay term, one more layer occurs on the rectangle domain. A numerical method comprising the standard finite difference scheme on a rectangular piecewise uniform mesh (Shishkin mesh) of $N_r \times N_t$ elements condensing in the boundary layers is suggested, and it is proved to be parameter-uniform. Also, the order of convergence is proved to be almost two in space variable and almost one in time variable. Numerical examples are proposed to validate the theory.

Keywords: Parabolic delay differential equations; Singular perturbation problem; Integral boundary condition; Shishkin mesh; Finite difference scheme; Boundary layers

1 Introduction

Singularly perturbed parabolic partial differential equations have received an increasing research attention over the past few decades (see [1–9] and the references therein). Ansari et al. [10] discussed singularly perturbed time delay partial differential equations with Dirichlet boundary conditions. Avudai Selvi and Ramanujam [11] discussed singularly perturbed parabolic partial time delay differential equations with Robin type boundary condition. The authors in [10] and [11] studied a finite difference scheme for solving singularly perturbed parabolic time delay differential equation with Dirichlet and Robin boundary conditions on Shishkin mesh. They made use of the results of Miller et al. [12]. In [13–16] researchers discussed a fitted operator method to solve singularly perturbed time delay partial differential equations. By the way, there are many methods available in the literature for time delay problems [17] and [18], but the study of problems with delay in space variable are still in the initial stage.

In [19], the authors considered second order parabolic delay differential equations with integral boundary condition of the form

$$\begin{cases} -ku_{rr}(r,t) + u_t(r,t) = -d(r,t)u(r,t) + b(r,t)u(r,t-\tau), & (r,t) \in (0,\pi) \times [0,T], \\ u(r,t) = \phi(r,t), \quad t \in [-\tau,0], y \in (0,\pi), \\ u_r(0,t) = 0, \quad t \in [0,T], \\ \int_0^\pi u(r,t)\,dr = \psi(t), \quad t \in [0,T], \end{cases}$$

and proved the existence and uniqueness of the solution.

The above problem arises in the study of population density with time delay. There is no flux condition considered in the left, and the average population size is being controlled by the function $\psi(t)$ in the right. In the past few years interest has substantially increased in solving singularly perturbed differential equations with integral boundary conditions (see [20–25]). Motivated by the above works, we have designed a finite difference scheme for a singularly perturbed reaction diffusion problem of partial delay differential equation with nonlocal boundary condition.

The remaining article is structured as follows: In Sect. 2, the model problem is stated and some preliminaries are presented. In Sect. 3, uniqueness and stability of the solution are established. Also we prove that the derivatives of the solution are bounded. A finite difference method for the continuous problem, the discrete maximum principle, and the stability result are discussed in Sect. 4. The convergence analysis of the finite difference method on Shishkin mesh is given in Sect. 5. The theoretical results are verified by numerical examples in Sect. 6. Finally, discussion is given in Sect. 7.

2 Preliminaries

We consider singularly perturbed parabolic partial delay differential equations with integral boundary condition described by

$$\begin{cases} \mathcal{L}u_\varepsilon(r,t) = (-\varepsilon \frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial t} + a(r,t))u_\varepsilon(r,t) + b(r,t)u_\varepsilon(r-1,t) = f(r,t), & (r,t) \in D, \\ u_\varepsilon(r,t) = \phi_l(r,t), \quad \phi_l(r,t) \in \Gamma_l = \{(r,t); -1 \le r \le 0 \text{ and } 0 \le t \le T\}, \\ \mathcal{K}u_\varepsilon(r,t) = u_\varepsilon(2,t) - \varepsilon \int_0^2 g(r)u_\varepsilon(r,t)\,dr = \phi_{\bar{r}}(r,t), \\ \quad \phi_{\bar{r}}(r,t) \in \Gamma_{\bar{r}} = \{(2,t); 0 \le t \le T\}, \\ u_\varepsilon(r,t) = \phi_b(r,t), \quad \phi_b(r,t) \in \Gamma_b = \{(r,0); 0 \le r \le 2\}, \end{cases} \tag{1}$$

where $(r,t) \in D = (0,2) \times (0,T]$, $\bar{D} = [0,2] \times [0,T]$, $D_1 = (0,1) \times [0,T]$, $D_2 = (1,2) \times [0,T]$, $D^* = D_1 \cup D_2$ and ε is a small positive parameter ($0 < \varepsilon << 1$). Assume that $a(r,t) \ge \alpha > 0$, $b(r,t) \le \beta < 0, \alpha + \beta > 0, f(r,t), \phi_l, \phi_{\bar{r}}, \phi_b$ are sufficiently smooth and $g(r)$ is a monotonically nonnegative function and satisfies $\int_0^2 g(r)\,dr < 1$.

Problem (1) is equivalent to

$$\mathcal{L}u_\varepsilon(r,t) = F(r,t),$$

where

$$\mathcal{L}u_\varepsilon(r,t) = \begin{cases} \mathcal{L}_1 u_\varepsilon(r,t) = (-\varepsilon\frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial t} + a(r,t))u_\varepsilon(r,t), & (r,t)\in D_1, \\ \mathcal{L}_2 u_\varepsilon(r,t) = (-\varepsilon\frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial t} + a(r,t))u_\varepsilon(r,t) + b(r,t)u_\varepsilon(r-1,t), & (r,t)\in D_2, \end{cases} \tag{2}$$

$$F(r,t) = \begin{cases} f(r,t) - b(r,t)\phi_l(r-1,t), & (r,t)\in D_1, \\ f(r,t), & (r,t)\in D_2, \end{cases} \tag{3}$$

with boundary conditions

$$\begin{cases} u_\varepsilon(r,t) = \phi_l(r,t), \quad \phi_l(r,t)\in\Gamma_l = \{(r,t); -1\le r\le 0 \text{ and } 0\le t\le T\}, \\ u_\varepsilon(1^-,t) = u_\varepsilon(1^+,t), \quad \frac{\partial u_\varepsilon}{\partial z}(1^-,t) = \frac{\partial u_\varepsilon}{\partial z}(1^+,t), \\ \mathcal{K}u_\varepsilon(r,t) = u_\varepsilon(2,t) - \varepsilon\int_0^2 g(r)u_\varepsilon(r,t)\,dr = \phi_{\bar r}(r,t), \\ \quad \phi_{\bar r}(r,t)\in\Gamma_{\bar r} = \{(2,t); 0\le t\le T\}, \\ u_\varepsilon(r,t) = \phi_b(r,t), \quad \phi_b(r,t)\in\Gamma_b = \{(r,0); 0\le r\le 2\}. \end{cases} \tag{4}$$

3 The analytical problem

Lemma 1 (Maximum principle) *If* $\Psi(r,t)\in C^{(0,0)}(\bar D)\cap C^{(1,0)}(D)\cap C^{(2,1)}(D_1\cup D_2)$ *such that* $\Psi(0,t)\ge 0$, $\Psi(r,0)\ge 0$, $\mathcal{K}\Psi(2,t)\ge 0$, $\mathcal{L}_1\Psi(r,t)\ge 0$, $\forall(r,t)\in D_1$, $\mathcal{L}_2\Psi(r,t)\ge 0$, $\forall(r,t)\in D_2$ *and* $[\Psi_r](1,t) = \Psi_r(1^+,t) - \Psi_r(1^-,t)\le 0$, *then* $\Psi(r,t)\ge 0$ *for all* $(r,t)\in\bar D$.

Proof Define a test function

$$s(r,t) = \begin{cases} \frac{1}{8} + \frac{r}{2}, & (r,t)\in D_1, \\ \frac{3}{8} + \frac{r}{4}, & (r,t)\in D_2. \end{cases} \tag{5}$$

Note that $s(r,t) > 0$, $\forall(r,t)\in\bar D$, $\mathcal{L}s(r,t) > 0$, $\forall(r,t)\in D_1\cup D_2$, $s(0,t) > 0$, $s(r,0) > 0$, $\mathcal{K}s(2,t) > 0$, and $[s_r](1,t) < 0$. Let

$$\mu_1 = \max\left\{\frac{-\Psi(r,t)}{s(r,t)} : (r,t)\in\bar D\right\}.$$

Then there exists $(r_0,t_0)\in\bar D$ such that $\Psi(r_0,t_0) + \mu_1 s(r_0,t_0) = 0$ and $\Psi(r,t) + \mu_1 s(r,t)\ge 0$, $\forall(r,t)\in\bar D$. Then the function is minimum at $(r,t) = (r_0,t_0)$, and the proof is completed. Suppose $\mu_1 > 0$, we need a contradiction.

Case (i): $(r_0,t_0) = (0,t_0)$

$$0 < (\Psi + \mu_1 s)(0,t) = \Psi(0,t_0) + \mu_1 s(0,t_0) = 0.$$

Case (ii): $(r_0,t_0)\in D_1$

$$\begin{aligned} 0 &< \mathcal{L}_1(\Psi + \mu_1 s)(r_0,t_0) \\ &= -\varepsilon(\Psi + \mu_1 s)_{rr}(r_0,t_0) + (\Psi + \mu_1 s)_t(r_0,t_0) + a(r_0,t_0)(\Psi + \mu_1 s)(r_0,t_0)\le 0. \end{aligned}$$

Case (iii): $(r_0, t_0) = (1, t_0)$

$$0 \le \left[(\Psi + \mu_1 s)'\right](1, t_0) = \left[\Psi'\right](1, t_0) + \mu_1\left[s'\right](1, t_0) < 0.$$

Case (iv): $(r_0, t_0) \in D_2$

$$\begin{aligned} 0 &< \mathcal{L}_2(\Psi + \mu_1 s)(r_0, t_0) \\ &= -\varepsilon(\Psi + \mu_1 s)_{rr}(r_0, t_0) + (\Psi + \mu_1 s)_t(r_0, t_0) \\ &\quad + a(r_0, t_0)(\Psi + \mu_1 s)(r_0, t_0) + b(r_0, t_0)(\Psi + \mu_1 s)(r_0 - 1, t_0) \le 0. \end{aligned}$$

Case (v): $(r_0, t_0) = (2, t_0)$

$$0 < \mathcal{K}(\Psi + \mu_1 s)(2, t_0) = (\Psi + \mu_1 s)(2, t_0) - \varepsilon \int_0^2 g(r)(\Psi + \mu_1 s)(r, t)\, dr \le 0.$$

In all the cases we arrived at a contradiction, and thus the proof is completed. □

Lemma 2 (Stability result) *If $u(r,t)$ satisfies problem (2)–(4), then the bound of $u(r,t)$ is*

$$\|u\|_{\bar{D}} \le C \max\left\{\|u\|_{\Gamma_l}, \|u\|_{\Gamma_b}, \|\mathcal{K}u\|_{\Gamma_{\bar{r}}}, \|\mathcal{L}u\|_{D^*}\right\}, \quad (r,t) \in \bar{D}.$$

Proof It can be easily proved by using the maximum principle (Lemma 1) and the barrier functions $\theta^{\pm}(r,t) = CMs(r,t) \pm u(r,t)$, $(r,t) \in \bar{D}$, where $M = \max\{\|u\|_{\Gamma_l}, \|u\|_{\Gamma_b}, \|\mathcal{K}u\|_{\Gamma_{\bar{r}}}, \|\mathcal{L}u\|_{D^*}\}$ and $s(r,t)$ is the test function as in (5). □

3.1 Compatibility conditions

For problem (2)–(4) to have existence and uniqueness of a solution, we assume that the coefficients of the problem are *Hölder* continuous and also impose proper compatibility conditions at $(0,0)$, $(2,0)$, $(-1,0)$, and $(1,0)$ [26]. Necessarily the following conditions are satisfied:

$$\phi_b(0,0) = \phi_l(0,0), \qquad \phi_b(2,0) = \phi_{\bar{r}}(2,0) \tag{6}$$

and

$$\begin{aligned} &-\varepsilon \frac{\partial^2 \phi_b(0,0)}{\partial r^2} + a(0,0)\phi_b(0,0) + \frac{\partial \phi_l(0,0)}{\partial t} + b(0,0)\phi_l(-1,0) = f(0,0), \\ &-\varepsilon \frac{\partial^2 \phi_b(2,0)}{\partial r^2} + a(2,0)\phi_b(2,0) + \frac{\partial \phi_{\bar{r}}(2,0)}{\partial t} + b(2,0)\phi_b(1,0) = f(2,0). \end{aligned} \tag{7}$$

Note that $\phi_l(r,t)$, $\phi_{\bar{r}}(r,t)$, and $\phi_b(r,t)$ are assumed to be smooth for (7) to make sense, namely $\phi_l(r,t) \in C^1([0,T])$, $\phi_{\bar{r}}(r,t) \in C^1([0,T])$, and $\phi_b(r,t) \in C^{(2,1)}(\Gamma_b)$.

The following classical theorem gives sufficient conditions for the existence of a unique solution of problem (1).

Theorem 1 *Suppose $a(r,t), b(r,t), f(r,t) \in C^{(\alpha_1, \alpha_1/2)}(\bar{D})$ and $\phi_l \in C^{1+\alpha_1/2}([0,T])$, $\phi_b \in C^{(2+\alpha_1, 1+\alpha_1/2)}(\Gamma_b)$, $\phi_{\bar{r}} \in C^{1+\alpha_1/2}([0,T])$, $\alpha_1 \in (0,1)$, and assume that the compatibility* *conditions* (6) *and* (7) *are satisfied. Then problem* (2)–(4) *has a unique solution* $u_\varepsilon \in C^{(2+\alpha_1,1+\alpha_1/2)}(\bar{D})$.

Proof See [26]. □

3.2 The solution and its derivatives are bounded

Theorem 2 *Suppose* $a(r,t), b(r,t), f(r,t) \in C^{(2+\alpha_1,1+\alpha_1/2)}(\bar{D})$ *and* $\phi_l \in C^{(2+\alpha_1/2)}([0,T])$, $\phi_b \in C^{(4+\alpha_1,2+\alpha_1/2)}(\Gamma_b)$, $\phi_{\bar{r}} \in C^{(2+\alpha_1/2)}([0,T])$, *where* $\alpha_1 \in (0,1)$. *Then problem* (2)–(4) *has a unique solution, which satisfies* $u_\varepsilon \in C^{(4+\alpha_1,2+\alpha_1/2)}(\bar{D})$. *Also, the derivatives of the solution* u_ε *are bounded*, $\forall i,j \in \mathbb{Z} \geq 0$ *such that* $0 \leq i + 2j \leq 4$,

$$\left\| \frac{\partial^{i+j} u_\varepsilon}{\partial r^i \partial t^j} \right\| \leq C\varepsilon^{\frac{-i}{2}}.$$

Proof The first part of the proof is given in Ladyzhenskaya [26, Chap. IV, p. 320]. The solution and its derivatives are bounded as follows. Under the stretched transformation $\tilde{r} = \frac{r}{\sqrt{\varepsilon}}$, problem (1) can be rewritten as follows:

$$\begin{cases} \mathcal{L}\tilde{u}(\tilde{r},t) = (-\frac{\partial^2}{\partial \tilde{r}^2} + \frac{\partial}{\partial t} + \tilde{a}(\tilde{r},t))\tilde{u}(\tilde{r},t) + \tilde{b}(\tilde{r},t)\tilde{u}(\tilde{r}-1,t) = f(\tilde{r},t), & (\tilde{r},t) \in \tilde{D}_\varepsilon, \\ \tilde{u}(\tilde{r},t) = \phi_l(\tilde{r},t), \quad (\tilde{r},t) \in \tilde{\Gamma}_l, \\ \mathcal{K}\tilde{u}(\tilde{r},t) = \tilde{u}(2,t) - \varepsilon \int_0^2 g(r)\tilde{u}(\tilde{r},t)\,dr = \phi_{\bar{r}}(\tilde{r},t), \quad (\tilde{r},t) \in \tilde{\Gamma}_{\bar{r}}, \\ \tilde{u}(\tilde{r},t) = \phi_b(\tilde{r},t), \quad (\tilde{r},t) \in \tilde{\Gamma}_b. \end{cases} \tag{8}$$

The equivalent problem is

$$\begin{cases} \mathcal{L}_1\tilde{u}(\tilde{r},t) = (-\frac{\partial^2}{\partial \tilde{r}^2} + \frac{\partial}{\partial t} + \tilde{a}(\tilde{r},t))\tilde{u}(\tilde{r},t) = f(\tilde{x},t) - \tilde{b}(\tilde{r},t)\tilde{\phi}_l(\tilde{r}-1,t), & (\tilde{r},t) \in \tilde{D}_{1,\varepsilon}, \\ \mathcal{L}_2\tilde{u}(\tilde{r},t) = (-\frac{\partial^2}{\partial \tilde{r}^2} + \frac{\partial}{\partial t} + \tilde{a}(\tilde{r},t))\tilde{u}(\tilde{r},t) + \tilde{b}(\tilde{r},t)\tilde{u}(\tilde{r}-1,t) = f(\tilde{x},t), & (\tilde{r},t) \in \tilde{D}_{2,\varepsilon}, \end{cases} \tag{9}$$

$$\begin{cases} \tilde{u}(\tilde{r},t) = \phi_l(\tilde{r},t), \quad (\tilde{r},t) \in \tilde{\Gamma}_l, \\ \tilde{u}(1^-,t) = \tilde{u}(1^+,t), \qquad \frac{\partial \tilde{u}}{\partial z}(1^-,t) = \frac{\partial \tilde{u}}{\partial z}(1^+,t), \\ \mathcal{K}\tilde{u}(\tilde{r},t) = \tilde{u}(2,t) - \varepsilon \int_0^2 g(r)\tilde{u}(\tilde{r},t)\,dr = \phi_{\bar{r}}(\tilde{r},t), \quad (\tilde{r},t) \in \tilde{\Gamma}_{\bar{r}}, \\ \tilde{u}(\tilde{r},t) = \phi_b(\tilde{r},t), \quad (\tilde{r},t) \in \tilde{\Gamma}_b, \end{cases} \tag{10}$$

where $\tilde{D}_\varepsilon = (0, \frac{2}{\sqrt{\varepsilon}}) \times (0,T]$ and the boundary conditions $\tilde{\Gamma}$ to Γ.

Note that (8) is independent of ε. Then, using the idea of estimation (10.6) from [26, p. 352], we get

$$\left\| \frac{\partial^{i+j} \tilde{u}}{\partial \tilde{r}^i \partial t^j} \right\|_{\tilde{N}_\delta} \leq C\big(1 + \|\tilde{u}\|_{\tilde{N}_{2\delta}}\big)$$

for all $\tilde{N}_\delta$ in $\tilde{D}_\varepsilon$. Here, $\tilde{N}_\delta$, $\delta > 0$ is a neighborhood with diameter δ in $\tilde{D}_\varepsilon$. Returning to the original variable

$$\left\| \frac{\partial^{i+j} u_\varepsilon}{\partial r^i \partial t^j} \right\|_{\bar{D}} \leq C\varepsilon^{\frac{-i}{2}}\big(1 + \|u_\varepsilon\|_{\bar{D}}\big).$$

The proof is completed by using the bound on u_ε in Lemma 2. □

3.3 Decomposition of the solution

The solution u_ε of (2)–(4) is decomposed into v_ε-smooth and w_ε-singular components.

Further, v_ε is separated into

$$v_\varepsilon = v_0 + \varepsilon v_1,$$

where v_0 and v_1 are solutions of the following differential equations respectively:

$$\begin{cases} \frac{\partial v_0}{\partial t}(r,t) + a(r,t)v_0(r,t) + b(r,t)v_0(r-1,t) = f(r,t), & (r,t) \in D, \\ v_0(r,t) = 0, & (r,t) \in \Gamma_b, \end{cases} \tag{11}$$

and

$$\begin{cases} \mathcal{L}v_1(r,t) = \frac{\partial^2 v_0}{\partial r^2}(r,t), & (r,t) \in D, \\ v_1(r,t) = 0, & (r,t) \in \Gamma_l, \\ \mathcal{K}v_1(r,t) = 0, & (r,t) \in \Gamma_{\bar{r}}, \\ v_1(r,t) = 0, & (r,t) \in \Gamma_b. \end{cases} \tag{12}$$

Then v_ε satisfies the condition

$$\begin{cases} \mathcal{L}v_\varepsilon(r,t) = f(r,t), & (r,t) \in D, \\ v_\varepsilon(r,t) = v_0(r,t), & (r,t) \in \Gamma_l, \\ \mathcal{K}v_\varepsilon(r,t) = \mathcal{K}v_0(2,t), & (r,t) \in \Gamma_{\bar{r}}, \\ v_\varepsilon(r,t) = \phi_b(r,t), & (r,t) \in \Gamma_b. \end{cases} \tag{13}$$

The singular component w_ε is determined from

$$\begin{cases} \mathcal{L}_1 w_\varepsilon(r,t) = 0, & (r,t) \in D_1, \\ w_\varepsilon(r,t) = \phi_l(r,t) - v_0(r,t), & (r,t) \in \Gamma_l, \\ [w](r,t) = -[v](r,t), & (r,t) \in \Gamma_{\bar{r}}, \\ \text{i,e) } w_r(1^+,t) - w_r(1^-,t) = -(v_r(1^+,t) - v_r(1^-,t)), \\ w_\varepsilon(r,t) = 0, & (r,t) \in \Gamma_b, \end{cases} \tag{14}$$

and

$$\begin{cases} \mathcal{L}_2 w_\varepsilon(r,t) = 0, & (r,t) \in D_2, \\ [w](r,t) = -[v](r,t), & (r,t) \in \Gamma_l, \\ \text{i,e) } w_r(1^+,t) - w_r(1^-,t) = -(v_r(1^+,t) - v_r(1^-,t)), \\ \mathcal{K}w_\varepsilon(r,t) = \mathcal{K}u_\varepsilon(r,t) - \mathcal{K}v_0(r,t), & (r,t) \in \Gamma_{\bar{r}}, \\ w_\varepsilon(r,t) = 0, & (r,t) \in \Gamma_b. \end{cases} \tag{15}$$

The reaction diffusion problem has boundary layers on both boundaries Γ_l and $\Gamma_{\tilde{r}}$. Separate w_ε as $w_\varepsilon = w_l + w_{\tilde{r}}$, where w_l and $w_{\tilde{r}}$ satisfy the following problems:

$$\begin{cases} \mathcal{L}_1 w_l(r,t) = 0, & (r,t) \in D_1, \\ w_l(r,t) = \phi_l(r,t) - v_0(r,t), & (r,t) \in \Gamma_l, \\ w_l(r,t) = 0, & (r,t) \in \Gamma_{\tilde{r}}, \\ w_l(r,t) = 0, & (r,t) \in \Gamma_b, \end{cases} \tag{16}$$

$$\begin{cases} \mathcal{L}_2 w_l(r,t) = 0, & (r,t) \in D_2, \\ w_l(r,t) = A, & (r,t) \in \Gamma_l, \\ w_l(r,t) = 0, & (r,t) \in \Gamma_{\tilde{r}}, \\ w_l(r,t) = 0, & (r,t) \in \Gamma_b, \end{cases} \tag{17}$$

and

$$\begin{cases} \mathcal{L}_1 w_{\tilde{r}}(r,t) = 0, & (r,t) \in D_1, \\ w_{\tilde{r}}(r,t) = 0, & (r,t) \in \Gamma_l, \\ w_{\tilde{r}}(r,t) = A, & (r,t) \in \Gamma_{\tilde{r}}, \\ w_{\tilde{r}}(r,t) = 0, & (r,t) \in \Gamma_b, \end{cases} \tag{18}$$

$$\begin{cases} \mathcal{L}_2 w_{\tilde{r}}(r,t) = 0, & (r,t) \in D_2, \\ w_{\tilde{r}}(r,t) = 0, & (r,t) \in \Gamma_l, \\ \mathcal{K} w_{\tilde{r}}(r,t) = \mathcal{K} w(r,t), & (r,t) \in \Gamma_{\tilde{r}}, \\ w_{\tilde{r}}(r,t) = 0, & (r,t) \in \Gamma_b. \end{cases} \tag{19}$$

Here, we choose A to be a suitable constant, then the solution is continuous at $r = 1$.

Theorem 3 *Suppose $a(r,t), b(r,t), f(r,t) \in C^{(4+\alpha_1, 2+\alpha_1/2)}(\bar{D})$ and $\phi_l \in C^{(3+\alpha_1/2)}([0,T])$, $\phi_b \in C^{(6+\alpha_1, 3+\alpha_1/2)}(\Gamma_b)$, $\phi_{\tilde{r}} \in C^{(3+\alpha_1/2)}([0,T])$, where $\alpha_1 \in (0,1)$. Then we have*

$$\left\| \frac{\partial^{i+j} v_\varepsilon}{\partial r^i \partial t^j} \right\|_{\bar{D}} \le C\bigl(1 + \varepsilon^{1-i/2}\bigr), \tag{20}$$

$$\left| \frac{\partial^{i+j} w_l(r,t)}{\partial r^i \partial t^j} \right| \le \begin{cases} C\varepsilon^{\frac{-i}{2}} e^{\frac{-r}{\sqrt{\varepsilon}}}, & (r,t) \in D_1, \\ C\varepsilon^{\frac{-i}{2}} e^{\frac{-(r-1)}{\sqrt{\varepsilon}}}, & (r,t) \in D_2, \end{cases} \tag{21}$$

$$\left| \frac{\partial^{i+j} w_{\tilde{r}}(r,t)}{\partial r^i \partial t^j} \right| \le \begin{cases} C\varepsilon^{\frac{-i}{2}} e^{\frac{-(1-r)}{\sqrt{\varepsilon}}}, & (r,t) \in D_1, \\ C\varepsilon^{\frac{-i}{2}} e^{\frac{-(2-r)}{\sqrt{\varepsilon}}}, & (r,t) \in D_2, \end{cases} \tag{22}$$

where C is a constant independent of the parameter ε, $(r,t) \in \bar{D}$, $i,j \ge 0$, $0 \le i + 2j \le 4$.

Proof The existence and smooth component result follow from [26, Chap. 4, p. 320]. The derivatives of smooth component functions are bounded and derived as follows.

First, we estimate that the reduced problem solution v_0 is bounded and its derivative is bounded

$$\left\| \frac{\partial^{i+j} v_0}{\partial r^i \partial t^j} \right\|_{\bar{D}} \leq C. \tag{23}$$

Using Theorem 2, we estimate that the derivative of the solution v_1 becomes

$$\left\| \frac{\partial^{i+j} v_1}{\partial r^i \partial t^j} \right\|_{\bar{D}} \leq C\varepsilon^{\frac{-i}{2}} \tag{24}$$

since

$$\frac{\partial^{i+j} v_\varepsilon}{\partial r^i \partial t^j} = \frac{\partial^{i+j} v_0}{\partial r^i \partial t^j} + \varepsilon \frac{\partial^{i+j} v_1}{\partial r^i \partial t^j}.$$

By (23) and (24), we establish smooth component v_ε estimates.

To prove inequality (21), following the procedure adapted in [12], it is easy to find that

$$\left| w_l(r,t) \right| \leq Ce^{\frac{-r}{\sqrt{\varepsilon}}}, \quad (r,t) \in D_1.$$

Now, we derive the bound on w_l on D_2. From the defining equations for w_l, we have

$$\mathcal{L}_2 w_l(r,t) = -\varepsilon w_{rr}(r,t) + w_t(r,t) + a(r,t)w_l(r,t) + b(r,t)w_l(r-1,t) = 0$$

or

$$\begin{aligned}
&\mathcal{L}_2 w_l(r,t) = -\varepsilon w_{rr}(r,t) + w_t(r,t) + a(r,t)w_l(r,t) = -b(r,t)w_l(r-1,t), \\
&\mathcal{L}_2 w_l(r,t) = \mathcal{L}_1 w_l(r,t) = -b(r,t)w_l(r-1,t), \\
&\left| \mathcal{L}_2 w_l(r,t) \right| \leq Ce^{\frac{-(r-1)}{\sqrt{\varepsilon}}}
\end{aligned}$$

for a choice of suitable $C > 0$. Using the stability result (Lemma 2), we have $|w_l(r,t)| \leq Ce^{\frac{-(r-1)}{\sqrt{\varepsilon}}}$.

Hence,

$$\left| w_l(r,t) \right| \leq \begin{cases} Ce^{\frac{-(r)}{\sqrt{\varepsilon}}}, & (r,t) \in D_1, \\ Ce^{\frac{-(r-1)}{\sqrt{\varepsilon}}}, & (r,t) \in D_2. \end{cases}$$

Let $\tilde{r} = \frac{r}{\sqrt{\varepsilon}}$. Clearly, under the transformation, stretched variable $\tilde{r}$ domain is $(0, \frac{2}{\sqrt{\varepsilon}})$, note that $(r,t) \Rightarrow (\tilde{r},t)$ and the parameter ε is independent of problem (16)–(17), then suitable estimates in [26, Sect. 4.10] are the solution of $\tilde{w}_l$. The position of $\tilde{r}$ argument is divided into two cases. In the first case, for a neighborhood of $\tilde{N}_\delta$ in $(0, \frac{2}{\sqrt{\varepsilon}}) \times (0,T]$, apply [26, Sect. 4.10], then

$$\left\| \frac{\partial^{i+j} \tilde{w}_l}{\partial \tilde{r}^i \partial t^j} \right\| \leq C\left(\|\tilde{w}_l\|_{\tilde{N}_{2\delta}} \right)$$

and derive the essential bound of r, the variable $\tilde{r}$ changing into variable r, using the bound of w_l.

Similarly, for any neighborhood $\tilde{N}_\delta$ in $(0,2] \times (0,T]$, apply [26, Sect. 4.10], then

$$\left\| \frac{\partial^{i+j}\tilde{w}_l}{\partial \tilde{r}^i \partial t^j} \right\| \leq C\bigl(1 + \|\tilde{w}_l\|_{\tilde{N}_{2\delta}}\bigr),$$

and derive the essential bound of r, the variable $\tilde{r}$ changing into variable r, using the w_l bound, and observe $e^{\frac{-r}{\sqrt{\varepsilon}}} \geq e^{-2} = C$ for $\tilde{r} \leq 2$.

Now to prove (22), consider

$$\theta^{\pm}(r,t) = C \begin{cases} e^{\alpha^* t} e^{\frac{-(1-r)}{\sqrt{\varepsilon}}}, & (r,t) \in D_1 \\ e^{\beta^* t} e^{\frac{-(2-r)}{\sqrt{\varepsilon}}}, & (r,t) \in D_2 \end{cases} \pm w_{\tilde{r}}(r,t),$$

where $\alpha^* = \max\{0, 1 - \min_{(r,t)\in\bar{D}} a(r,t)\}$ and $\beta^* = \max\{0, 1 - \min_{(r,t)\in\bar{D}}(a(r,t) + b(r,t))\}$.

Note that $\theta^{\pm}(0,t) \geq 0$,

$$\begin{aligned}
\mathcal{L}_1\theta^{\pm}(r,t) &= -\varepsilon\theta^{\pm}_{rr}(r,t) + \theta^{\pm}_t(r,t) + a(r,t)\theta^{\pm}(r,t) \\
&= C\bigl(\alpha^* - 1 + a(r,t)\bigr)e^{\frac{-(1-r)}{\sqrt{\varepsilon}}} e^{\alpha^* t} \\
&\geq 0, \\
\mathcal{L}_2\theta^{\pm}(r,t) &= -\varepsilon\theta^{\pm}_{rr}(r,t) + \theta^{\pm}_t(r,t) + a(r,t)\theta^{\pm}(r,t) + b(r,t)\theta^{\pm}(r-1,t) \\
&= C\bigl(\beta^* - 1 + a(r,t) + b(r,t)e^{\frac{-1}{\sqrt{\varepsilon}}}\bigr)e^{\frac{-(2-r)}{\sqrt{\varepsilon}}} e^{\beta^* t} \\
&\geq C\bigl(\alpha^* - 1 + a(r,t) + b(r,t)\bigr)e^{\frac{-(2-r)}{\sqrt{\varepsilon}}} e^{\alpha^* t} \\
&\geq 0,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{K}\theta^{\pm}(2,t) &= \theta^{\pm}(2,t) - \varepsilon \int_0^2 g(r)\theta^{\pm}(r,t)\,dr \\
&= C - C\varepsilon \int_0^1 g(r)\theta^{\pm}(r,t)\,dr - C\varepsilon \int_1^2 g(r)\theta^{\pm}(r,t)\,dr \\
&= C - C\varepsilon \int_0^1 g(r)e^{\frac{-(1-r)}{\sqrt{\varepsilon}}}\,dr - C\varepsilon \int_1^2 g(r)e^{\frac{-(2-r)}{\sqrt{\varepsilon}}}\,dr \\
&\geq C\left(1 - \varepsilon \int_0^2 g(r)\,dr\right) \\
&\geq 0
\end{aligned}$$

for a choice of suitable $C > 0$. Using the maximum principle (Lemma 1), we have

$$\bigl|w_{\tilde{r}}(r,t)\bigr| \leq \begin{cases} Ce^{\frac{-(1-r)}{\sqrt{\varepsilon}}}, & (r,t) \in D_1, \\ Ce^{\frac{-(2-r)}{\sqrt{\varepsilon}}}, & (r,t) \in D_2. \end{cases}$$

Let $\tilde{r} = \frac{r}{\sqrt{\varepsilon}}$. Clearly, under the transformation, the stretched variable $\tilde{r}$ domain is $(0, \frac{2}{\sqrt{\varepsilon}})$, note that $(r,t) \Rightarrow (\tilde{r},t)$ and the parameter ε is independent of problem (18)–(19), then suitable estimates in [26, Sect. 4.10] are the solution of $\tilde{w}_{\tilde{r}}$. The position of $\tilde{r}$ argument is divided into two cases. In the first case, for the neighborhood of $\tilde{N}_\delta$ in $(0, \frac{2}{\sqrt{\varepsilon}}) \times (0,T]$, applying [26, Sect. 4.10], we get

$$\left\| \frac{\partial^{i+j}\tilde{w}_{\tilde{r}}}{\partial \tilde{r}^i \partial t^j} \right\| \le C\bigl(\|\tilde{w}_{\tilde{r}}\|_{\tilde{N}_{2\delta}}\bigr)$$

and derive the essential bound of r, variable $\tilde{r}$ changing into variable r, using the $w_{\tilde{r}}$ bound.

Similarly, for any neighborhood $\tilde{N}_\delta$ in $(0,2] \times (0,T]$ applying [26, Sect. 4.10], we get

$$\left\| \frac{\partial^{i+j}\tilde{w}_{\tilde{r}}}{\partial \tilde{r}^i \partial t^j} \right\| \le C\bigl(1 + \|\tilde{w}_{\tilde{r}}\|_{\tilde{N}_{2\delta}}\bigr)$$

and derive the essential bound of r; moreover, the variable $\tilde{r}$ changing into variable r and using bound of $w_{\tilde{r}}$, observe $e^{\frac{-r}{\sqrt{\varepsilon}}} \ge e^{-2} = C$ for $\tilde{r} \le 2$. Hence proved. □

4 The discretised problem

In this part, we apply the finite difference method to the continuous problem (2)–(4) on a piecewise uniform mesh. The second order space derivative (u_{rr}) is replaced by the central difference scheme ($\delta_r^2 U$), and the first order time derivative (u_t) is replaced by the backward difference scheme ($D_t^- U$). In r-direction the interval $\Theta = [0,2]$ is divided into $\bar{\Theta}_1 = [0,1]$ and $\bar{\Theta}_2 = [1,2]$ with $\frac{N}{2}$ equal mesh points. Furthermore, the piecewise uniform mesh (Shishkin mesh) $\bar{\Theta}_1 = [0,1]$ is divided into three subintervals

$$\bar{\Theta}_1 = \bar{\Theta}_l \cup \bar{\Theta}_c \cup \bar{\Theta}_{\tilde{r}},$$

where $\Theta_l = (0,\mu)$, $\Theta_c = (\mu, 1-\mu)$, $\Theta_{\tilde{r}} = (1-\mu, 1)$.

Similarly, the interval $\bar{\Theta}_2 = [1,2]$ is divided into three subintervals

$$\bar{\Theta}_2 = \bar{\Theta}_l \cup \bar{\Theta}_c \cup \bar{\Theta}_{\tilde{r}},$$

where $\Theta_l = (1, 1+\mu)$, $\Theta_c = (1+\mu, 2-\mu)$, $\Theta_{\tilde{r}} = (2-\mu, 2)$. Furthermore, μ is called a fitting factor, and it satisfies the following condition:

$$\mu = \min\left\{\frac{1}{4}, 2\sqrt{\varepsilon}\ln(N_r)\right\},$$

where N_r denotes the number of mesh elements in the r-direction.

A piecewise uniform mesh $\Theta_\mu^{N_r}$ is on Θ with N_r mesh elements. Both intervals Θ_l and $\Theta_{\tilde{r}}$ are uniform meshes with $\frac{N_r}{8}$ elements and Θ_c is also a uniform mesh with $\frac{N_r}{4}$ mesh element, and thus we divide r-direction. Uniform meshes on t-direction on step size Δt and the number mesh denotes N_t in t-direction. $D_\mu^N = \Theta_\mu^{N_r} \times \Theta^{N_t}$ and the boundary analogues Γ_μ^N of D_μ^N are $\Gamma_\mu^N = \bar{D}_\mu^N \cap \Gamma$. We put $\Gamma_{l,\mu}^N = \Gamma_\mu^N \cap \Gamma_l$, $\Gamma_{r,\mu}^N = \Gamma_\mu^N \cap \Gamma_{\tilde{r}}$, and $\Gamma_{b,\mu}^N = \Gamma_\mu^N \cap \Gamma_b$.

Using the finite difference method for the continuous problem (2)–(4).

$$\mathcal{L}^N U_\varepsilon(r_i,t_j) = F(r_i,t_j),$$

where

$$\mathcal{L}^N U_\varepsilon(r_i,t_j) = \begin{cases} \mathcal{L}_1^N U_\varepsilon(r_i,t_j) = (-\varepsilon\delta_r^2 U_\varepsilon + D_t^- U_\varepsilon + aU_\varepsilon)(r_i,t_j), & (r_i,t_j)\in D_1^N, \\ \mathcal{L}_2^N U_\varepsilon(r_i,t_j) = (-\varepsilon\delta_r^2 U_\varepsilon + D_t^- U_\varepsilon + aU_\varepsilon)(r_i,t_j) & \\ \qquad + bU_\varepsilon(r_{i-\frac{N}{2}},t_j), & (r_i,t_j)\in D_2^N, \end{cases} \tag{25}$$

$$F(r_i,t_j) = \begin{cases} f(r_i,t_j) - b(r_i,t_j)\phi_l(r_{i-\frac{N}{2}},t_j), & (r_i,t_j)\in D_1^N, \\ f(r_i,t_j), & (r_i,t_j)\in D_2^N, \end{cases} \tag{26}$$

with boundary conditions

$$\begin{cases} U_\varepsilon(r_i,t_j) = \phi_l(r_i,t_j), \quad \phi_l(r_i,t_j)\in\Gamma_l^N, \\ \mathcal{K}^N U_\varepsilon(r_N,t_j) = U_\varepsilon(r_N,t_j) - \varepsilon\sum_{i=1}^N \frac{g(r_{i-1})U_\varepsilon(r_{i-1},t_j)+g(r_i)U_\varepsilon(r_i,t_j)}{2}h_i \\ \qquad = \phi_{\tilde{r}}, \quad \phi_{\tilde{r}}(r_i,t_j)\in\Gamma_{\tilde{r}}^N, \\ D_r^- U_\varepsilon(r_{\frac{N}{2}},t_j) = D_r^+ U_\varepsilon(r_{\frac{N}{2}},t_j), \\ U_\varepsilon(r_i,t_j) = \phi_b(r_i,t_j), \quad \phi_b(r_i,t_j)\in\Gamma_b^N, \text{where } i,j = 1,2,\ldots,N. \end{cases} \tag{27}$$

The differential operator notation is

$$\begin{cases} \delta_r^2 U_{i,j} = \frac{(D_r^+ - D_r^-)U_{i,j}}{(r_{i+1}-r_{i-1})/2} \\ \text{with} \\ D_r^+ U_{i,j} = \frac{U_{i+1,j}-U_{i,j}}{r_{i+1}-r_i}, \qquad D_r^- U_{i,j} = \frac{U_{i,j}-U_{i-1,j}}{r_i - r_{i-1}} \end{cases}$$

and an analogous definition of D_t^-.

Lemma 3 (Discrete maximum principle) *Let Z be any mesh function satisfying $Z(r_0,t_j)\geq 0$, $Z(r_i,t_0)\geq 0$, $\mathcal{K}^N Z(r_N,t_j)\geq 0$, $\mathcal{L}_1^N Z(r_i,t_j)\geq 0$, $\forall (r_i,t_j)\in D_1^N$, $\mathcal{L}_2^N Z(r_i,t_j)\geq 0$, $\forall (r_i,t_j)\in D_2^N$, and $[D_r]Z(r_{\frac{N}{2}},t_j) = D_r^+ Z(r_{\frac{N}{2}},t_j) - D_r^- Z(r_{\frac{N}{2}},t_j)\leq 0$. Then $Z(r_i,t_j)\geq 0$ for all $(r_i,t_j)\in\bar{D}^N$.*

Proof Define a test function $S(r_i,t_j)$ as

$$S(r_i,t_j) = \begin{cases} \frac{1}{8} + \frac{r_i}{2}, & (r_i,t_j)\in D_1^N, \\ \frac{3}{8} + \frac{r_i}{4}, & (r_i,t_j)\in D_2^N. \end{cases} \tag{28}$$

Note that $S(r_i,t_j) > 0$, $\forall (r_i,t_j)\in\bar{D}^N$, $\mathcal{L}^N S(r_i,t_j) > 0$, $\forall (r_i,t_j)\in D_1^N\cup D_2^N$, $S(r_0,t_j) > 0$, $S(r_i,t_0) > 0$, $\mathcal{K}^N S(r_N,t_j) > 0$, and $[D_r]S(r_{\frac{N}{2}},t_j) < 0$. Let

$$\eta = \max\left\{\frac{-r(r_i,t_j)}{S(r_i,t_j)} : (r_i,t_j)\in\bar{D}^N\right\}.$$

Then there exists $(r^*,t^*)\in\bar{D}^N$ such that $Z(r^*,t^*) + \eta S(r^*,t^*) = 0$ and $Z(r_i,t_j) + \eta S(r_i,t_j)\geq 0$, $\forall (r_i,t_j)\in\bar{D}^N$. Then the function is minimum at $(r,t) = (r^*,t^*)$ and this proof is completed. Suppose $\eta > 0$, we need a contradiction.

Case (i): $(r^*,t^*) = (r_0,t^*)$

$$0 < (r + \eta S)(r_0,t^*) = Z(r_0,t^*) + \eta S(r_0,t^*) = 0.$$

Case (ii): $(r^*, t^*) \in D_1^N$

$$0 < \mathcal{L}_1^N (r + \eta S)\big(r^*, t^*\big) = \big(-\varepsilon\delta_r^2 + D_t + a\big)(r + \eta S)\big(r^*, t^*\big) \leq 0.$$

Case (iii): $(r^*, t^*) = (r_{\frac{N}{2}}, t^*)$

$$0 \leq \big[D_r(\Psi + \eta s)\big]\big(r_{\frac{N}{2}}, t^*\big) < 0.$$

Case (iv): $(r^*, t^*) \in D_2^N$

$$\begin{aligned} 0 &< \mathcal{L}_2^N (r + \eta S)\big(r^*, t^*\big) \\ &= \big(-\varepsilon\delta_r^2 + D_t + a\big)(r + \eta S)\big(r^*, t^*\big) + b(r + \eta S)\big(r^* - r_{\frac{N}{2}}, t^*\big) \leq 0. \end{aligned}$$

Case (v): $(r^*, t^*) = (r_N, t^*)$

$$0 < \mathcal{K}^N (r + \eta S)\big(r_N, t^*\big) = (r + \eta S)\big(r_N, t^*\big) - \varepsilon \sum_{i=1}^{N} \frac{g_{i-1}(r + \eta S)_{i-1,j} + g_i(r + \eta S)_{i,j}}{2} h_i \leq 0.$$

In all the cases we arrived at a contradiction. Hence proved. □

Lemma 4 *Prove that*

$$\|Z\|_{\bar{D}^N} \leq C \max\Big\{\|Z\|_{\Gamma_l^N}, \|Z\|_{\Gamma_b^N}, \big\|\mathcal{K}^N Z\big\|_{\Gamma_{\tilde{r}}^N}, \max_{(r_i,t_j)\in D_1^N \cup D_2^N} \big\|\mathcal{L}^N Z\big\|\Big\}, \quad (r_i, t_j) \in \bar{D}^N.$$

Proof It can be easily proved using the discrete maximum principle (Lemma 3) and the barrier functions $\Xi^{\pm}(r_i, t_j) = CMS(r_i, t_j) \pm Z(r_i, t_j)$, $(r_i, t_j) \in \bar{D}^N$, where

$$M = \max\Big\{\|Z\|_{\Gamma_l^N}, \|Z\|_{\Gamma_b^N}, \big\|\mathcal{K}^N Z\big\|_{\Gamma_{\tilde{r}}^N}, \max_{(r_i,t_j)\in D_1^N \cup D_2^N} \big\|\mathcal{L}^N Z\big\|\Big\}, \tag{29}$$

and $S(r_i, t_j)$ is the test function as in Lemma 3. □

5 Error estimate

In this section, we prove parameter uniform convergence of the numerical solution.

Theorem 4 *Let u_ε and U_ε be solutions of problems (2)–(4) and (25)–(27). Assume that the coefficients $a(r,t), b(r,t), f(r,t) \in C^{(4+\alpha_1, 2+\alpha_1/2)}(\bar{D})$ and the boundary conditions satisfy $\phi_l \in C^{(3+\alpha_1/2)}([0,T])$, $\phi_b \in C^{(6+\alpha_1, 3+\alpha_1/2)}(\Gamma_b)$, $\phi_{\tilde{r}} \in C^{(3+\alpha_1/2)}([0,T])$, where $\alpha_1 \in (0,1)$. Then we have*

$$\sup_{0<\varepsilon\leq 1} \|U_\varepsilon - u_\varepsilon\|_{\bar{D}^N} \leq C\big(N_r^{-2} \ln^2 N_r + N_t^{-1}\big).$$

Proof Since the continuous problem solution u_ε of $\mathcal{L}_\varepsilon$ is decomposed into smooth and singular components, by the same process, the discrete problem solution U_ε of $\mathcal{L}_\varepsilon^N$ is decomposed into smooth and singular components as

$$U_\varepsilon = V_\varepsilon + W_\varepsilon,$$

where the V_ε-smooth component is the solution of the following problem:

$$\begin{cases} \mathcal{L}^N V_\varepsilon(r_i,t_j) = f(r_i,t_j), & (r_i,t_j) \in D^N, \\ V_\varepsilon(r_i,t_j) = \phi_0(r_i,t_j), & (r_i,t_j) \in \Gamma_l^N, \\ \mathcal{K}^N V_\varepsilon(r_N,t_j) = \mathcal{K}^N V_0(r_N,t_j), & (r_i,t_j) \in \Gamma_{\bar{r}}^N, \\ V_\varepsilon(r_i,t_j) = \phi_b(r_i,t_j), & (r_i,t_j) \in \Gamma_b^N, \end{cases}$$

and therefore W_ε must satisfy

$$\begin{cases} \mathcal{L}^N W_\varepsilon(r_i,t_j) = 0, & (r_i,t_j) \in D^N, \\ W_\varepsilon(r_i,t_j) = U_\varepsilon(r_i,t_j) - V_\varepsilon(r_i,t_j), & (r_i,t_j) \in \Gamma_l^N, \\ \mathcal{K}^N W_\varepsilon(r_N,t_j) = \mathcal{K}^N U_\varepsilon(r_N,t_j) - \mathcal{K}^N V_0(r_N,t_j), & (r_i,t_j) \in \Gamma_{\bar{r}}^N, \\ W_\varepsilon(r_i,t_j) = 0, & (r_i,t_j) \in \Gamma_b^N. \end{cases}$$

The error can be written in the form

$$U_\varepsilon - u_\varepsilon = (V_\varepsilon - v_\varepsilon) + (W_\varepsilon - w_\varepsilon),$$

and we prove separately the error estimates of smooth and singular components.

First, we derive the error estimate for the smooth component using the following classical argument.

At the point $r_i = r_N$,

$$\begin{aligned} \mathcal{K}^N(V_\varepsilon - v_\varepsilon)(r_N,t_j) &= \mathcal{K}^N V_\varepsilon(r_N,t_j) - \mathcal{K}^N v_\varepsilon(r_N,t_j) \\ &= \phi_{\bar{r}} - \mathcal{K}^N v_\varepsilon(r_N,t_j) \\ &= \mathcal{K} v_\varepsilon(r_N,t) - \mathcal{K}^N v_\varepsilon(r_N,t_j) \\ &= v_\varepsilon(r_N,t) - \int_{r_0}^{r_N} g(r) v_\varepsilon(r,t)\,dr - v_\varepsilon(r_N,t_j) + \sum_{i=1}^{N} \frac{g_{i-1}v_{i-1,j} + g_i v_{i,j}}{2} h_i \\ &= \frac{g_0 v_{0,j} + g_1 v_{1,j}}{2} h_1 + \cdots + \frac{g_{N-1}v_{N-1,j} + g_N v_{N,j}}{2} h_N \\ &\quad - \int_{r_0}^{r_1} g(r) v_\varepsilon(r,t)\,dr - \cdots - \int_{r_{N-1}}^{r_N} g(r) v_\varepsilon(r,t)\,dr \\ &= -\frac{h_1^3}{12} g''(\chi_1) v_\varepsilon''(\chi_1,t) - \cdots - \frac{h_{2N}^3}{12} g''(\chi_N) v_\varepsilon''(\chi_N,t), \\ \bigl|\mathcal{K}^N(V_\varepsilon - v_\varepsilon)(r_N,t_j)\bigr| &= \bigl|C\bigl(h_1^3 v_\varepsilon''(\chi_1,t) + \cdots + h_N^3 v_\varepsilon''(\chi_N,t)\bigr)\bigr|, \\ \bigl|\mathcal{K}^N(V_\varepsilon - v_\varepsilon)(r_N,t_j)\bigr| &\le C\varepsilon\bigl(\bigl(h_1^3 v_\varepsilon''(\chi_1,t) + \cdots + h_N^3 v_\varepsilon''(\chi_N,t)\bigr) \\ &\le C\varepsilon^{-1}\bigl(e^{-\chi_1\sqrt{\frac{\alpha}{\varepsilon}}} h_1^3 + \cdots + e^{-\chi_N\sqrt{\frac{\alpha}{\varepsilon}}} h_N^3\bigr) \\ &\le C\varepsilon\bigl(h_1^3 + \cdots + h_N^3\bigr) \\ &\le C N_r^{-2}, \end{aligned}$$

where $r_{i-1} \le \chi_i \le r_i$, $1 \le i \le N$, for a proper choice of C.

From the difference and differential equations, it is not hard to see that

$$\mathcal{L}_1^N(V_\varepsilon - v_\varepsilon) = f - \mathcal{L}_1^N v_\varepsilon = \left(\mathcal{L}_1 - \mathcal{L}_1^N\right) v_\varepsilon.$$

Then

$$\mathcal{L}_1^N(V_\varepsilon - v_\varepsilon) = -\varepsilon\left(\frac{\partial^2}{\partial r^2} - \delta_r^2\right) v_\varepsilon + \left(\frac{\partial}{\partial t} - D_t^-\right) v_\varepsilon.$$

It follows from classical estimates [12], at each point (r_i, t_j) in D_1^N,

$$\left|\mathcal{L}_1^N(V_\varepsilon - v_\varepsilon)(r_i, t_j)\right| \le C \begin{cases} (\sqrt{\varepsilon} N_r^{-1} + N_t^{-1}), & \text{if } r_i = \mu \text{ or } r_i = 1 - \mu, \\ (N_r^{-2} + N_t^{-1}), & \text{otherwise.} \end{cases}$$

Similarly,

$$\left|\mathcal{L}_2^N(V_\varepsilon - v_\varepsilon)(r_i, t_j)\right| \le C \begin{cases} (\sqrt{\varepsilon} N_r^{-1} + N_t^{-1}), & \text{if } r_i = 1 + \mu \text{ or } r_i = 2 - \mu, \\ (N_r^{-2} + N_t^{-1}), & \text{otherwise.} \end{cases}$$

Define

$$\phi(r_i, t_j) = C \begin{cases} \frac{\mu}{\sqrt{\varepsilon}} \theta_1(r_i) N_r^{-2} + (1 + t_j) N_r^{-2} + t_j N_t^{-1}, & (r_i, t_j) \in D_1^N, \\ \frac{(1+\mu)}{\sqrt{\varepsilon}} \theta_2(r_i) N_r^{-2} + (1 + t_j) N_r^{-2} + t_j N_t^{-1}, & (r_i, t_j) \in D_2^N, \end{cases}$$

where θ_1 and θ_2 are piecewise linear polynomials

$$\theta_1 = \begin{cases} \frac{r}{\mu} & \text{for } 0 \le z \le \mu, \\ 1 & \text{for } \mu \le z \le 1 - \mu, \\ \frac{1-r}{\mu} & \text{for } 1 - \mu \le z \le 1, \end{cases}$$

and

$$\theta_2 = \begin{cases} \frac{r}{1+\mu} & \text{for } 1 \le z \le 1 + \mu, \\ 1 & \text{for } 1 + \mu \le z \le 2 - \mu, \\ \frac{3-r}{1+\mu} & \text{for } 2 - \mu \le z \le 2. \end{cases}$$

Then, for all $(r_i, t_j) \in \bar{D}^N$,

$$0 \le \phi(r_i, t_j) \le C\left(N_r^{-2} \ln N_r + N_t^{-1}\right),$$

and also

$$\mathcal{L}_1^N \phi(r_i, t_j) \ge \begin{cases} C(\sqrt{\varepsilon} N_r^{-1} + N_r^{-2} + N_t^{-1}), & \text{if } r_i = \mu \text{ or } 1 - \mu, \\ C(N_r^{-2} + N_t^{-1}), & \text{otherwise,} \end{cases}$$

note that $\frac{\mu}{\sqrt{\varepsilon}} \leq 2\ln N$ and

$$\mathcal{L}_1^N \theta(r_i) = \begin{cases} \frac{\varepsilon N_r}{\mu} + a(r_i), & \text{if } r_i = \mu \text{ or } r_i = 1-\mu, \\ a(r_i)\theta(r_i), & \text{otherwise.} \end{cases}$$

Similarly,

$$\mathcal{L}_2^N \phi(r_i,t_j) \geq \begin{cases} C(\sqrt{\varepsilon}N_r^{-1} + N_r^{-2} + N_t^{-1}), & \text{if } r_i = 1+\mu \text{ or } 2-\mu, \\ C(N_r^{-2} + N_t^{-1}), & \text{otherwise,} \end{cases}$$

note that $\frac{1+\mu}{\sqrt{\varepsilon}} \leq 2\ln N$ and

$$\mathcal{L}_2^N \theta(r_i) = \begin{cases} \frac{\varepsilon N_r}{1+\mu} + a(r_i) + b(r_i), & \text{if } r_i = 1+\mu \text{ or } r_i = 2-\mu, \\ a(r_i)\theta(r_i) + b(r_i)\theta(r_{i-\frac{N}{2}}), & \text{otherwise,} \end{cases}$$

and $\forall(r_i,t_j) \in \Gamma^N$, then $\phi(r_i,t_j) \geq 0$.

Observe that $\mathcal{K}^N\theta(r_N,t_j) \geq 0$.

Define the barrier functions

$$\Xi^{\pm}(r_i,t_j) = \phi(r_i,t_j) \pm (V_\varepsilon - v_\varepsilon)(r_i,t_j),$$

it follows from $\forall(r_i,t_j) \in D_1^N$, $\forall(r_i,t_j) \in D_2^N$,

$$\mathcal{L}_1^N \Xi^{\pm}(r_i,t_j) \geq 0 \quad \text{and} \quad \mathcal{L}_2^N \Xi^{\pm}(r_i,t_j) \geq 0.$$

Then, from the discrete maximum principle,

$$\Xi^{\pm}(r_i,t_j) \geq 0, \quad (r_i,t_j) \in \bar{D}^N.$$

Then we have

$$\begin{aligned} &\left|(V_\varepsilon - v_\varepsilon)(r_i,t_j)\right| \leq \phi(r_i,t_j) \leq C\big(N_r^{-2}\ln N_r + N_t^{-1}\big), \\ &\left|(V_\varepsilon - v_\varepsilon)(r_i,t_j)\right| \leq C\big(N_r^{-2}\ln N_r + N_t^{-1}\big). \end{aligned}$$

Next, we derive an error estimate for the singular component. We decompose W_ε into W_l and $W_{\tilde{r}}$, where

$$\begin{cases} \mathcal{L}_1^N W_l(r_i,t_j) = 0, \quad (r_i,t_j) \in D_1^N, \\ W_l(r_i,t_j) = \phi_l(r_i,t_j) - v_0(r_i,t_j), \quad (r_i,t_j) \in \Gamma_l^N, \\ W_l(r_i,t_j) = 0, \quad (r_i,t_j) \in \Gamma_{\tilde{r}}^N, \\ W_l(r_i,t_j) = 0, \quad (r_i,t_j) \in \Gamma_b^N, \end{cases}$$

$$\begin{cases} \mathcal{L}_2^N W_l(r_i,t_j) = 0, & (r_i,t_j) \in D_2^N, \\ W_l(r_i,t_j) = A, & (r_i,t_j) \in \Gamma_l^N, \\ \mathcal{K}^N W_l(r_i,t_j) = 0, & (r_i,t_j) \in \Gamma_{\tilde{r}}^N, \\ W_l(r_i,t_j) = 0, & (r_i,t_j) \in \Gamma_b^N, \end{cases}$$

and

$$\begin{cases} \mathcal{L}_1^N W_{\tilde{r}}(r_i,t_j) = 0, & (r_i,t_j) \in D_1^N, \\ W_{\tilde{r}}(r_i,t_j) = 0, & (r_i,t_j) \in \Gamma_l^N, \\ W_{\tilde{r}}(r_i,t_j) = A, & (r_i,t_j) \in \Gamma_{\tilde{r}}^N, \\ W_{\tilde{r}}(r_i,t_j) = 0, & (r_i,t_j) \in \Gamma_b^N, \end{cases}$$

$$\begin{cases} \mathcal{L}_2^N W_{\tilde{r}}(r,t) = 0, & (r_i,t_j) \in D_2^N, \\ W_{\tilde{r}}(r_i,t_j) = 0, & (r_i,t_j) \in \Gamma_l^N, \\ \mathcal{K}^N W_{\tilde{r}}(r_i,t_j) = \mathcal{K}^N W(r_i,t_j), & (r_i,t_j) \in \Gamma_{\tilde{r}}^N, \\ W_{\tilde{r}}(r_i,t_j) = 0, & (r_i,t_j) \in \Gamma_b^N. \end{cases}$$

The singular component error is equivalent to

$$\begin{aligned} W_\varepsilon - w_\varepsilon &= (W_l - w_l) + (W_{\tilde{r}} - w_{\tilde{r}}), \\ \mathcal{L}^N(W_\varepsilon - w_\varepsilon) &= -\varepsilon\left(\frac{\partial^2}{\partial r^2} - \delta_r^2\right) w_\varepsilon + \left(\frac{\partial}{\partial t} - D_t^-\right) w_\varepsilon. \end{aligned}$$

It follows from classical estimates [12], at each point (r_i,t_j) in D_1^N,

$$\left|\mathcal{L}^N(W_\varepsilon - w_\varepsilon)(r_i,t_j)\right| \le C \begin{cases} ((N_r^{-1}\ln N_r)^2 + N_t^{-1}), & \text{if } (r_i,t_j) \in D_1^N, \\ ((N_r^{-1}\ln N_r)^2 + N_t^{-1}), & \text{if } (r_i,t_j) \in D_2^N. \end{cases}$$

First, the estimate for $W_l - w_l$ is given. The argument depends on whether $\mu = \frac{1}{4}$ or $\mu = 2\sqrt{\varepsilon}\ln N$

Case(i): $\mu = \frac{1}{4}$.

In this case the mesh is uniform and $2\sqrt{\varepsilon}\ln N \ge \frac{1}{4}$. It is clear that $r_i - r_{i-1} = N^{-1}$ and $\varepsilon^{-\frac{1}{2}} \le C\ln N$. By [12] we have

$$\begin{aligned} \mathcal{K}^N(W_l - w_l)(r_N,t_j) &= \mathcal{K}^N W_l(r_N,t_j) - \mathcal{K}^N w_l(r_N,t_j) \\ &= \phi_{\tilde{r}} - \mathcal{K}^N w_l(r_N,t_j) \\ &= \mathcal{K} w_l(r_N,t) - \mathcal{K}^N w_l(r_N,t_j), \\ \left|\mathcal{K}^N(W_l - w_l)(r_N)\right| &\le C\varepsilon\left(\left(h_1^3 w_l''(\chi_1,t_j) + \cdots + h_N^3 w_l''(\chi_N,t_j)\right)\right. \\ &\le C\varepsilon^{-1}\left(h_1^3 + \cdots + h_N^3\right) \\ &\le C N_r^{-2}, \end{aligned}$$

where $r_{i-1} \leq \chi_i \leq r_i$. Using Lemma 4 to the function $(W_l - w_l)(r_i)$ gives

$$\left|(W_l - w_l)(r_i)\right| \leq C\left(N_r^{-2}\ln^2 N_r\right).$$

Case(ii): $\mu < \frac{1}{4}$.

Since the mesh is piecewise uniform with the subinterval $[\mu, 1-\mu]$, mesh elements are $4(1-2\mu)/N$ and the rest of the intervals $[0,\mu]$ and $[1-\mu, 1]$ with $8\mu/N$ mesh elements.

By [12], we have

$$\left|\mathcal{K}^N(W_l - w_l)(r_i)\right| \leq C\left(N_r^{-2}\ln^2 N_r\right)$$

and

$$\begin{aligned}\left|\mathcal{K}^N(W_l - w_l)(r_N)\right| &\leq \varepsilon\left|C\left(h_1^3 w''(\chi_1) + \cdots + h_N^3 w''(\chi_N)\right)\right| \\ &\leq C\left(h_1^3 + \cdots + h_N^3\right) \\ &\leq CN_r^{-2},\end{aligned}$$

where $r_{i-1} \leq \chi_i \leq r_i$. Using Lemma 4 to the function $(W_l - w_l)(r_i)$ gives

$$\left|(W_l - w_l)(r_i)\right| \leq C\left(N_r^{-2}\ln^2 N_r + N_t^{-1}\right).$$

Analogous arguments are used to establish the error estimate for W_R. Hence proved. □

6 Numerical examples

As the exact solutions of these problems are not known, to compute the error estimate, the double mesh principle is used, which is stated as follows:

$$E_\varepsilon^{N,\Delta t} = \max_{(r_i,t_j)\in\bar{D}^N}\left|U^{N,\Delta t}(r_i,t_j) - U^{2N,\Delta t/2}(r_i,t_j)\right|.$$

We determine the uniform error $E^{N,\Delta t} = \max_\varepsilon E^{N,\Delta t}$ and the rate of convergence as $p^{N,\Delta t} = \log_2(\frac{E^{N,\Delta t}}{E^{2N,\Delta t/2}})$.

Example 6.1

$$\begin{cases}(-\varepsilon\frac{\partial^2 u_\varepsilon}{\partial r^2} + \frac{\partial u_\varepsilon}{\partial t} + 5u_\varepsilon)(r,t) - u_\varepsilon(r-1,t) = e^{-r}, & (r,t) \in (0,2)\times(0,2], \\ u_\varepsilon(r,t) = 0, \quad \forall(r,t) \in \Gamma_l, \\ \mathcal{K}u_\varepsilon(2,t) = u_\varepsilon(2,t) - \varepsilon\int_0^2 \frac{r}{3}u_\varepsilon(r,t)\,dr = 0, \quad \forall(r,t) \in \Gamma_{\bar{r}}, \\ u_\varepsilon(r,t) = 0, \quad \forall(r,t) \in \Gamma_b.\end{cases}$$

Example 6.2

$$\begin{cases}(-\varepsilon\frac{\partial^2 u_\varepsilon}{\partial r^2} + \frac{\partial u_\varepsilon}{\partial t} + 5u_\varepsilon)(r,t) - ru_\varepsilon(r-1,t) = 1, & (r,t) \in (0,2)\times(0,2], \\ u_\varepsilon(r,t) = 0, \quad \forall(r,t) \in \Gamma_l, \\ \mathcal{K}u_\varepsilon(2,t) = u_\varepsilon(2,t) - \varepsilon\int_0^2 \frac{1}{6}u_\varepsilon(r,t)\,dr = 0, \quad \forall(r,t) \in \Gamma_{\bar{r}}, \\ u_\varepsilon(r,t) = \sin(\pi r), \quad \forall(r,t) \in \Gamma_b.\end{cases}$$

7 Discussion

In the literature, so far, no one has considered singularly perturbed parabolic partial differential equation with nonlocal boundary condition. We considered a class of singularly perturbed partial differential equations with delay in space and integral boundary condition. A finite difference method and a trapezoidal rule were proposed. We showed that the order of convergence and the error is of order $O(N_r^{-2}\ln N_r^2 + N_t^{-1})$. The theory has been validated with two examples. Our numerical results reflect the theoretical estimates. Maximum point-wise errors and the order of convergence of Example 6.1 and Example 6.2 are given in Tables 1 and 2 respectively. The numerical solution of Examples 6.1 and 6.2 are plotted in Figs. 1 and 2.

Table 1 Example 6.1 corresponding maximum pointwise errors and the rate of convergence such as $E_\varepsilon^{N,\Delta t}$, $E^{N,\Delta t}$, $p^{N,\Delta t}$

ε	$N=8$	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$
	$\Delta t=0.1$	$\Delta t=0.1/2$	$\Delta t=0.1/2^2$	$\Delta t=0.1/2^3$	$\Delta t=0.1/2^4$	$\Delta t=0.1/2^5$	$\Delta t=0.1/2^6$
1	1.1975e–02	9.7873e–03	6.8849e–03	4.3534e–03	2.5602e–03	1.4343e–03	7.7788e-04
10^{-1}	1.9191e–02	1.4963e–02	9.7686e–03	5.7576e–03	3.1867e–03	1.6990e–03	8.8579e-04
10^{-2}	2.5748e–02	1.8063e–02	1.1329e–02	6.4557e–03	3.4830e–03	1.8181e–03	9.3259e-04
10^{-3}	2.8666e–02	1.9764e–02	1.2063e–02	6.7764e–03	3.6094e–03	1.8676e–03	9.5132e-04
10^{-4}	2.9667e–02	2.0342e–02	1.2358e–02	6.9012e–03	3.6589e–03	1.8864e–03	9.5841e-04
10^{-5}	2.9991e–02	2.0528e–02	1.2473e–02	6.9477e–03	3.6771e–03	1.8933e–03	9.6099e-04
10^{-6}	3.0094e–02	2.0588e–02	1.2515e–02	6.9646e–03	3.6838e–03	1.8958e–03	9.6192e-04
10^{-7}	3.0127e–02	2.0607e–02	1.2528e–02	6.9707e–03	3.6861e–03	1.8967e–03	9.6224e-04
10^{-8}	3.0137e–02	2.0613e–02	1.2532e–02	6.9728e–03	3.6869e–03	1.8970e–03	9.6236e-04
10^{-9}	3.0140e–02	2.0614e–02	1.2534e–02	6.9736e–03	3.6872e–03	1.8971e–03	9.6240e-04
10^{-10}	3.0142e–02	2.0615e–02	1.2534e–02	6.9738e–03	3.6873e–03	1.8972e–03	9.6241e-04
$E^{N,\Delta t}$	3.0142e–02	2.0615e–02	1.2534e–02	6.9738e–03	3.6873e–03	1.8972e–03	9.6241e-04
$p^{N,\Delta t}$	**5.4806e–01**	**7.1783e–01**	**8.4584e–01**	**9.1937e–01**	**9.5873e–01**	**9.7912e–01**	–

Table 2 Example 6.2 corresponding maximum pointwise errors and the rate of convergence such as $E_\varepsilon^{N,\Delta t}$, $E^{N,\Delta t}$, $p^{N,\Delta t}$

ε	$N=8$	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$
	$\Delta t=0.1$	$\Delta t=0.1/2$	$\Delta t=0.1/2^2$	$\Delta t=0.1/2^3$	$\Delta t=0.1/2^4$	$\Delta t=0.1/2^5$	$\Delta t=0.1/2^6$
1	1.9169e–01	1.8765e–01	1.4776e–01	9.7571e–02	5.7092e–02	3.1057e–02	1.6222e-02
10^{-1}	2.0157e–01	1.5386e–01	9.9026e–02	5.6998e–02	3.0701e–02	1.5952e–02	8.1328e-03
10^{-2}	1.9835e–01	1.4615e–01	9.1730e–02	5.1998e–02	2.7761e–02	1.4356e–02	7.3013e-03
10^{-3}	1.9792e–01	1.4531e–01	9.0969e–02	5.1441e–02	2.7447e–02	1.4189e–02	7.2176e-03
10^{-4}	1.9789e–01	1.4523e–01	9.0893e–02	5.1389e–02	2.7427e–02	1.4176e–02	7.2080e-03
10^{-5}	1.9788e–01	1.4522e–01	9.0885e–02	5.1384e–02	2.7429e–02	1.4176e–02	7.2082e-03
10^{-6}	1.9788e–01	1.4522e–01	9.0885e–02	5.1384e–02	2.7430e–02	1.4177e–02	7.2085e-03
10^{-7}	1.9788e–01	1.4522e–01	9.0885e–02	5.1384e–02	2.7430e–02	1.4177e–02	7.2086e-03
10^{-8}	1.9788e–01	1.4522e–01	9.0885e–02	5.1384e–02	2.7431e–02	1.4177e–02	7.2087e-03
10^{-9}	1.9788e–01	1.4522e–01	9.0885e–02	5.1384e–02	2.7431e–02	1.4177e–02	7.2087e-03
10^{-10}	1.9788e–01	1.4522e–01	9.0885e–02	5.1384e–02	2.7431e–02	1.4177e–02	7.2087e-03
$E^{N,\Delta t}$	2.0157e–01	1.8765e–01	1.4776e–01	9.7571e–02	5.7092e–02	3.1057e–02	1.6222e-02
$p^{N,\Delta t}$	**1.0324e–01**	**3.4479e–01**	**5.9873e–01**	**7.7316e–01**	**8.7837e–01**	**9.3697e–01**	–

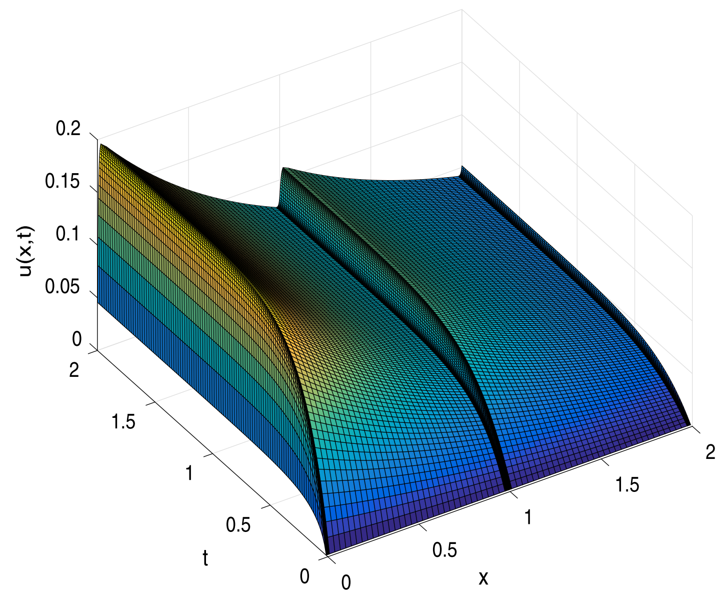

Figure 1 Solution graph of Example 6.1

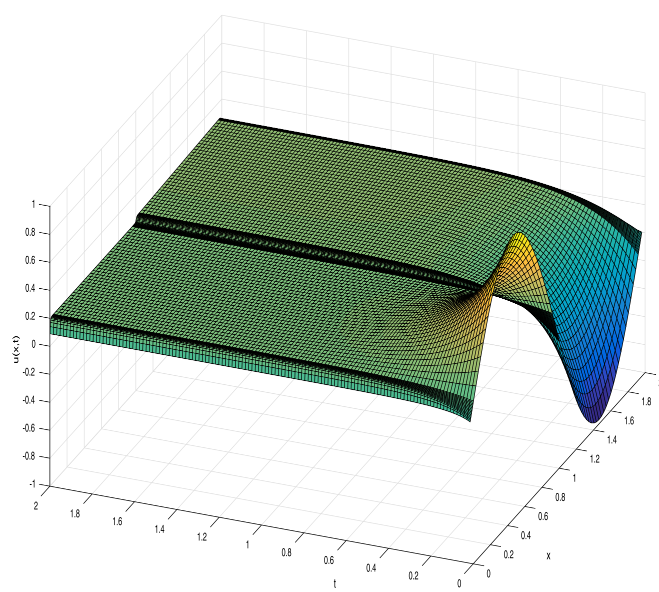

Figure 2 Solution graph of Example 6.2

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Authors' contributions

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