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Existence, uniqueness and global stability of Clifford-valued neutral-type neural networks with time delays

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Abstract

In this paper, we analyze the global asymptotic stability and global exponential stability with respect to the Clifford-valued neutral-type neural network (NN) models with time delays. By considering the neutral term, a Clifford-valued NN model with time delays is formulated, which encompasses real-valued, complex-valued, and quaternion-valued NN models as special cases. In order to achieve our main results, the n-dimensional Clifford-valued NN model is decomposed into $2^m n$-dimensional real-valued models. Moreover, a proper function is constructed to handle the neutral term and prove that the equilibrium point exists. Utilizing the homeomorphism theory, linear matrix inequality as well as Lyapunov functional methods, we derive the sufficient conditions corresponding to the existence, uniqueness, and global asymptotic stability with respect to the equilibrium point of the Clifford-valued neutral-type NN model. Numerical examples to demonstrate the effectiveness of the results are provided, and the simulations results are analyzed and discussed.

Keywords: Clifford-valued neural network; Asymptotic stability; Exponential stability; Lyapunov functional; Neutral delay

1. Introduction

Over the last few decades, there have been many developments in NN models and the analysis of NNs dynamics has received considerable attention [7,14,15,33]. Indeed, many NN models have been successfully applied to solve real-world problems associated with pattern recognition [15,33], optimization issues [31,43], signal and image processing [11], and associative memory [12,42]. In such applications, it is usually desirable for the NN models to exhibit certain behaviors, depending on the characteristics of the problem to be solved [1,35]. In this context, the stability of NN models becomes an essential requirement [48,58]. On the other hand, complex signals are present in most NN applications [12,19,34]. In fact, NN models with the capability of handling complex signals to properly represent geometric transformation and to interpret multidimensional signals makes them promising for applications in various fields. Because of this, complex-valued and quaternion-valued NN models have received

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increasing research interest over the last few years [1,19,32,34,48,58]. Some of the common applications of these NN models include image compression, chaotic time series prediction, color night vision, polarized signal classification, $3D$ wind forecasting, and others. Many results with respect to the stability of complex-valued and quaternion-valued NN models with time delays are available in the literature, e.g. [19,25,32,34,48,50,53,57,58].

From the theoretical perspective, Clifford algebra provides a strong foundation for solving geometric problems. The principle of Clifford algebra is useful for addressing a variety of science and engineering problems, which include robotic, signal and image processing and neural computing [6,13,20,36,37]. Clifford-valued NN models are the generalization of real-valued, complex-valued, and quaternion-valued NN models. These NN models are effective for tackling high-dimensional data and spatial geometric transformation problems [5,6,13,20]. Theoretical and applied studies of Clifford-valued NN models have recently become a new research subject. However, the dynamic properties of Clifford-valued NN models are usually more complex than those of the real-valued and complex-valued and quaternion-valued NN models. As such, studies on Clifford-valued NN dynamics are still limited due to the non-commutativity of multiplication with respect to Clifford numbers [2,3,21,22,26–28,30,49,59].

Based on the linear matrix inequality (LMI) method, global exponential stability criteria pertaining to delayed Clifford-valued recurrent NN models were studied in [59]. Pertaining to Clifford-valued NN models with time delays, their global asymptotic stability issues were examined in [30]. The sufficient conditions are obtained by decomposing the n-dimensional Clifford-valued NN model into $2^m n$-dimensional real-valued models. In [22], the existence and global exponential stability with respect to almost periodic solutions has been derived for Clifford-valued neutral high-order Hopfield NN models with leakage delays. Leveraging the Banach fixed point theorem as well as Lyapunov principles, the global asymptotic almost periodic synchronization issues have been derived for Clifford-valued cellular NN models [27]. In [2], the study on weighted pseudo almost automorphic solutions pertaining to neutral type fuzzy cellular NN models with mixed delays and D operator in Clifford algebra has been conducted. The existence of anti-periodic solutions with respect to a class of Clifford-valued inertial Cohen–Grossberg NN models utilizing Lyapunov functionals has been investigated in [26].

Due to the limited speed of signal propagation, time delays (either constant or varying) are often encountered in NN models operating in real-world applications [4,38–41]. Time delays are the main source of various dynamics such as chaos, divergence, poor functionality, instability [8,18,45,51]. As such, dynamics of recurrent NN models with time delays have gained growing attention, and many results have been reported [9,10,16,17,44,46]. In NN models, we study two general types of time delays: neural-type and retarded-type delays [47,50,52–54,57]. In retarded-type NN models, time delays in the states are formulated, which are not adequate to describe the precise dynamic characteristics with respect to real neurons. As such, neutral-type NN models have become important, whereby delays corresponding to the time derivatives of states are formulated [23,24,29,55,56]. This constitutes the motivation for the current study.

To the best of our knowledge, there are hardly any papers that deal with the problem of global asymptotical stability and global exponential stability of Clifford-valued neutral-type NN models. Indeed, this interesting topic is still an open challenge. Therefore, our study focuses on the sufficient conditions to ascertain the global asymptotical stability and globally exponentially stability of Clifford-valued neutral-type NN model. Firstly, the original n-dimensional Clifford-valued model is decomposed into $2^m n$-dimensional real-valued models. Next, we investigate the global asymptotic and exponential stability characteristics of Clifford-valued neutral-type NN models. In comparison with related studies in the literature, our research has the following key contribution. For the first time, we investigate both global asymptotic stability and global exponential stability of Clifford-valued NN models that include the neutral term. In comparison with other results, the outcome of our study is new and is more general even when the considered Clifford-valued NN model is decomposing into real-valued, complex-valued, and quaternion-valued models. In addition, the proposed technique is applicable to other dynamic behaviors of various types of Clifford-valued NN models with time delays.

We organize this paper as follows. The Clifford-valued NN model with neutral term is formally defined in Section 2. In Section 3, the new stability criterion is presented. Numerical examples and the associated results are provided in Section 4. The research findings are concluded in Section 5.

2. Problem definition and mathematical fundamentals

2.1. Notations

Let $\mathbb{R}^n$ and $\mathbb{A}^n$ denote the n-dimensional real vector space and n-dimensional real Clifford vector space, respectively; $\mathbb{R}^{n\times n}$ denotes the set of all $n\times n$ real matrices and $\mathbb{A}^{n\times n}$ denotes the set of all $n\times n$ real Clifford matrices, respectively. Superscripts T and $*$ indicate matrix transposition and matrix involution transposition, respectively. Any matrix $\mathcal{P} > 0$ (< 0) denotes a positive (negative) definite matrix. We define $\mathbb{A}$ as the Clifford algebra with m generators over the real number $\mathbb{R}$ and the norm of $\mathbb{R}^n$ as $\|p\| = \sum_{i=1}^{n} |p_i|$. In addition, $p = \sum_A p^A e_A \in \mathbb{A}$ denotes $\|p\|_{\mathbb{A}} = \sum_A |p^A|$. $\lambda_{\max}(\mathcal{P})$ and $\lambda_{\min}(\mathcal{P})$, respectively, denote the maximum and minimum eigenvalues of the matrix $\mathcal{P}$. For $\varphi \in \mathscr{C}([-\eta, 0], \mathbb{A}^n)$, and the norm $\|\varphi\|_\eta \le \sup_{-\eta \le s \le 0} \|\varphi(t+s)\|$ is introduced.

2.2. Clifford algebra

This subsection provides some Clifford algebra results. For more details, we refer the papers [21,49].

Define the Clifford real algebra over $\mathbb{R}^m$ as

$$\mathbb{A} = \left\{ \sum_{A \subseteq \{1,2,\dots,m\}} a^A e_A,\ a^A \in \mathbb{R} \right\},$$

where $e_A = e_{r_1} e_{r_2} \dots e_{r_\nu}$ with $A = \{r_1, r_2, \dots, r_\nu\}$, $1 \le r_1 < r_2 < \dots < r_\nu \le m$.

Moreover, the Clifford generators are denoted as $e_\emptyset = e_0 = 1$ and $e_r = e_{\{r\}}$, $r = 1, 2, \dots, m$, and they fulfill the following conditions

$$\begin{cases} e_i e_j + e_j e_i = 0, & i \ne j, \quad i, j = 1, 2, \dots, m, \\ e_i^2 = -1, & i = 1, 2, \dots, m. \end{cases}$$

For simplicity, we combine the related subscripts when an element represents the product of multiple Clifford generators, e.g. $e_4 e_5 e_6 e_7 = e_{4567}$.

Let $\Lambda = \{\emptyset, 1, 2, \dots, A, \dots, 12\dots m\}$, and we have

$$\mathbb{A} = \left\{ \sum_A a^A e_A,\ a^A \in \mathbb{R} \right\},$$

where $\sum_A$ denotes $\sum_{A \in \Lambda}$ and $\mathbb{A}$ is isomorphic to $\mathbb{R}^{2^m}$.

For any Clifford number $p = \sum_A p^A e_A$, the involution of p is defined by

$$\bar{p} = \sum_A p^A \bar{e}_A,$$

where $\bar{e}_A = (-1)^{\frac{\sigma[A](\sigma[A]+1)}{2}} e_A$, and

$$\sigma[A] = \begin{cases} 0, & \text{if} \quad A = \emptyset, \\ \nu, & \text{if} \quad A = r_1 r_2 \dots r_\nu. \end{cases}$$

From the definition, we can directly deduce that $e_A \bar{e}_A = \bar{e}_A e_A = 1$. For a Clifford-valued function $p = \sum_A p^A e_A : \mathbb{R} \to \mathbb{A}$, where $p^A : \mathbb{R} \to \mathbb{R}$, $A \in \Lambda$, and its derivative is represented by $\frac{dp(t)}{dt} = \sum_A \frac{dp^A(t)}{dt} e_A$.

Since $e_B \bar{e}_A = (-1)^{\frac{\sigma[A](\sigma[A]+1)}{2}} e_B e_A$, we can write $e_B \bar{e}_A = e_C$ or $e_B \bar{e}_A = -e_C$, where e_C is a basis of Clifford algebra $\mathbb{A}$. As an example, $e_{r_1 r_2} \bar{e}_{r_2 r_3} = -e_{r_1 r_2} e_{r_2 r_3} = -e_{r_1} e_{r_2} e_{r_2} e_{r_3} = -e_{r_1}(-1) e_{r_3} = e_{r_1} e_{r_3} = e_{r_1 r_3}$. Therefore, we can identify a unique corresponding basis e_C pertaining to a given $e_B \bar{e}_A$. Define

$$\sigma[B.\bar{A}] = \begin{cases} 0, & \text{if } e_B \bar{e}_A = e_C, \\ 1, & \text{if } e_B \bar{e}_A = -e_C, \end{cases}$$

and then, $e_B \bar{e}_A = (-1)^{\sigma[B.\bar{A}]} e_C$.

Moreover, for any $\mathscr{G} \in \mathbb{A}$, there is a unique $\mathscr{G}^C$ that satisfies $\mathscr{G}^{B.\bar{A}} = (-1)^{\sigma[B.\bar{A}]} \mathscr{G}^C$ for $e_B \bar{e}_A = (-1)^{\sigma[B.\bar{A}]} e_C$. Therefore

$$\mathscr{G}^{B.\bar{A}} e_B \bar{e}_A = \mathscr{G}^{B.\bar{A}} (-1)^{\sigma[B.\bar{A}]} e_C = (-1)^{\sigma[B.\bar{A}]} \mathscr{G}^C (-1)^{\sigma[B.\bar{A}]} e_C = \mathscr{G}^C e_C.$$

and $\mathscr{G} = \sum_C \mathscr{G}^C e_C \in \mathbb{A}$.

2.3. Problem definition

Consider a Clifford-valued NN model with discrete time delays, as follows:

$$\begin{cases} \dot{p}_i(t) = -d_i p_i(t) + \sum_{j=1}^{n} a_{ij} f_j(p_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(p_j(t-\tau)) + \sum_{j=1}^{n} c_{ij} \dot{p}_j(t-h) + u_i, \ t \geq 0, \\ p_i(s) = \varphi_i(s), \ s \in [-\eta, 0], \ i = 1, 2, \ldots, n, \end{cases} \tag{1}$$

where $i, j = 1, 2, \ldots, n$, and the number of neurons is represented by n. In addition, $p_i(t) \in \mathbb{A}$ represents the state vector of the ith unit; $0 < d_i \in \mathbb{R}$ indicates the rate used by the ith unit to reset its potential to the resting state upon disconnection from the NN model; $a_{ij} \in \mathbb{A}, b_{ij} \in \mathbb{A}$ are the strengths of the neuron interconnections without and with time delay between cells i and j; $c_{ij} \in \mathbb{A}$ denotes coefficients of the time derivative of the delayed states; $u_i \in \mathbb{A}$ is an external input for the ith unit; $f_j(\cdot) : \mathbb{A} \to \mathbb{A}$ and $g_j(\cdot) : \mathbb{A} \to \mathbb{A}$ represent the activation functions; $h > 0$ is the neutral delay, while $\tau > 0$ is the constant time delay, respectively. Furthermore, φ_i is continuously differential on $s \in [-\eta, 0]$, and $\eta = \max\{\tau, h\}$.

For the convenience of discussion, (1) is re-formulated in the following vector form

$$\begin{cases} \dot{p}(t) = -\mathcal{D}p(t) + \mathcal{A}f(p(t)) + \mathcal{B}g(p(t-\tau)) + \mathcal{C}\dot{p}(t-h) + u, \ t \geq 0, \\ p(s) = \varphi(s), \ s \in [-\eta, 0], \end{cases} \tag{2}$$

where $p(t) = (p_1(t), p_2(t), \ldots, p_n(t))^T \in \mathbb{A}^n$; $\mathcal{D} = diag\{d_1, d_2, \ldots, d_n\} \in \mathbb{R}^n$ with $d_i > 0, i = 1, 2, \ldots, n$; and $\mathcal{A} = (a_{ij})_{n\times n} \in \mathbb{A}^{n\times n}$; $\mathcal{B} = (b_{ij})_{n\times n} \in \mathbb{A}^{n\times n}$; $\mathcal{C} = (c_{ij})_{n\times n} \in \mathbb{A}^{n\times n}$; $u = (u_1, u_2, \ldots, u_n)^T \in \mathbb{A}^n$; $f(p(t)) = (f_1(p_1(t)), f_2(p_2(t)), \ldots, f_n(p_n(t)))^T \in \mathbb{A}^n$; $g(p(t-\tau)) = (g_1(p_1(t-\tau)), g_2(p_2(t-\tau)), \ldots, g_n(p_n(t-\tau)))^T \in \mathbb{A}^n$.
(H1) For each $j = 1, 2, \ldots, n$, $f_j(\cdot), g_j(\cdot) \in \mathscr{C}(\mathbb{A}, \mathbb{A})$ and there exist positive constants k_j and l_j such that

$$|f_j(x) - f_j(y)|_{\mathbb{A}} \leq k_j |x - y|_{\mathbb{A}}, \ j = 1, 2, \ldots, n, \tag{3}$$

$$|g_j(x) - g_j(y)|_{\mathbb{A}} \leq l_j |x - y|_{\mathbb{A}}, \ j = 1, 2, \ldots, n, \tag{4}$$

and there exist positive constants $\mathcal{K}$ and $\mathcal{L}$, such that $|f(x)|_{\mathbb{A}} \leq \mathcal{K}, \ |g(y)|_{\mathbb{A}} \leq \mathcal{L}$, for any $x, y \in \mathbb{A}$.
By means of assumption **(H1)**, it is clear that

$$(f(x) - f(y))^*(f(x) - f(y)) \leq (x-y)^* \mathcal{K}^T \mathcal{K}(x-y), \tag{5}$$

$$(g(x) - g(y))^*(g(x) - g(y)) \leq (x-y)^* \mathcal{L}^T \mathcal{L}(x-y), \tag{6}$$

where $\mathcal{K} = diag\{k_1, k_2, \ldots, k_n\}$ and $\mathcal{L} = diag\{l_1, l_2, \ldots, l_n\}$ and $*$ represents the matrix involution transposition.

Remark 2.1. Since the commutative law is not applicable to multiplication of Clifford numbers, there are limited results on Clifford-valued NN models. Most of the existing results are derived based on the decomposition of Clifford-valued NN models into real-valued NN models. Therefore, the use of decomposition methods to analyze Clifford-valued NNs is highly useful.

3. Main results

We transform the Clifford-valued NN model (2) into the real-valued NN models, in order to undertake issues on non-commutativity of multiplication of Clifford numbers. This can be achieved with the help of $e_A \bar{e}_A = \bar{e}_A e_A = 1$ and $e_B \bar{e}_A e_A = e_B$. Given any $\mathscr{G} \in \mathbb{A}$, a unique $\mathscr{G}^C$ that is able to satisfy $\mathscr{G}^C e_C g^A e_A = (-1)^{\sigma[B.\bar{A}]} \mathscr{G}^C g^A e_B = \mathscr{G}^{B.\bar{A}} g^A e_B$ can be identified. By decomposing (2) into $\dot{p} = \sum_A \dot{p}^A e_A$, we have

$$\begin{cases} \dot{p}^A(t) = & -\mathcal{D}p^A(t) + \sum_{B\in\Lambda} \mathcal{A}^{A.\bar{B}} f^B(p(t)) + \sum_{B\in\Lambda} \mathcal{B}^{A.\bar{B}} g^B(p(t-\tau)) \\ & + \sum_{B\in\Lambda} \mathcal{C}^{A.\bar{B}} \dot{p}^A(t-h) + u^A, \ t \geq 0, \\ p^A(s) = & \varphi^A(s), \ s \in [-\eta, 0], \end{cases} \tag{7}$$

where

$$
\begin{aligned}
p^A(t) &= (p_1^A(t), p_2^A(t), \dots, p_n^A(t))^T,\ \ p(t) = \sum_{A\in\Lambda} p^A(t)e_A,\\
u^A &= (u_1^A, u_2^A, \dots, u_n^A)^T,\ \ u = \sum_{A\in\Lambda} u^A e_A,\\
\dot{p}^A(t-h) &= (\dot{p}_1^A(t-h), \dot{p}_2^A(t-h), \dots, \dot{p}_n^A(t-h))^T,\ \ \dot{p}(t-h) = \sum_{A\in\Lambda} \dot{p}^A(t-h)e_A,\\
f^B(p(t)) &= (f_1^B(p_1^{C_1}(t), p_1^{C_2}(t), \dots, p_1^{C_{2m}}(t)), f_2^B(p_2^{C_1}(t), p_2^{C_2}(t), \dots, p_2^{C_{2m}}(t)),\\
&\quad \dots, f_n^B(p_n^{C_1}(t), p_n^{C_2}(t), \dots, p_n^{C_{2m}}(t)))^T,\\
g^B(p(t-\tau)) &= (g_1^B(p_1^{C_1}(t-\tau), p_1^{C_2}(t-\tau), \dots, p_1^{C_{2m}}(t-\tau)),\\
&\quad g_2^B(p_2^{C_1}(t-\tau), p_2^{C_2}(t-\tau), \dots, p_2^{C_{2m}}(t-\tau)),\\
&\quad \dots, g_n^B(p_n^{C_1}(t-\tau), p_n^{C_2}(t-\tau), \dots, p_n^{C_{2m}}(t-\tau)))^T,\\
\mathcal{A} &= \sum_{C\in\Lambda} \mathcal{A}^C e_C,\ \ \mathcal{A}^{A.\bar{B}} = (-1)^{\sigma[A.\bar{B}]}\mathcal{A}^C,\\
\mathcal{B} &= \sum_{C\in\Lambda} \mathcal{B}^C e_C,\ \ \mathcal{B}^{A.\bar{B}} = (-1)^{\sigma[A.\bar{B}]}\mathcal{B}^C,\\
\mathcal{C} &= \sum_{C\in\Lambda} \mathcal{C}^C e_C,\ \ \mathcal{C}^{A.\bar{B}} = (-1)^{\sigma[A.\bar{B}]}\mathcal{C}^C,\\
\mathcal{A}^{A.\bar{B}} &= (a_{ij}^{A.\bar{B}})_{n\times n},\ \ \mathcal{B}^{A.\bar{B}} = (b_{ij}^{A.\bar{B}})_{n\times n},\ \ \mathcal{C}^{A.\bar{B}} = (c_{ij}^{A.\bar{B}})_{n\times n},\\
e_A\bar{e}_B &= (-1)^{\sigma[A.\bar{B}]}e_C.
\end{aligned}
$$

Remark 3.1. If $p(t) = (p^0(t), p^1(t), \dots, p^A(t), \dots, p^{12\dots m}(t))^T \triangleq \{p_i^A(t)\}$ is a solution to the NN model (7), then $p(t) = (p_1(t), p_2(t), \dots, p_n(t))^T$ must be a solution to the NN model (2), where $p_i(t) = \sum_A p_i^A(t)e_A,\ i = 1, 2, \dots, n,\ A \in \Lambda$.

According to Clifford algebra, the Clifford-valued NN model can be re-written in novel real-valued ones. Define

$$
\begin{aligned}
q(t) &= \left((p^0(t))^T, (p^1(t))^T, \dots, (p^A(t))^T, \dots, (p^{12\dots m}(t))^T\right)^T \in \mathbb{R}^{2^m n},\\
\bar{f}(q(t)) &= \left((f^0(p(t)))^T, (f^1(p(t)))^T, \dots, (f^A(p(t)))^T, \dots, (f^{12\dots m}(p(t)))^T\right)^T \in \mathbb{R}^{2^m n},\\
\bar{g}(q(t-\tau)) &= \left((g^0(p(t-\tau)))^T, (g^1(p(t-\tau)))^T, \dots, (g^A(p(t-\tau)))^T, \dots,\right.\\
&\quad \left.(g^{12\dots m}(p(t-\tau)))^T\right)^T \in \mathbb{R}^{2^m n},\\
\dot{q}(t-h) &= \left((\dot{p}^0(t-h))^T, (\dot{p}^1(t-h))^T, \dots, (\dot{p}^A(t-h))^T, \dots, (\dot{p}^{12\dots m}(t-h))^T\right)^T \in \mathbb{R}^{2^m n},\\
\bar{u} &= \left((u^0)^T, (u^1)^T, \dots, (u^A)^T, \dots, (u^{12\dots m})^T\right)^T \in \mathbb{R}^{2^m n},\\
\tilde{\mathcal{D}} &= \begin{pmatrix} \mathcal{D} & 0 & \dots & 0\\ 0 & \mathcal{D} & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \mathcal{D}\end{pmatrix}_{2^m n\times 2^m n},\\
\tilde{\mathcal{A}} &= \begin{pmatrix} \mathcal{A}^0 & \mathcal{A}^{\bar{1}} & \dots & \mathcal{A}^{\bar{A}} & \dots & \mathcal{A}^{\overline{12\dots m}}\\ \mathcal{A}^1 & \mathcal{A}^{1\cdot\bar{1}} & \dots & \mathcal{A}^{1\cdot\bar{A}} & \dots & \mathcal{A}^{1\cdot\overline{12\dots m}}\\ \vdots & \vdots & \dots & \vdots & \dots & \vdots\\ \mathcal{A}^{12\dots m} & \mathcal{A}^{12\dots m\cdot\bar{1}} & \dots & \mathcal{A}^{12\dots m\cdot\bar{A}} & \dots & \mathcal{A}^{12\dots m\cdot\overline{12\dots m}}\end{pmatrix}_{2^m n\times 2^m n},
\end{aligned}
$$

$$\tilde{\mathcal{B}} = \begin{pmatrix} \mathcal{B}^0 & \mathcal{B}^{\overline{1}} & \dots & \mathcal{B}^{\overline{A}} & \dots & \mathcal{B}^{\overline{12\dots m}} \\ \mathcal{B}^1 & \mathcal{B}^{1\cdot\overline{1}} & \dots & \mathcal{B}^{1\cdot\overline{A}} & \dots & \mathcal{B}^{1\cdot\overline{12\dots m}} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ \mathcal{B}^{12\dots m} & \mathcal{B}^{12\dots m\cdot\overline{1}} & \dots & \mathcal{B}^{12\dots m\cdot\overline{A}} & \dots & \mathcal{B}^{12\dots m\cdot\overline{12\dots m}} \end{pmatrix}_{2^m n\times 2^m n},$$

$$\tilde{\mathcal{C}} = \begin{pmatrix} \mathcal{C}^0 & \mathcal{C}^{\overline{1}} & \dots & \mathcal{C}^{\overline{A}} & \dots & \mathcal{C}^{\overline{12\dots m}} \\ \mathcal{C}^1 & \mathcal{C}^{1\cdot\overline{1}} & \dots & \mathcal{C}^{1\cdot\overline{A}} & \dots & \mathcal{C}^{1\cdot\overline{12\dots m}} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ \mathcal{C}^{12\dots m} & \mathcal{C}^{12\dots m\cdot\overline{1}} & \dots & \mathcal{C}^{12\dots m\cdot\overline{A}} & \dots & \mathcal{C}^{12\dots m\cdot\overline{12\dots m}} \end{pmatrix}_{2^m n\times 2^m n},$$

and we can represent (7) as

$$\dot{q}(t) = -\tilde{\mathcal{D}}q(t) + \tilde{\mathcal{A}}\bar{f}(q(t)) + \tilde{\mathcal{B}}\bar{g}(q(t-\tau)) + \tilde{\mathcal{C}}\dot{q}(t-h) + \bar{u},\ \ t \geq 0, \tag{8}$$

with the initial value,

$$q(s) = \bar{\varphi}(s),\ \ s \in [-\eta, 0], \tag{9}$$

where $\bar{\varphi}(s) = ((\varphi^0(s))^T, (\varphi^1(s))^T, \dots, (\varphi^A(s))^T, \dots, (\varphi^{12\dots m}(s))^T)^T \in \mathbb{R}^{2^m n}$.
In addition, notice that (5) and (6) can be expressed with the following inequalities

$$(\bar{f}(q) - \bar{f}(\acute{q}))^T(\bar{f}(q) - \bar{f}(\acute{q})) \leq (q-\acute{q})^T\tilde{\mathcal{K}}^T(q-\acute{q}), \tag{10}$$

$$(\bar{g}(q) - \bar{g}(\acute{q}))^T(\bar{g}(q) - \bar{g}(\acute{q})) \leq (q-\acute{q})^T\tilde{\mathcal{L}}^T(q-\acute{q}), \tag{11}$$

where $\tilde{\mathcal{K}} = \begin{pmatrix} \mathcal{K}^T\mathcal{K} & 0 & \dots & 0 \\ 0 & \mathcal{K}^T\mathcal{K} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathcal{K}^T\mathcal{K} \end{pmatrix}_{2^m n\times 2^m n}$ and $\tilde{\mathcal{L}} = \begin{pmatrix} \mathcal{L}^T\mathcal{L} & 0 & \dots & 0 \\ 0 & \mathcal{L}^T\mathcal{L} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathcal{L}^T\mathcal{L} \end{pmatrix}_{2^m n\times 2^m n}$.

Notice that since the equilibrium point of both (2) and (8) are the same, stability of model (2) is equivalent to that of (8). As a result, we examine the real-valued NN models in our subsequent analysis.

Lemma 3.2 ([30]). *Let $\mathcal{H}(q) : \mathbb{R}^{2^m n} \times \mathbb{R}^{2^m n}$ be a continuous map which fulfills the following conditions: (i) $\mathcal{H}(q)$ is injective on $\mathbb{R}^{2^m n}$, (ii) $\|\mathcal{H}(q)\| \to \infty$ as $\|q\| \to \infty$. As such, $\mathcal{H}(q)$ is a homeomorphism of $\mathbb{R}^{2^m n}$.*

Lemma 3.3. *If $-\mathcal{I} + \tilde{\mathcal{C}}^T\tilde{\mathcal{C}} < 0$, then $\mathcal{I} - \tilde{\mathcal{C}}$ is a nonsingular matrix (or an invertible matrix).*

Proof. Utilizing contradiction, suppose $\mathcal{I} - \tilde{\mathcal{C}}$ is a singular matrix, and vector $\mathcal{X} \neq 0$ exists such that $(\mathcal{I} - \tilde{\mathcal{C}})\mathcal{X} = 0$. As a result, $\tilde{\mathcal{C}}\mathcal{X} = \mathcal{X},\ \ \mathcal{X}^T\tilde{\mathcal{C}}^T = \mathcal{X}^T$, which yield

$$\mathcal{X}^T\tilde{\mathcal{C}}^T\tilde{\mathcal{C}}\mathcal{X} = \mathcal{X}^T\mathcal{X},$$

then

$$\mathcal{X}^T(\tilde{\mathcal{C}}^T\tilde{\mathcal{C}} - \mathcal{I})\mathcal{X} = 0,\ \ (\mathcal{X} \neq 0),$$

which presents a contradiction to $-\mathcal{I} + \tilde{\mathcal{C}}^T\tilde{\mathcal{C}} < 0$. This completes the proof.

Definition 3.4 ([47]). With the existence of $\beta > 0$ and $\Upsilon(\beta) > 0$ such that

$$\|q(t)\| \leq \Upsilon(\beta)e^{-\beta t},\ \ \forall\, t \geq 0,$$

NN model (8) is said to be exponentially stable with convergence rate β at the equilibrium point.

3.1. Global asymptotic stability

Theorem 3.5. *Suppose (**H1**) is satisfied, model (8) has a unique equilibrium point and it is globally asymptotically stable if there exist positive definite matrices $\mathcal{P}$, $\mathcal{Q}$ and positive scalars ϵ_1 and ϵ_2 such that the following LMI is* *feasible:*

$$\Xi = \begin{pmatrix} -\mathcal{P}\tilde{\mathcal{D}} - \tilde{\mathcal{D}}^T\mathcal{P} - \tilde{\mathcal{D}}^T\tilde{\mathcal{D}} + \epsilon_1\tilde{\mathcal{K}} + \mathcal{Q} & \mathcal{P}\tilde{\mathcal{A}} & \mathcal{P}\tilde{\mathcal{B}} & \mathcal{P}\tilde{\mathcal{C}} \\ \star & -\epsilon_1\mathcal{I} + \tilde{\mathcal{A}}^T\tilde{\mathcal{A}} & \tilde{\mathcal{A}}^T\tilde{\mathcal{B}} & \tilde{\mathcal{A}}^T\tilde{\mathcal{C}} \\ \star & \star & -\epsilon_2\mathcal{I} + \tilde{\mathcal{B}}^T\tilde{\mathcal{B}} & \tilde{\mathcal{B}}^T\tilde{\mathcal{C}} \\ \star & \star & \star & -\mathcal{I} + \tilde{\mathcal{C}}^T\tilde{\mathcal{C}} \end{pmatrix} < 0, \tag{12}$$

$$\Psi = -\mathcal{Q} + \epsilon_2\tilde{\mathcal{L}} \leq 0. \tag{13}$$

Proof. Step-1: By using Lemma 3.2, the existence and uniqueness of the equilibrium point with respect to NN model (8) is proven.

From (12), by Schur complement, one has $-\mathcal{I}+\tilde{\mathcal{C}}^T\tilde{\mathcal{C}} < 0$. Then, $\mathcal{I}-\tilde{\mathcal{C}}$ is nonsingular with respect to Lemma 3.3.

Referring to (8), consider the following mapping:

$$\mathcal{H}(q) = (\mathcal{I} - \tilde{\mathcal{C}})^{-1}[-\tilde{\mathcal{D}}q + \tilde{\mathcal{A}}\bar{f}(q) + \tilde{\mathcal{B}}\bar{g}(q) + \bar{u}]. \tag{14}$$

As such,

$$\mathcal{H}(q) = -\tilde{\mathcal{D}}q + \tilde{\mathcal{A}}\bar{f}(q) + \tilde{\mathcal{B}}\bar{g}(q) + \tilde{\mathcal{C}}\mathcal{H}(q) + \bar{u}. \tag{15}$$

It is obvious that $q^* = (q_1^*, q_2^*, \ldots, q_n^*)^T$ is an equilibrium point of (8) subject to q^* meets the following equation:

$$\mathcal{H}(q^*) = -\tilde{\mathcal{D}}(q^*) + \tilde{\mathcal{A}}\bar{f}(q^*) + \tilde{\mathcal{B}}\bar{g}(q^*) + \tilde{\mathcal{C}}\mathcal{H}(q^*) + \bar{u} = 0. \tag{16}$$

Consequently, based on Lemma 3.2, it can be concluded that, corresponding to the model defined in (8), a unique equilibrium point exists for every input vector $\bar{u}$ if $\mathcal{H}(q)$ is homeomorphism of $\mathbb{R}^{2^m n}$.

The proof pertaining to map $\mathcal{H}(q)$ is injective is first shown. Given the existence of q and q́ with $q \neq \acute{q}$ and in accordance with (16), one has

$$\mathcal{H}(q) - \mathcal{H}(\acute{q}) = -\tilde{\mathcal{D}}(q - \acute{q}) + \tilde{\mathcal{A}}(\bar{f}(q) - \bar{f}(\acute{q})) + \tilde{\mathcal{B}}(\bar{g}(q) - \bar{g}(\acute{q})) + \tilde{\mathcal{C}}(\mathcal{H}(q) - \mathcal{H}(\acute{q})). \tag{17}$$

By multiplying both sides of (17) with $[2(q - \acute{q})\mathcal{P} + 2(q - \acute{q})\tilde{\mathcal{D}} + (\mathcal{H}(q) - \mathcal{H}(\acute{q}))]^T$, one has

$$\begin{aligned} &[2(q - \acute{q})\mathcal{P} + 2(q - \acute{q})\tilde{\mathcal{D}} + (\mathcal{H}(q) - \mathcal{H}(\acute{q}))]^T[\mathcal{H}(q) - \mathcal{H}(\acute{q})] \\ &= 2(q - \acute{q})^T\mathcal{P}[-\tilde{\mathcal{D}}(q - \acute{q}) + \tilde{\mathcal{A}}(\bar{f}(q) - \bar{f}(\acute{q})) + \tilde{\mathcal{B}}(\bar{g}(q) - \bar{g}(\acute{q})) \\ &\quad + \tilde{\mathcal{C}}(\mathcal{H}(q) - \mathcal{H}(\acute{q}))] + [\tilde{\mathcal{D}}(q - \acute{q}) + \tilde{\mathcal{A}}(\bar{f}(q) - \bar{f}(\acute{q})) + \tilde{\mathcal{B}}(\bar{g}(q) - \bar{g}(\acute{q})) \\ &\quad + \tilde{\mathcal{C}}(\mathcal{H}(q) - \mathcal{H}(\acute{q}))]^T[-\tilde{\mathcal{D}}(q - \acute{q}) + \tilde{\mathcal{A}}(\bar{f}(q) - \bar{f}(\acute{q})) + \tilde{\mathcal{B}}(\bar{g}(q) - \bar{g}(\acute{q})) \\ &\quad + \tilde{\mathcal{C}}(\mathcal{H}(q) - \mathcal{H}(\acute{q}))]. \end{aligned}$$

The above equation is equivalent to

$$\begin{aligned} &2(q - \acute{q})^T(\mathcal{P} + \tilde{\mathcal{D}})(\mathcal{H}(q) - \mathcal{H}(\acute{q})) \\ &= -[\mathcal{H}(q) - \mathcal{H}(\acute{q})]^T[\mathcal{H}(q) - \mathcal{H}(\acute{q})] - 2(q - \acute{q})^T(\mathcal{P}\tilde{\mathcal{D}})(q - \acute{q}) \\ &\quad + 2(q - \acute{q})^T(\mathcal{P}\tilde{\mathcal{A}})(\bar{f}(q) - \bar{f}(\acute{q})) + 2(q - \acute{q})^T(\mathcal{P}\tilde{\mathcal{B}})(\bar{g}(q) - \bar{g}(\acute{q})) \\ &\quad + 2(q - \acute{q})^T(\mathcal{P}\tilde{\mathcal{C}})(\mathcal{H}(q) - \mathcal{H}(\acute{q})) - (q - \acute{q})^T(\tilde{\mathcal{D}}^T\tilde{\mathcal{D}})(q - \acute{q}) \\ &\quad + (q - \acute{q})^T(\tilde{\mathcal{D}}\tilde{\mathcal{A}})(\bar{f}(q) - \bar{f}(\acute{q})) + (q - \acute{q})^T(\tilde{\mathcal{D}}\tilde{\mathcal{B}})(\bar{g}(q) - \bar{g}(\acute{q})) \\ &\quad + (q - \acute{q})^T(\tilde{\mathcal{D}}\tilde{\mathcal{C}})(\mathcal{H}(q) - \mathcal{H}(\acute{q})) - (\bar{f}(q) - \bar{f}(\acute{q}))^T(\tilde{\mathcal{A}}^T\tilde{\mathcal{D}})(q - \acute{q}) \\ &\quad + (\bar{f}(q) - \bar{f}(\acute{q}))^T(\tilde{\mathcal{A}}^T\tilde{\mathcal{A}})(\bar{f}(q) - \bar{f}(\acute{q})) + (\bar{f}(q) - \bar{f}(\acute{q}))^T(\tilde{\mathcal{A}}^T\tilde{\mathcal{B}})(\bar{g}(q) - \bar{g}(\acute{q})) \\ &\quad + (\bar{f}(q) - \bar{f}(\acute{q}))^T(\tilde{\mathcal{A}}^T\tilde{\mathcal{C}})(\mathcal{H}(q) - \mathcal{H}(\acute{q})) - (\bar{g}(q) - \bar{g}(\acute{q}))^T(\tilde{\mathcal{B}}^T\tilde{\mathcal{D}})(q - \acute{q}) \\ &\quad + (\bar{g}(q) - \bar{g}(\acute{q}))^T(\tilde{\mathcal{B}}^T\tilde{\mathcal{A}})(\bar{f}(q) - \bar{f}(\acute{q})) + (\bar{g}(q) - \bar{g}(\acute{q}))^T(\tilde{\mathcal{B}}^T\tilde{\mathcal{B}})(\bar{g}(q) - \bar{g}(\acute{q})) \\ &\quad + (\bar{g}(q) - \bar{g}(\acute{q}))^T(\tilde{\mathcal{B}}^T\tilde{\mathcal{C}})(\mathcal{H}(q) - \mathcal{H}(\acute{q})) - (\mathcal{H}(q) - \mathcal{H}(\acute{q}))^T(\tilde{\mathcal{C}}^T\tilde{\mathcal{D}})(q - \acute{q}) \\ &\quad + (\mathcal{H}(q) - \mathcal{H}(\acute{q}))^T(\tilde{\mathcal{C}}^T\tilde{\mathcal{A}})(\bar{f}(q) - \bar{f}(\acute{q})) + (\mathcal{H}(q) - \mathcal{H}(\acute{q}))^T(\tilde{\mathcal{C}}^T\tilde{\mathcal{B}})(\bar{g}(q) - \bar{g}(\acute{q})) \\ &\quad + (\mathcal{H}(q) - \mathcal{H}(\acute{q}))^T(\tilde{\mathcal{C}}^T\tilde{\mathcal{C}})(\mathcal{H}(q) - \mathcal{H}(\acute{q})). \end{aligned} \tag{18}$$

In addition,

$$\begin{aligned}(q-\acute{q})^T(\tilde{\mathcal{D}}\tilde{\mathcal{A}})(\bar{f}(q)-\bar{f}(\acute{q})) &= (\bar{f}(q)-\bar{f}(\acute{q}))^T(\tilde{\mathcal{A}}^T\tilde{\mathcal{D}})(q-\acute{q}),\\ (q-\acute{q})^T(\tilde{\mathcal{D}}\tilde{\mathcal{B}})(\bar{g}(q)-\bar{g}(\acute{q})) &= (\bar{g}(q)-\bar{g}(\acute{q}))^T(\tilde{\mathcal{B}}^T\tilde{\mathcal{D}})(q-\acute{q}),\\ (q-\acute{q})^T(\tilde{\mathcal{D}}\tilde{\mathcal{C}})(\mathcal{H}(q)-\mathcal{H}(\acute{q})) &= (\mathcal{H}(q)-\mathcal{H}(\acute{q}))^T(\tilde{\mathcal{C}}^T\tilde{\mathcal{D}})(q-\acute{q}),\\ (\bar{f}(q)-\bar{f}(\acute{q}))^T(\tilde{\mathcal{A}}^T\tilde{\mathcal{B}})(\bar{g}(q)-\bar{g}(\acute{q})) &= (\bar{g}(q)-\bar{g}(\acute{q}))^T(\tilde{\mathcal{B}}^T\tilde{\mathcal{A}})(\bar{f}(q)-\bar{f}(\acute{q})),\\ (\bar{f}(q)-\bar{f}(\acute{q}))^T(\tilde{\mathcal{A}}^T\tilde{\mathcal{C}})(\mathcal{H}(q)-\mathcal{H}(\acute{q})) &= (\mathcal{H}(q)-\mathcal{H}(\acute{q}))^T(\tilde{\mathcal{C}}^T\tilde{\mathcal{A}})(\bar{f}(q)-\bar{f}(\acute{q})),\\ (\bar{g}(q)-\bar{g}(\acute{q}))^T(\tilde{\mathcal{B}}^T\tilde{\mathcal{C}})(\mathcal{H}(q)-\mathcal{H}(\acute{q})) &= (\mathcal{H}(q)-\mathcal{H}(\acute{q}))^T(\tilde{\mathcal{C}}^T\tilde{\mathcal{B}})(\bar{g}(q)-\bar{g}(\acute{q})).\end{aligned} \tag{19}$$

Using equalities (19) in (18), one has

$$\begin{aligned}&2(q-\acute{q})^T(\mathcal{P}+\tilde{\mathcal{D}})(\mathcal{H}(q)-\mathcal{H}(\acute{q}))\\ &=-[\mathcal{H}(q)-\mathcal{H}(\acute{q})]^T[\mathcal{H}(q)-\mathcal{H}(\acute{q})]-(q-\acute{q})^T(2\mathcal{P}\tilde{\mathcal{D}}+\tilde{\mathcal{D}}^T\tilde{\mathcal{D}})(q-\acute{q})\\ &\quad+2(q-\acute{q})^T(\mathcal{P}\tilde{\mathcal{A}})(\bar{f}(q)-\bar{f}(\acute{q}))+2(q-\acute{q})^T(\mathcal{P}\tilde{\mathcal{B}})(\bar{g}(q)-\bar{g}(\acute{q}))\\ &\quad+2(q-\acute{q})^T(\mathcal{P}\tilde{\mathcal{C}})(\mathcal{H}(q)-\mathcal{H}(\acute{q}))+(\bar{f}(q)-\bar{f}(\acute{q}))^T(\tilde{\mathcal{A}}^T\tilde{\mathcal{A}})(\bar{f}(q)-\bar{f}(\acute{q}))\\ &\quad+2(\bar{f}(q)-\bar{f}(\acute{q}))^T(\tilde{\mathcal{A}}^T\tilde{\mathcal{B}})(\bar{g}(q)-\bar{g}(\acute{q}))+2(\bar{f}(q)-\bar{f}(\acute{q}))^T(\tilde{\mathcal{A}}^T\tilde{\mathcal{C}})(\mathcal{H}(q)-\mathcal{H}(\acute{q}))\\ &\quad+(\bar{g}(q)-\bar{g}(\acute{q}))^T(\tilde{\mathcal{B}}^T\tilde{\mathcal{B}})(\bar{g}(q)-\bar{g}(\acute{q}))+2(\bar{g}(q)-\bar{g}(\acute{q}))^T(\tilde{\mathcal{B}}^T\tilde{\mathcal{C}})(\mathcal{H}(q)-\mathcal{H}(\acute{q}))\\ &\quad+(\mathcal{H}(q)-\mathcal{H}(\acute{q}))^T(\tilde{\mathcal{C}}^T\tilde{\mathcal{C}})(\mathcal{H}(q)-\mathcal{H}(\acute{q})).\end{aligned} \tag{20}$$

Moreover, from (10) and (11) it follows that

$$\epsilon_1[(q-\acute{q})^T\tilde{\mathcal{K}}(q-\acute{q})-(\bar{f}(q)-\bar{f}(\acute{q}))^T(\bar{f}(q)-\bar{f}(\acute{q}))]\geq 0, \tag{21}$$

$$\epsilon_2[(q-\acute{q})^T\tilde{\mathcal{L}}(q-\acute{q})-(\bar{g}(q)-\bar{g}(\acute{q}))^T(\bar{g}(q)-\bar{g}(\acute{q}))]\geq 0. \tag{22}$$

Combining (20)–(22), one has

$$\begin{aligned}&2(q-\acute{q})^T(\mathcal{P}+\tilde{\mathcal{D}})(\mathcal{H}(q)-\mathcal{H}(\acute{q}))\\ &\leq(q-\acute{q})^T(-2\mathcal{P}\tilde{\mathcal{D}}-\tilde{\mathcal{D}}^T\tilde{\mathcal{D}}+\epsilon_1\tilde{\mathcal{K}}+\epsilon_2\tilde{\mathcal{L}})(q-\acute{q})\\ &\quad+2(q-\acute{q})^T(\mathcal{P}\tilde{\mathcal{A}})(\bar{f}(q)-\bar{f}(\acute{q}))+2(q-\acute{q})^T(\mathcal{P}\tilde{\mathcal{B}})(\bar{g}(q)-\bar{g}(\acute{q}))\\ &\quad+2(q-\acute{q})^T(\mathcal{P}\tilde{\mathcal{C}})(\mathcal{H}(q)-\mathcal{H}(\acute{q}))+(\bar{f}(q)-\bar{f}(\acute{q}))^T(-\epsilon_1\mathcal{I}+\tilde{\mathcal{A}}^T\tilde{\mathcal{A}})(\bar{f}(q)-\bar{f}(\acute{q}))\\ &\quad+2(\bar{f}(q)-\bar{f}(\acute{q}))^T(\tilde{\mathcal{A}}^T\tilde{\mathcal{B}})(\bar{g}(q)-\bar{g}(\acute{q}))+2(\bar{f}(q)-\bar{f}(\acute{q}))^T(\tilde{\mathcal{A}}^T\tilde{\mathcal{C}})(\mathcal{H}(q)-\mathcal{H}(\acute{q}))\\ &\quad+(\bar{g}(q)-\bar{g}(\acute{q}))^T(-\epsilon_2\mathcal{I}+\tilde{\mathcal{B}}^T\tilde{\mathcal{B}})(\bar{g}(q)-\bar{g}(\acute{q}))+2(\bar{g}(q)-\bar{g}(\acute{q}))^T(\tilde{\mathcal{B}}^T\tilde{\mathcal{C}})\\ &\quad\times(\mathcal{H}(q)-\mathcal{H}(\acute{q}))+(\mathcal{H}(q)-\mathcal{H}(\acute{q}))^T(-\mathcal{I}+\tilde{\mathcal{C}}^T\tilde{\mathcal{C}})(\mathcal{H}(q)-\mathcal{H}(\acute{q})), \qquad (23)\\ &=\zeta^T(t)\Theta\zeta(t), \qquad (24)\end{aligned}$$

where $\zeta(t)=[(q-\acute{q})^T,(\bar{f}(q)-\bar{f}(\acute{q}))^T,(\bar{g}(q)-\bar{g}(\acute{q}))^T,(\mathcal{H}(q)-\mathcal{H}(\acute{q}))^T]^T$ and

$$\Theta=\begin{pmatrix}\Theta_{11} & \Theta_{12} & \Theta_{13} & \Theta_{14}\\ \star & \Theta_{22} & \Theta_{23} & \Theta_{24}\\ \star & \star & \Theta_{33} & \Theta_{34}\\ \star & \star & \star & \Theta_{44}\end{pmatrix},$$

where $\Theta_{11}=-2\mathcal{P}\tilde{\mathcal{D}}-\tilde{\mathcal{D}}^T\tilde{\mathcal{D}}+\epsilon_1\tilde{\mathcal{K}}+\epsilon_2\tilde{\mathcal{L}}$, $\Theta_{12}=\mathcal{P}\tilde{\mathcal{A}}$, $\Theta_{13}=\mathcal{P}\tilde{\mathcal{B}}$, $\Theta_{14}=\mathcal{P}\tilde{\mathcal{C}}$, $\Theta_{22}=-\epsilon_1\mathcal{I}+\tilde{\mathcal{A}}^T\tilde{\mathcal{A}}$, $\Theta_{23}=\tilde{\mathcal{A}}^T\tilde{\mathcal{B}}$, $\Theta_{24}=\tilde{\mathcal{A}}^T\tilde{\mathcal{C}}$, $\Theta_{33}=-\epsilon_2\mathcal{I}+\tilde{\mathcal{B}}^T\tilde{\mathcal{B}}$, $\Theta_{34}=\tilde{\mathcal{B}}^T\tilde{\mathcal{C}}$, $\Theta_{44}=-\mathcal{I}+\tilde{\mathcal{C}}^T\tilde{\mathcal{C}}$.

Since $\Xi<0$ is true and based on the relation between Θ_{ij} and Ξ_{ij} $(i,j=1,2,3,4)$, one has

$$\begin{aligned}&\mathcal{X}_1^T(\Theta_{11}+\mathcal{Q}-\epsilon_2\tilde{\mathcal{L}})\mathcal{X}_1+(\mathcal{X}_2^T\Theta_{21}+\mathcal{X}_3^T\Theta_{31}+\mathcal{X}_4^T\Theta_{41})\mathcal{X}_1\\ &\quad+(\mathcal{X}_1^T\Theta_{12}+\mathcal{X}_2^T\Theta_{22}+\mathcal{X}_3^T\Theta_{32}+\mathcal{X}_4^T\Theta_{42})\mathcal{X}_2\\ &\quad+(\mathcal{X}_1^T\Theta_{13}+\mathcal{X}_2^T\Theta_{23}+\mathcal{X}_3^T\Theta_{33}+\mathcal{X}_4^T\Theta_{43})\mathcal{X}_3\\ &\quad+(\mathcal{X}_1^T\Theta_{14}+\mathcal{X}_2^T\Theta_{24}+\mathcal{X}_3^T\Theta_{34}+\mathcal{X}_4^T\Theta_{44})\mathcal{X}_4<0,\end{aligned}$$

for $\forall\ \mathcal{X} = [\mathcal{X}_1^T, \mathcal{X}_2^T, \mathcal{X}_3^T, \mathcal{X}_4^T]^T$, and $\mathcal{X} \neq 0$. From (13), we have

$$\begin{aligned}
&(\mathcal{X}_1^T \Theta_{11}\mathcal{X}_1 + (\mathcal{X}_2^T \Theta_{21}) + \mathcal{X}_3^T \Theta_{31} + \mathcal{X}_4^T \Theta_{41})\mathcal{X}_1 \\
&\quad + (\mathcal{X}_1^T \Theta_{12} + \mathcal{X}_2^T \Theta_{22} + \mathcal{X}_3^T \Theta_{32} + \mathcal{X}_4^T \Theta_{42})\mathcal{X}_2 \\
&\quad + (\mathcal{X}_1^T \Theta_{13} + \mathcal{X}_2^T \Theta_{23} + \mathcal{X}_3^T \Theta_{33} + \mathcal{X}_4^T \Theta_{43})\mathcal{X}_3 \\
&\quad + (\mathcal{X}_1^T \Theta_{14} + \mathcal{X}_2^T \Theta_{24} + \mathcal{X}_3^T \Theta_{34} + \mathcal{X}_4^T \Theta_{44})\mathcal{X}_4 < \mathcal{X}_1^T(-\mathcal{Q} + \epsilon_2\tilde{\mathcal{L}})\mathcal{X}_1 \leq 0.
\end{aligned}$$

As such, $\mathcal{X}^T\Theta\mathcal{X} < 0$ holds for $\forall\ \mathcal{X} \neq 0$. Therefore, one has

$$\Theta < 0. \tag{25}$$

Based on (24)–(25) and $q \neq \acute{q}$, the following inequality holds

$$2(q - \acute{q})^T(\mathcal{P} + \tilde{\mathcal{D}})(\mathcal{H}(q) - \mathcal{H}(\acute{q})) < 0, \tag{26}$$

and one can conclude that $\mathcal{H}(q) \neq \mathcal{H}(\acute{q})$ for all $q \neq \acute{q}$. As such, map $\mathcal{H}(q)$ is injective.

Next, $\|\mathcal{H}(q)\| \to \infty$ as $\|q\| \to \infty$ is proven. Let $\acute{q} = 0$. Then, from (24) one can deduce that

$$-2q^T(\mathcal{P} + \tilde{\mathcal{D}})(\mathcal{H}(q) - \mathcal{H}(0)) \geq \lambda_{\min}(-\Theta)\|q\|^2. \tag{27}$$

Based on the Schwartz inequality, one has

$$2\|q\|\|\mathcal{P} + \tilde{\mathcal{D}}\|(\|\mathcal{H}(q)\| - \|\mathcal{H}(0)\|) \geq \|\lambda_{\min}(-\Theta)\|\|q\|^2. \tag{28}$$

That is

$$2(\|\mathcal{H}(q)\| - \|\mathcal{H}(0)\|) \geq \frac{\|\lambda_{\min}(-\Theta)\|}{\|\mathcal{P} + \tilde{\mathcal{D}}\|}\|q\|^2. \tag{29}$$

As such, $\|\mathcal{H}(q)\| \to \infty$ as $\|q\| \to \infty$. By Lemma 3.2, map $\mathcal{H}(q)$ is a homeomorphism of $\mathbb{R}^{2^m n}$. As a result, a unique point q^* exists such that $\mathcal{H}(q^*) = 0$. In other words, (8) has a unique equilibrium point q^*.

Step-2: The globally asymptotical stability corresponding to the equilibrium point with respect to model (8) is proven. Utilizing the transformation $\tilde{q} = q - q^*$, the equilibrium point with respect to model (8) can be shifted to the origin of the following model. Then, model (8) as

$$\begin{aligned}
\dot{\tilde{q}}(t) &= -\tilde{\mathcal{D}}\tilde{q}(t) + \tilde{\mathcal{A}}\tilde{f}(\tilde{q}(t)) + \tilde{\mathcal{B}}\tilde{g}(\tilde{q}(t-\tau)) + \tilde{\mathcal{C}}\dot{\tilde{q}}(t-h),\ \ t \geq 0, \\
\tilde{q}(s) &= \tilde{\varphi}(s),\ \ s \in [-\eta, 0],
\end{aligned} \tag{30}$$

where $\tilde{f}(\tilde{q}) = \bar{f}(\tilde{q} + q^*) - \bar{f}(q^*)$ and $\tilde{g}(\tilde{q} - \tau) = \bar{g}((\tilde{q} - \tau) + q^*) - \bar{g}(q^*)$, and $\tilde{\varphi}(s) = \bar{\varphi}(s) - q^*$.

We construct the following Lyapunov functional:

$$\mathcal{V}(\tilde{q}(t)) = \tilde{q}^T(t)\mathcal{P}\tilde{q}(t) + \tilde{q}^T(t)\tilde{\mathcal{D}}\tilde{q}(t) + \int_{t-\tau}^{t} \tilde{q}^T(s)\mathcal{Q}\tilde{q}(s)ds + \int_{t-h}^{t} \dot{\tilde{q}}^T(s)\dot{\tilde{q}}(s)ds. \tag{31}$$

The time derivative of $\mathcal{V}(\tilde{q}(t))$ along the trajectory of (30) yields

$$\begin{aligned}
\dot{\mathcal{V}}(\tilde{q}(t)) &= 2\tilde{q}^T(t)\mathcal{P}\dot{\tilde{q}}(t) + 2\dot{\tilde{q}}^T(t)\tilde{\mathcal{D}}\tilde{q}(t) + \tilde{q}^T(t)\mathcal{Q}\tilde{q}(t) - \tilde{q}^T(t-\tau)\mathcal{Q}\tilde{q}(t-\tau) \\
&\quad + \dot{\tilde{q}}^T(t)\dot{\tilde{q}}(t) - \dot{\tilde{q}}^T(t-h)\dot{\tilde{q}}(t-h), \\
&= 2\tilde{q}^T(t)\mathcal{P}\dot{\tilde{q}}(t) + \dot{\tilde{q}}^T(t)(2\tilde{\mathcal{D}}\tilde{q}(t) + \dot{\tilde{q}}(t)) + \tilde{q}^T(t)\mathcal{Q}\tilde{q}(t) \\
&\quad - \tilde{q}^T(t-\tau)\mathcal{Q}\tilde{q}(t-\tau) - \dot{\tilde{q}}^T(t-h)\dot{\tilde{q}}(t-h), \\
&= 2\tilde{q}^T(t)\mathcal{P}[-\tilde{\mathcal{D}}\tilde{q}(t) + \tilde{\mathcal{A}}\tilde{f}(\tilde{q}(t)) + \tilde{\mathcal{B}}\tilde{g}(\tilde{q}(t-\tau)) + \tilde{\mathcal{C}}\dot{\tilde{q}}(t-h)] \\
&\quad + [-\tilde{\mathcal{D}}\tilde{q}(t) + \tilde{\mathcal{A}}\tilde{f}(\tilde{q}(t)) + \tilde{\mathcal{B}}\tilde{g}(\tilde{q}(t-\tau)) + \tilde{\mathcal{C}}\dot{\tilde{q}}(t-h)]^T \\
&\quad \times [\tilde{\mathcal{D}}\tilde{q}(t) + \tilde{\mathcal{A}}\tilde{f}(\tilde{q}(t)) + \tilde{\mathcal{B}}\tilde{g}(\tilde{q}(t-\tau)) + \tilde{\mathcal{C}}\dot{\tilde{q}}(t-h)] \\
&\quad + \tilde{q}^T(t)\mathcal{Q}\tilde{q}(t) - \tilde{q}^T(t-\tau)\mathcal{Q}\tilde{q}(t-\tau) - \dot{\tilde{q}}^T(t-h)\dot{\tilde{q}}(t-h),
\end{aligned}$$

which gives

$$\begin{aligned}\dot{\mathcal{V}}(\tilde{q}(t)) &= \tilde{q}^T(t)(-2\mathcal{P}\tilde{\mathcal{D}})\tilde{q}(t) + 2\tilde{q}^T(t)(\mathcal{P}\tilde{\mathcal{A}})\tilde{f}(\tilde{q}(t)) + 2\tilde{q}^T(t)(\mathcal{P}\tilde{\mathcal{B}})\tilde{g}(\tilde{q}(t-\tau)) \\ &+ 2\tilde{q}^T(t)(\mathcal{P}\tilde{\mathcal{C}})\dot{\tilde{q}}(t-h) - \tilde{q}^T(t)(\tilde{\mathcal{D}}^T\tilde{\mathcal{D}})\tilde{q}(t) - \tilde{q}^T(t)(\tilde{\mathcal{D}}\tilde{\mathcal{A}})\tilde{f}(\tilde{q}(t)) \\ &- \tilde{q}^T(t)(\tilde{\mathcal{D}}\tilde{\mathcal{B}})\tilde{g}(\tilde{q}(t-\tau)) - \tilde{q}^T(t)(\tilde{\mathcal{D}}\tilde{\mathcal{C}})\dot{\tilde{q}}(t-h) \\ &+ \tilde{f}^T(\tilde{q}(t))(\tilde{\mathcal{A}}^T\tilde{\mathcal{D}})\tilde{q}(t) + \tilde{f}^T(\tilde{q}(t))(\tilde{\mathcal{A}}^T\tilde{\mathcal{A}})\tilde{f}(\tilde{q}(t)) \\ &+ \tilde{f}^T(\tilde{q}(t))(\tilde{\mathcal{A}}^T\tilde{\mathcal{B}})\tilde{g}(\tilde{q}(t-\tau)) + \tilde{f}^T(\tilde{q}(t))(\tilde{\mathcal{A}}^T\tilde{\mathcal{C}})\dot{\tilde{q}}(t-h) \\ &+ \tilde{g}^T(\tilde{q}(t-\tau))(\tilde{\mathcal{B}}^T\tilde{\mathcal{D}})\tilde{q}(t) + \tilde{g}^T(\tilde{q}(t-\tau))(\tilde{\mathcal{B}}^T\tilde{\mathcal{A}})\tilde{f}(\tilde{q}(t)) \\ &+ \tilde{g}^T(\tilde{q}(t-\tau))(\tilde{\mathcal{B}}^T\tilde{\mathcal{B}})\tilde{g}(\tilde{q}(t-\tau)) + \tilde{g}^T(\tilde{q}(t-\tau))(\tilde{\mathcal{B}}^T\tilde{\mathcal{C}})\dot{\tilde{q}}(t-h) \\ &+ \dot{\tilde{q}}^T(t-h)(\tilde{\mathcal{C}}^T\tilde{\mathcal{D}})\tilde{q}(t) + \dot{\tilde{q}}^T(t-h)(\tilde{\mathcal{C}}^T\tilde{\mathcal{A}})\tilde{f}(\tilde{q}(t)) \\ &+ \dot{\tilde{q}}^T(t-h)(\tilde{\mathcal{C}}^T\tilde{\mathcal{B}})\tilde{g}(\tilde{q}(t-\tau)) + \dot{\tilde{q}}^T(t-h)(\tilde{\mathcal{C}}^T\tilde{\mathcal{C}})\dot{\tilde{q}}(t-h) \\ &+ \tilde{q}^T(t)\mathcal{Q}\tilde{q}(t) - \tilde{q}^T(t-\tau)\mathcal{Q}\tilde{q}(t-\tau) - \dot{\tilde{q}}^T(t-h)\dot{\tilde{q}}(t-h). \end{aligned} \tag{32}$$

In addition, the following equalities hold

$$\begin{aligned} \tilde{q}^T(t)(\tilde{\mathcal{D}}\tilde{\mathcal{A}})\tilde{f}(\tilde{q}(t)) &= \tilde{f}^T(\tilde{q}(t))(\tilde{\mathcal{A}}^T\tilde{\mathcal{D}})\tilde{q}(t), \\ \tilde{q}^T(t)(\tilde{\mathcal{D}}\tilde{\mathcal{B}})\tilde{g}(\tilde{q}(t-\tau)) &= \tilde{g}^T(\tilde{q}(t-\tau))(\tilde{\mathcal{B}}^T\tilde{\mathcal{D}})\tilde{q}(t), \\ \tilde{q}^T(t)(\tilde{\mathcal{D}}\tilde{\mathcal{C}})\dot{\tilde{q}}(t-h) &= \dot{\tilde{q}}^T(t-h)(\tilde{\mathcal{C}}^T\tilde{\mathcal{D}})\tilde{q}(t), \\ \tilde{f}^T(\tilde{q}(t))(\tilde{\mathcal{A}}^T\tilde{\mathcal{B}})\tilde{g}(\tilde{q}(t-\tau)) &= \tilde{g}^T(\tilde{q}(t-\tau))(\tilde{\mathcal{B}}^T\tilde{\mathcal{A}})\tilde{f}(\tilde{q}(t)), \\ \tilde{f}^T(\tilde{q}(t))(\tilde{\mathcal{A}}^T\tilde{\mathcal{C}})\dot{\tilde{q}}(t-h) &= \dot{\tilde{q}}^T(t-h)(\tilde{\mathcal{C}}^T\tilde{\mathcal{A}})\tilde{f}(\tilde{q}(t)), \\ \tilde{g}^T(\tilde{q}(t-\tau))(\tilde{\mathcal{B}}^T\tilde{\mathcal{C}})\dot{\tilde{q}}(t-h) &= \dot{\tilde{q}}^T(t-h)(\tilde{\mathcal{C}}^T\tilde{\mathcal{B}})\tilde{g}(\tilde{q}(t-\tau)). \end{aligned} \tag{33}$$

Hence, (32) together with (33) give

$$\begin{aligned}\dot{\mathcal{V}}(\tilde{q}(t)) &= \tilde{q}^T(t)(-2\mathcal{P}\tilde{\mathcal{D}} - \tilde{\mathcal{D}}^T\tilde{\mathcal{D}})\tilde{q}(t) + 2\tilde{q}^T(t)(\mathcal{P}\tilde{\mathcal{A}})\tilde{f}(\tilde{q}(t)) \\ &+ 2\tilde{q}^T(t)(\mathcal{P}\tilde{\mathcal{B}})\tilde{g}(\tilde{q}(t-\tau)) + 2\tilde{q}^T(t)(\mathcal{P}\tilde{\mathcal{C}})\dot{\tilde{q}}(t-h) \\ &+ \tilde{f}^T(\tilde{q}(t))(\tilde{\mathcal{A}}^T\tilde{\mathcal{A}})\tilde{f}(\tilde{q}(t)) + 2\tilde{f}^T(\tilde{q}(t))(\tilde{\mathcal{A}}^T\tilde{\mathcal{B}})\tilde{g}(\tilde{q}(t-\tau)) \\ &+ 2\tilde{f}^T(\tilde{q}(t))(\tilde{\mathcal{A}}^T\tilde{\mathcal{C}})\dot{\tilde{q}}(t-h) + \tilde{g}^T(\tilde{q}(t-\tau))(\tilde{\mathcal{B}}^T\tilde{\mathcal{B}})\tilde{g}(\tilde{q}(t-\tau)) \\ &+ 2\tilde{g}^T(\tilde{q}(t-\tau))(\tilde{\mathcal{B}}^T\tilde{\mathcal{C}})\dot{\tilde{q}}(t-h) + \dot{\tilde{q}}^T(t-h)(\tilde{\mathcal{C}}^T\tilde{\mathcal{C}})\dot{\tilde{q}}(t-h) \\ &+ \tilde{q}^T(t)\mathcal{Q}\tilde{q}(t) - \tilde{q}^T(t-\tau)\mathcal{Q}\tilde{q}(t-\tau) - \dot{\tilde{q}}^T(t-h)\dot{\tilde{q}}(t-h). \end{aligned} \tag{34}$$

Moreover, from (10)–(11) it follows that

$$\epsilon_1[\tilde{q}^T(t)\mathcal{K}\tilde{q}(t) - \tilde{f}^T(\tilde{q}(t))\tilde{f}(\tilde{q}(t))] \geq 0, \tag{35}$$

$$\epsilon_2[\tilde{q}^T(t-\tau)\mathcal{L}\tilde{q}(t-\tau) - \tilde{g}^T(\tilde{q}(t-\tau))\tilde{g}(\tilde{q}(t-\tau))] \geq 0. \tag{36}$$

Combining (34)–(36), one has

$$\dot{\mathcal{V}}(\tilde{q}(t)) \leq \ \xi^T(t)\Xi\xi(t) + \tilde{q}^T(t-\tau)\Psi\tilde{q}(t-\tau), \tag{37}$$

where $\xi(t) = [\tilde{q}^T(t), \tilde{f}^T(\tilde{q}(t)), \tilde{g}^T(\tilde{q}(t-\tau)), \dot{\tilde{q}}^T(t-h)]^T$, and Ξ, Ψ are given in Theorem 3.5.

As a result, based on inequalities (12), (13) and (37), $\dot{\mathcal{V}}(\tilde{q}(t)) < 0$ and the origin of model (30), or equivalently the equilibrium point of model (8), is globally asymptotically stable. This completes the proof.

3.2. Global exponential stability

Theorem 3.6. *Assume that (**H1**) is satisfied. Subject to the existence of positive definite matrices $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ and positive scalars ϵ_1 and ϵ_2 such that the following LMI is feasible:*

$$\Omega = \begin{pmatrix} -\mathcal{P}\tilde{\mathcal{D}} - \tilde{\mathcal{D}}^T\mathcal{P} + \epsilon_1\tilde{\mathcal{K}} + \epsilon_2\tilde{\mathcal{L}} + 2\beta\mathcal{P} & \mathcal{P}\tilde{\mathcal{A}} & 0 & \mathcal{P}\tilde{\mathcal{B}} & \mathcal{P}\tilde{\mathcal{C}} & -\tilde{\mathcal{D}}^T\mathcal{R} \\ \star & -\epsilon_1\mathcal{I} & 0 & 0 & 0 & \tilde{\mathcal{A}}^T\mathcal{R} \\ \star & \star & -\epsilon_2\mathcal{I} + \mathcal{Q} & 0 & 0 & 0 \\ \star & \star & \star & -e^{-2\beta\tau}\mathcal{Q} & 0 & \tilde{\mathcal{B}}^T\mathcal{R} \\ \star & \star & \star & \star & -e^{-2\beta h}\mathcal{R} & \tilde{\mathcal{C}}^T\mathcal{R} \\ \star & \star & \star & \star & \star & -\mathcal{R} \end{pmatrix} < 0, \tag{38}$$

then the origin of model (30) is globally exponentially stable with exponential convergence rate β. Moreover,

$$\|\tilde{q}(t)\| \leq \sqrt{\frac{\lambda_{\max}(\mathcal{P})\|\bar{\varphi}\|^2 + \lambda_{\max}(\mathcal{Q})\frac{1-e^{-2\beta\tau}}{2\beta}\|\theta_1\|^2 + \lambda_{\max}(\mathcal{R})\frac{1-e^{-2\beta h}}{2\beta}\|\theta_2\|^2}{\lambda_{\min}(\mathcal{P})}}e^{-\beta t}, \tag{39}$$

where $\|\theta_1\| = \sup_{-\tau\leq s\leq 0}\|\tilde{g}(\tilde{q}(s))\|$ and $\|\theta_2\| = \sup_{-h\leq s\leq 0}\|\dot{\tilde{q}}(s)\|$.

Proof. For deriving the global exponential stability criteria, the following Lyapunov functional is formed:

$$\begin{aligned}\mathcal{V}(\tilde{q}(t)) &= e^{2\beta t}\tilde{q}^T(t)\mathcal{P}\tilde{q}(t) + \int_{t-\tau}^{t} e^{2\beta s}\tilde{g}^T(\tilde{q}(s))\mathcal{Q}\tilde{g}(\tilde{q}(s))ds \\ &\quad + \int_{t-h}^{t} e^{2\beta s}\dot{\tilde{q}}^T(s)\mathcal{R}\dot{\tilde{q}}(s)ds.\end{aligned} \tag{40}$$

Based on the time derivative of $\mathcal{V}(\tilde{q}(t))$ along the trajectory of (30), one has

$$\begin{aligned}\dot{\mathcal{V}}(\tilde{q}(t)) &= 2\beta e^{2\beta t}\tilde{q}^T(t)\mathcal{P}\tilde{q}(t) + 2e^{2\beta t}\tilde{q}^T(t)\mathcal{P}\dot{\tilde{q}}(t) + e^{2\beta t}\tilde{g}^T(\tilde{q}(t))\mathcal{Q}\tilde{g}(\tilde{q}(t)) \\ &\quad - e^{2\beta(t-\tau)}\tilde{g}^T(\tilde{q}(t-\tau))\mathcal{Q}\tilde{g}(\tilde{q}(t-\tau)) + e^{2\beta t}\dot{\tilde{q}}^T(t)\mathcal{R}\dot{\tilde{q}}(t) \\ &\quad - e^{2\beta(t-h)}\dot{\tilde{q}}^T(t-h)\mathcal{R}\dot{\tilde{q}}(t-h), \\ &= 2\beta e^{2\beta t}\tilde{q}^T(t)\mathcal{P}\tilde{q}(t) + 2e^{2\beta t}\tilde{q}^T(t)\mathcal{P}[-\tilde{\mathcal{D}}\tilde{q}(t) + \tilde{\mathcal{A}}\tilde{f}(\tilde{q}(t)) + \tilde{\mathcal{B}}\tilde{g}(\tilde{q}(t-\tau)) \\ &\quad + \tilde{\mathcal{C}}\dot{\tilde{q}}(t-h)] + e^{2\beta t}\tilde{g}^T(\tilde{q}(t))\mathcal{Q}\tilde{g}(\tilde{q}(t)) - e^{2\beta(t-\tau)}\tilde{g}^T(\tilde{q}(t-\tau))\mathcal{Q}\tilde{g}(\tilde{q}(t-\tau)) \\ &\quad + e^{2\beta t}[-\tilde{\mathcal{D}}\tilde{q}(t) + \tilde{\mathcal{A}}\tilde{f}(\tilde{q}(t)) + \tilde{\mathcal{B}}\tilde{g}(\tilde{q}(t-\tau)) + \tilde{\mathcal{C}}\dot{\tilde{q}}(t-h)]^T\mathcal{R} \\ &\quad \times [-\tilde{\mathcal{D}}\tilde{q}(t) + \tilde{\mathcal{A}}\tilde{f}(\tilde{q}(t)) + \tilde{\mathcal{B}}\tilde{g}(\tilde{q}(t-\tau)) + \tilde{\mathcal{C}}\dot{\tilde{q}}(t-h)] \\ &\quad - e^{2\beta(t-h)}\dot{\tilde{q}}^T(t-h)\mathcal{R}\dot{\tilde{q}}(t-h),\end{aligned}$$

$$\begin{aligned}\dot{\mathcal{V}}(\tilde{q}(t)) &= 2\beta e^{2\beta t}\tilde{q}^T(t)\mathcal{P}\tilde{q}(t) - 2e^{2\beta t}\tilde{q}^T(t)(\mathcal{P}\tilde{\mathcal{D}})\tilde{q}(t) + 2e^{2\beta t}\tilde{q}^T(t)(\mathcal{P}\tilde{\mathcal{A}})\tilde{f}(\tilde{q}(t)) \\ &\quad + 2e^{2\beta t}\tilde{q}^T(t)(\mathcal{P}\tilde{\mathcal{B}})\tilde{g}(\tilde{q}(t-\tau)) + 2e^{2\beta t}\tilde{q}^T(t)(\mathcal{P}\tilde{\mathcal{C}})\dot{\tilde{q}}(t-h) \\ &\quad + e^{2\beta t}\tilde{g}^T(\tilde{q}(t))\mathcal{Q}\tilde{g}(\tilde{q}(t)) - e^{2\beta(t-\tau)}\tilde{g}^T(\tilde{q}(t-\tau))\mathcal{Q}\tilde{g}(\tilde{q}(t-\tau)) \\ &\quad + e^{2\beta t}[-\tilde{\mathcal{D}}\tilde{q}(t) + \tilde{\mathcal{A}}\tilde{f}(\tilde{q}(t)) + \tilde{\mathcal{B}}\tilde{g}(\tilde{q}(t-\tau)) + \tilde{\mathcal{C}}\dot{\tilde{q}}(t-h)]^T\mathcal{R} \\ &\quad \times [-\tilde{\mathcal{D}}\tilde{q}(t) + \tilde{\mathcal{A}}\tilde{f}(\tilde{q}(t)) + \tilde{\mathcal{B}}\tilde{g}(\tilde{q}(t-\tau)) + \tilde{\mathcal{C}}\dot{\tilde{q}}(t-h)] \\ &\quad - e^{2\beta(t-h)}\dot{\tilde{q}}^T(t-h)\mathcal{R}\dot{\tilde{q}}(t-h).\end{aligned} \tag{41}$$

Moreover, from (10)–(11) it follows that

$$e^{2\beta t}\epsilon_1[\tilde{q}^T(t)\tilde{\mathcal{K}}\tilde{q}(t) - \tilde{f}^T(\tilde{q}(t))\tilde{f}(\tilde{q}(t))] \geq 0, \tag{42}$$

$$e^{2\beta t}\epsilon_2[\tilde{q}^T(t)\tilde{\mathcal{L}}\tilde{q}(t) - \tilde{g}^T(\tilde{q}(t))\tilde{g}(\tilde{q}(t))] \geq 0. \tag{43}$$

for $\epsilon_1 > 0$ and $\epsilon_2 > 0$. As such, by combining (41)–(43), one has

$$\dot{\mathcal{V}}(\tilde{q}(t)) = e^{2\beta t}\varpi^T(t)(\Phi + \Pi^T \mathcal{R}\Pi)\varpi(t), \tag{44}$$

where

$$\varpi(t) = [\tilde{q}^T(t)\ \bar{f}^T(\tilde{q}(t))\ \bar{g}^T(\tilde{q}(t))\ \bar{g}^T(\tilde{q}(t-\tau))\ \dot{\tilde{q}}^T(t-h)]^T,$$

$$\Phi = \begin{pmatrix} -2\mathcal{P}\tilde{\mathcal{D}} + \epsilon_1\tilde{\mathcal{K}} + \epsilon_2\tilde{\mathcal{L}} + 2\beta\mathcal{P} & \mathcal{P}\tilde{\mathcal{A}} & 0 & \mathcal{P}\tilde{\mathcal{B}} & \mathcal{P}\tilde{\mathcal{C}} \\ \star & -\epsilon_1\mathcal{I} & 0 & 0 & 0 \\ \star & \star & -\epsilon_2\mathcal{I} + \mathcal{Q} & 0 & 0 \\ \star & \star & \star & -e^{-2\beta\tau}\mathcal{Q} & 0 \\ \star & \star & \star & \star & -e^{-2\beta h}\mathcal{R} \end{pmatrix},$$

$$\Pi = [-\tilde{\mathcal{D}}\ \ \tilde{\mathcal{A}}\ \ 0\ \ \tilde{\mathcal{B}}\ \ \tilde{\mathcal{C}}].$$

By Schur Complement, it is obvious that $\Omega < 0$ in (38) is equivalent to $\Phi + \Pi^T\mathcal{R}\Pi < 0$. Thus, $\dot{\mathcal{V}}(\tilde{q}(t)) < 0$, then

$$\mathcal{V}(\tilde{q}(t)) \le \mathcal{V}(\tilde{q}(0)). \tag{45}$$

However,

$$\begin{aligned}\mathcal{V}(\tilde{q}(0)) &= \tilde{q}^T(0)\mathcal{P}\tilde{q}(0) + \int_{-\tau}^{0} e^{2\beta s}\bar{g}^T(\tilde{q}(s))\mathcal{Q}\bar{g}(\tilde{q}(s))ds \\ &\quad + \int_{-h}^{0} e^{2\beta s}\dot{\tilde{q}}^T(s)\mathcal{R}\dot{\tilde{q}}(s)ds, \\ &\le \lambda_{\max}(\mathcal{P})\|\bar{\varphi}\|^2 + \lambda_{\max}(\mathcal{Q})\int_{-\tau}^{0} e^{2\beta s}\bar{g}^T(\tilde{q}(s))\bar{g}(\tilde{q}(s))ds \\ &\quad + \lambda_{\max}(\mathcal{R})\int_{-h}^{0} e^{2\beta s}\dot{\tilde{q}}^T(s)\dot{\tilde{q}}(s)ds, \\ &= \lambda_{\max}(\mathcal{P})\|\bar{\varphi}\|^2 + \lambda_{\max}(\mathcal{Q})\int_{-\tau}^{0} e^{2\beta s}ds\|\theta_1\|^2 \\ &\quad + \lambda_{\max}(\mathcal{R})\int_{-h}^{0} e^{2\beta s}ds\|\theta_2\|^2, \\ &= \lambda_{\max}(\mathcal{P})\|\bar{\varphi}\|^2 + \lambda_{\max}(\mathcal{Q})\frac{1-e^{-2\beta\tau}}{2\beta}\|\theta_1\|^2 \\ &\quad + \lambda_{\max}(\mathcal{R})\frac{1-e^{-2\beta h}}{2\beta}\|\theta_2\|^2, \end{aligned} \tag{46}$$

and

$$\mathcal{V}(\tilde{q}(t)) \ge e^{2\beta t}\tilde{q}^T(t)\mathcal{P}\tilde{q}(t) \ge e^{2\beta t}\lambda_{\min}(\mathcal{P})\|\tilde{q}(t)\|^2. \tag{47}$$

From (45)–(47), one has

$$\|\tilde{q}(t)\| \le \sqrt{\frac{\lambda_{\max}(\mathcal{P})\|\bar{\varphi}\|^2 + \lambda_{\max}(\mathcal{Q})\frac{1-e^{-2\beta\tau}}{2\beta}\|\theta_1\|^2 + \lambda_{\max}(\mathcal{R})\frac{1-e^{-2\beta h}}{2\beta}\|\theta_2\|^2}{\lambda_{\min}(\mathcal{P})}}e^{-\beta t}. \tag{48}$$

From Definition 3.4, model (30) is exponentially stable with exponential convergence rate β. This completes the proof of Theorem 3.6.

Remark 3.7. The investigation of Clifford-valued neutral-type NN models has a higher degree of complexity than that of the usual recurrent NN models due to the neutral term. Corresponding to the neutral term and using Lemma 3.3 we formulate an important mapping, that is, $\mathcal{H}(q) = (\mathcal{I} - \tilde{\mathcal{C}})^{-1}[-\tilde{\mathcal{D}}q + \tilde{\mathcal{A}}\bar{f}(q) + \tilde{\mathcal{B}}\bar{g}(q) + \bar{u}]$ instead of $\mathcal{H}(q) = -\tilde{\mathcal{D}}q + \tilde{\mathcal{A}}\bar{f}(q) + \tilde{\mathcal{B}}\bar{g}(q) + \bar{u}$, which is often considered in the existing literature [23,52,54,55].

Remark 3.8. The Clifford-valued NN model (2) includes real-valued ($m = 0$), complex-valued ($m = 1$) and quaternion-valued ($m = 2$) NN models as its special cases.

Remark 3.9. In [2,21,22,27,28,49], the authors studied the dynamics of Clifford-valued NN models without the neutral term. Here, we formulate new sufficient conditions pertaining to the global asymptotic and exponential stability of Clifford-valued NN models with the neutral term. Therefore, our proposed method is different as compared with those in the existing literature.

Remark 3.10. The computational complexity depends primarily on the maximum number of LMI decision variables. As is well known, the number of decision variables increases when using the augmented LKFs and free matrix method. While, when the delay subintervals number becomes more, it might prompt the complexity and the computational burden of the main results. In order to handle this issue easily, we have introduced standard Lyapunov functional, and its derivative estimated without any integral inequalities and decomposition approach. Hence, the proposed stability results may provide a smaller computational burden.

4. Numerical examples

Numerical examples are presented to show the usefulness pertaining to the results in Section 3.

Example 1. Given that $m = 2$ and $n = 2$, the following Clifford-valued NN model with the neutral term is considered

$$\dot{p}(t) = -\mathcal{D}p(t) + \mathcal{A}f(p(t)) + \mathcal{B}g(p(t-\tau)) + \mathcal{C}\dot{p}(t-h) + u, \ \ t \geq 0, \tag{49}$$

The multiplication generators are: $e_1^2 = e_2^2 = e_{12}^2 = e_1e_2e_{12} = -1$, $e_1e_2 = -e_2e_1 = e_{12}$, $e_1e_{12} = -e_{12}e_1 = -e_2$, $e_2e_{12} = -e_{12}e_2 = e_1$, $\dot{p}_1(t) = \dot{p}_1^0(t)e_0 + \dot{p}_1^1(t)e_1 + \dot{p}_1^2(t)e_2 + \dot{p}_1^{12}(t)e_{12}$, $\dot{p}_2(t) = \dot{p}_2^0(t)e_0 + \dot{p}_2^1(t)e_1 + \dot{p}_2^2(t)e_2 + \dot{p}_2^{12}(t)e_{12}$. Furthermore, we take

$$\mathcal{D} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix},$$

$$\mathcal{A} = \begin{pmatrix} 0.2 \sin te_0 + \sin te_1 & 0.1e_0 + 0.3 \cos te_2 + 0.6e_{12} \\ 0.05e_0 - 0.2 \cos \sqrt{5}te_2 + 0.4e_{12} & 0.1e_0 + 0.2e_1 + 0.05 \sin \sqrt{3}te_{12} \end{pmatrix},$$

$$\mathcal{B} = \begin{pmatrix} 0.3 \sin \sqrt{3}te_0 + 0.01 \sin te_1 & 0.1e_0 + 0.02 \cos \sqrt{3}te_2 + 0.3e_{12} \\ 0.05e_0 - 0.2 \sin te_2 + 0.05e_{12} & 0.2e_0 + 0.2e_1 + 0.05 \sin \sqrt{3}te_{12} \end{pmatrix},$$

$$\mathcal{C} = \begin{pmatrix} 0.4 \sin \sqrt{3}te_0 + 0.02 \sin te_1 & 0.2e_0 + 0.03 \cos \sqrt{3}te_2 + 0.4e_{12} \\ 0.06e_0 - 0.3 \sin te_2 + 0.06e_{12} & 0.3e_0 + 0.3e_1 + 0.06 \sin \sqrt{3}te_{12} \end{pmatrix},$$

$$\mathcal{K} = \mathcal{L} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix},$$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{10}\sin p_1^0 e_0 + \frac{1}{15}\sin \sqrt{3}p_1^{12}e_{12} \\ \frac{1}{10}\sin p_2^0 e_0 + \frac{1}{15}\sin \sqrt{3}p_2^{12}e_{12} \end{pmatrix},$$

$$f_1(p_1) = \frac{1}{65}\sin p_1^0 e_0 - \frac{1}{70}\sin p_1^2 e_2,$$

$$f_2(p_2) = \frac{1}{65}\sin p_2^0 e_0 - \frac{1}{70}\sin p_2^2 e_2,$$

$$g_1(p_1) = \frac{1}{58}\cos \sqrt{3}p_1^2 e_2 - \frac{1}{55}\sin \sqrt{6}p_1^{12}e_{12},$$

$$g_2(p_2) = \frac{1}{58}\cos \sqrt{3}p_2^2 e_2 - \frac{1}{55}\sin \sqrt{6}p_2^{12}e_{12},$$

According to their definitions, we have

$$\mathcal{A}^0 = \begin{pmatrix} 0.2 \sin t & 0.1 \\ 0.05 & 0.1 \end{pmatrix},$$

$$\mathcal{A}^1 = \begin{pmatrix} \sin t & 0 \\ 0 & 0.2 \end{pmatrix},$$

$$\mathcal{A}^2 = \begin{pmatrix} 0 & 0.3 \cos t \\ -0.2 \cos \sqrt{5}t & 0 \end{pmatrix},$$

$$\mathcal{A}^{12} = \begin{pmatrix} 0 & 0.6 \\ 0.4 & 0.05 \sin \sqrt{3}t \end{pmatrix},$$

$$\mathcal{B}^0 = \begin{pmatrix} 0.3 \sin \sqrt{3}t & 0.1 \\ 0.05 & 0.2 \end{pmatrix},$$

$$\mathcal{B}^1 = \begin{pmatrix} 0.01 \sin t & 0 \\ 0 & 0.2 \end{pmatrix},$$

$$\mathcal{B}^2 = \begin{pmatrix} 0 & 0.02 \cos \sqrt{3}t \\ -0.2 \sin t & 0 \end{pmatrix},$$

$$\mathcal{B}^{12} = \begin{pmatrix} 0 & 0.3 \\ 0.05 & 0.05 \sin \sqrt{3}t \end{pmatrix},$$

$$\mathcal{C}^0 = \begin{pmatrix} 0.4 \sin \sqrt{3}t & 0.2 \\ 0.06 & 0.3 \end{pmatrix},$$

$$\mathcal{C}^1 = \begin{pmatrix} 0.02 \sin t & 0 \\ 0 & 0.3 \end{pmatrix},$$

$$\mathcal{C}^2 = \begin{pmatrix} 0 & 0.03 \cos \sqrt{3}t \\ -0.3 \sin t & 0 \end{pmatrix},$$

$$\mathcal{C}^{12} = \begin{pmatrix} 0 & 0.4 \\ 0.06 & 0.06 \sin \sqrt{3}t \end{pmatrix},$$

and

$$\begin{aligned}
\tilde{\mathcal{A}} &= \begin{pmatrix} \mathcal{A}^0 & \mathcal{A}^{\bar{1}} & \mathcal{A}^{\bar{2}} & \mathcal{A}^{\bar{12}} \\ \mathcal{A}^1 & \mathcal{A}^{1.\bar{1}} & \mathcal{A}^{1.\bar{2}} & \mathcal{A}^{1.\bar{12}} \\ \mathcal{A}^2 & \mathcal{A}^{2.\bar{1}} & \mathcal{A}^{2.\bar{2}} & \mathcal{A}^{2.\bar{12}} \\ \mathcal{A}^{12} & \mathcal{A}^{12.\bar{1}} & \mathcal{A}^{12.\bar{2}} & \mathcal{A}^{12.\bar{12}} \end{pmatrix}, \\
&= \begin{pmatrix} \mathcal{A}^0 & -\mathcal{A}^1 & -\mathcal{A}^2 & -\mathcal{A}^{12} \\ \mathcal{A}^1 & \mathcal{A}^0 & -\mathcal{A}^{12} & \mathcal{A}^2 \\ \mathcal{A}^2 & \mathcal{A}^{12} & \mathcal{A}^0 & -\mathcal{A}^1 \\ \mathcal{A}^{12} & -\mathcal{A}^2 & \mathcal{A}^1 & \mathcal{A}^0 \end{pmatrix}, \\
\tilde{\mathcal{B}} &= \begin{pmatrix} \mathcal{B}^0 & \mathcal{B}^{\bar{1}} & \mathcal{B}^{\bar{2}} & \mathcal{B}^{\bar{12}} \\ \mathcal{B}^1 & \mathcal{B}^{1.\bar{1}} & \mathcal{B}^{1.\bar{2}} & \mathcal{B}^{1.\bar{12}} \\ \mathcal{B}^2 & \mathcal{B}^{2.\bar{1}} & \mathcal{B}^{2.\bar{2}} & \mathcal{B}^{2.\bar{12}} \\ \mathcal{B}^{12} & \mathcal{B}^{12.\bar{1}} & \mathcal{B}^{12.\bar{2}} & \mathcal{B}^{12.\bar{12}} \end{pmatrix},
\end{aligned}$$

$$= \begin{pmatrix} \mathcal{B}^0 & -\mathcal{B}^1 & -\mathcal{B}^2 & -\mathcal{B}^{12} \\ \mathcal{B}^1 & \mathcal{B}^0 & -\mathcal{B}^{12} & \mathcal{B}^2 \\ \mathcal{B}^2 & \mathcal{B}^{12} & \mathcal{B}^0 & -\mathcal{B}^1 \\ \mathcal{B}^{12} & -\mathcal{B}^2 & \mathcal{B}^1 & \mathcal{B}^0 \end{pmatrix},$$

$$\tilde{\mathcal{C}} = \begin{pmatrix} \mathcal{C}^0 & \mathcal{C}^{\bar{1}} & \mathcal{C}^{\bar{2}} & \mathcal{C}^{\bar{12}} \\ \mathcal{C}^1 & \mathcal{C}^{1.\bar{1}} & \mathcal{C}^{1.\bar{2}} & \mathcal{C}^{1.\bar{12}} \\ \mathcal{C}^2 & \mathcal{C}^{2.\bar{1}} & \mathcal{C}^{2.\bar{2}} & \mathcal{C}^{2.\bar{12}} \\ \mathcal{C}^{12} & \mathcal{C}^{12.\bar{1}} & \mathcal{C}^{12.\bar{2}} & \mathcal{C}^{12.\bar{12}} \end{pmatrix},$$

$$= \begin{pmatrix} \mathcal{C}^0 & -\mathcal{C}^1 & -\mathcal{C}^2 & -\mathcal{C}^{12} \\ \mathcal{C}^1 & \mathcal{C}^0 & -\mathcal{C}^{12} & \mathcal{C}^2 \\ \mathcal{C}^2 & \mathcal{C}^{12} & \mathcal{C}^0 & -\mathcal{C}^1 \\ \mathcal{C}^{12} & -\mathcal{C}^2 & \mathcal{C}^1 & \mathcal{C}^0 \end{pmatrix}.$$

We choose the constant delay parameters $\tau = 0.6$ and $h = 0.4$. Utilizing the LMI toolbox in MATLAB, we ascertain that the LMI conditions of (12) and (13) in Theorem 3.5 are true with $t_{\min} = -0.2638$. The feasible solutions of the existing positive definite matrices $\mathcal{P}$, $\mathcal{Q}$ and positive scalars ϵ_1 and ϵ_2 are

$$\mathcal{P} = \begin{pmatrix} 38.2791 & -0.8887 & 0.0675 & -1.7130 & 2.0833 & -0.7219 & 0.1309 & 1.6582 \\ -0.8887 & 30.4308 & 1.9566 & -1.9600 & 1.4907 & -1.9715 & -1.5555 & 0.5288 \\ 0.0675 & 1.9566 & 38.1666 & -1.2884 & 0.1224 & 1.5489 & -0.3699 & 0.8036 \\ -1.7130 & -1.9600 & -1.2884 & 32.5698 & 0.9321 & 0.1034 & -0.5295 & 0.3151 \\ 2.0833 & 1.4907 & 0.1224 & 0.9321 & 43.1672 & -0.6722 & 3.0597 & -1.9214 \\ -0.7219 & -1.9715 & 1.5489 & 0.1034 & -0.6722 & 33.5505 & 2.8693 & 0.0062 \\ 0.1309 & -1.5555 & -0.3699 & -0.5295 & 3.0597 & 2.8693 & 40.6472 & -0.9730 \\ 1.6582 & 0.5288 & 0.8036 & 0.3151 & -1.9214 & 0.0062 & -0.9730 & 33.6483 \end{pmatrix},$$

$$\mathcal{Q} = \begin{pmatrix} 97.5437 & -6.1196 & 0.2006 & -7.1902 & 9.6137 & -0.5725 & 0.5496 & 9.3294 \\ -6.1196 & 255.3968 & 8.2195 & -6.3808 & 6.0384 & -5.9193 & -7.7196 & 1.9666 \\ 0.2006 & 8.2195 & 97.1600 & -6.1034 & 0.3674 & 8.8631 & -3.0002 & 0.6301 \\ -7.1902 & -6.3808 & -6.1034 & 262.7868 & 2.8300 & -0.0693 & 0.6800 & 1.3257 \\ 9.6137 & 6.0384 & 0.3674 & 2.8300 & 117.4192 & -4.3146 & 14.8075 & -7.0619 \\ -0.5725 & -5.9193 & 8.8631 & -0.0693 & -4.3146 & 265.2626 & 11.0428 & 0.2659 \\ 0.5496 & -7.7196 & -3.0002 & 0.6800 & 14.8075 & 11.0428 & 104.2020 & -5.2613 \\ 9.3294 & 1.9666 & 0.6301 & 1.3257 & -7.0619 & 0.2659 & -5.2613 & 266.1515 \end{pmatrix},$$

$\epsilon_1 = 258.0273$, $\epsilon_2 = 173.5523$.

It is straightforward to show the fulfillment of all conditions with respect to Theorem 3.5. Through numerical simulation, we verify that the unique equilibrium point of model (49) is globally asymptotically stable. Under the initial conditions $\varphi_1(t) = -2.5e_0 + 0.9e_1 - 0.3e_2 - 2.2e_{12}$ and $\varphi_2(t) = 1.6e_0 - 0.4e_1 + 0.2e_2 + 2e_{12}$, the time curves of the state trajectories of model (49) are presented in Figs. 1–5.

Example 2. Consider the two neuron Clifford-valued NN models with neutral term (49) with the parameters $\mathcal{D}$, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ as defined in Example 1.

We choose the constant delay parameters $\tau = 0.5$, $h = 0.4$ and $\beta = 0.3012$. With the help of the Matlab LMI toolbox, LMI (38) of Theorem 3.6 is feasible with $t_{\min} = -0.0041$. The feasible solutions of the existing positive

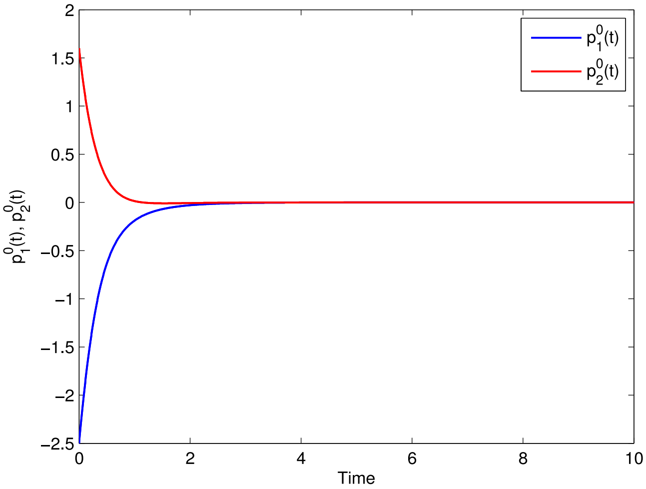

Fig. 1. The time curves of the states $p_1^0(t)$, $p_2^0(t)$ of the NN model (49).

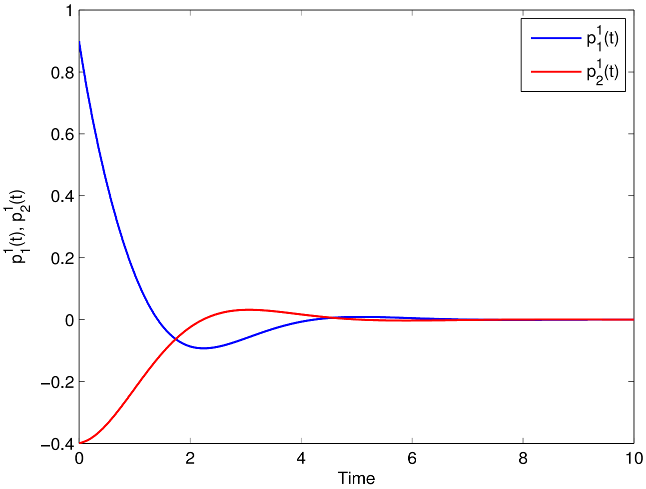

Fig. 2. The time curves of the states $p_1^1(t)$, $p_2^1(t)$ of the NN model (49).

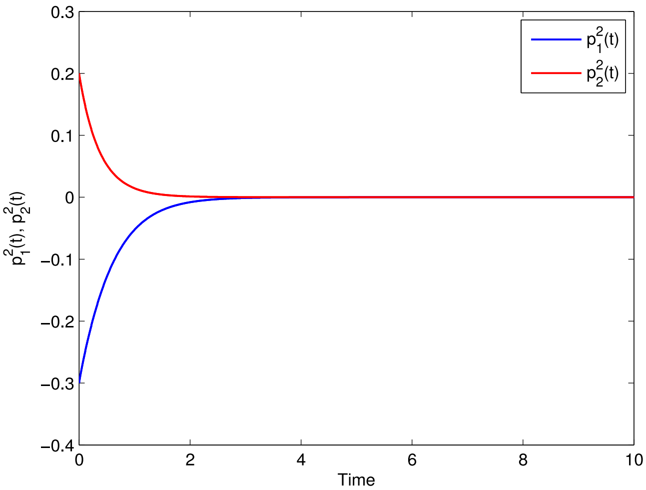

Fig. 3. The time curves of the states $p_1^2(t)$, $p_2^2(t)$ of the NN model (49).

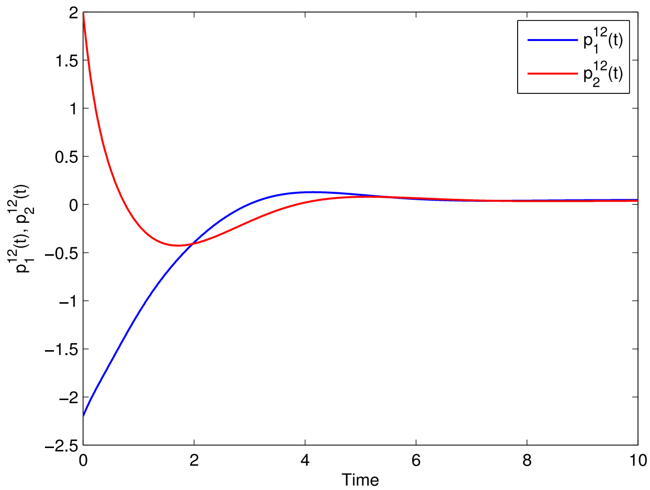

Fig. 4. The time curves of the states $p_1^{12}(t)$, $p_2^{12}(t)$ of the NN model (49).

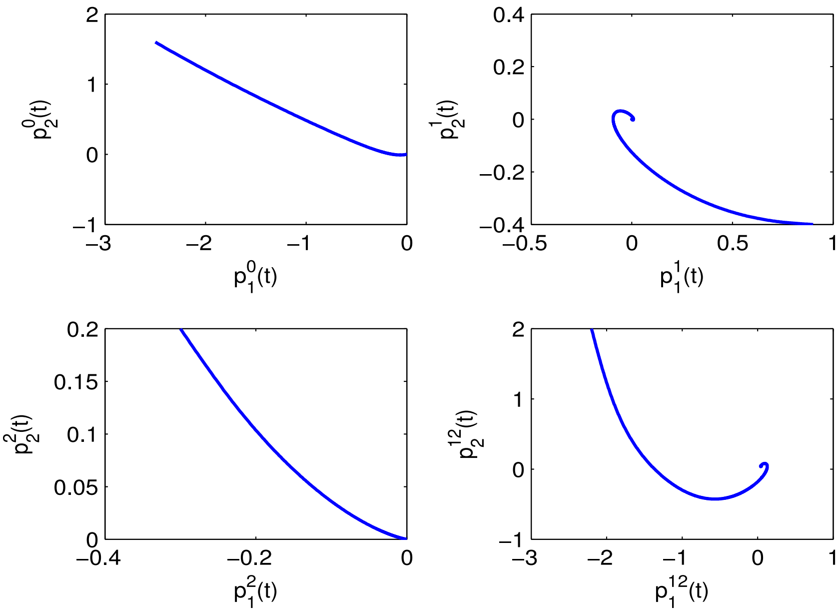

Fig. 5. The time curves of the states corresponding to four parts of $p_1(t)$ and $p_2(t)$ in 2-dimensional space.

definite matrices $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ and positive scalars ϵ_1 and ϵ_2 are

$$\mathcal{P} = \begin{pmatrix} 23.4850 & -1.1421 & 0.0453 & 0.1693 & -1.9659 & 2.6122 & -0.4756 & 2.5301 \\ -1.1421 & 13.8770 & -0.3456 & 0.6360 & 0.4564 & -0.4761 & -2.2512 & 0.9021 \\ 0.0453 & -0.3456 & 23.5748 & -1.6670 & 0.2379 & 2.5966 & 0.6167 & -2.5062 \\ 0.1693 & 0.6360 & -1.6670 & 15.5336 & -2.6502 & 0.7351 & 1.5919 & 0.3889 \\ -1.9659 & 0.4564 & 0.2379 & -2.6502 & 20.7754 & -2.2779 & -1.7230 & -0.0589 \\ 2.6122 & -0.4761 & 2.5966 & 0.7351 & -2.2779 & 16.2712 & 0.4579 & -0.1116 \\ -0.4756 & -2.2512 & 0.6167 & 1.5919 & -1.7230 & 0.4579 & 22.6933 & -2.6519 \\ 2.5301 & 0.9021 & -2.5062 & 0.3889 & -0.0589 & -0.1116 & -2.6519 & 15.9879 \end{pmatrix},$$

$$\mathcal{Q} = \begin{pmatrix} 15.1204 & 0.9142 & -0.0465 & -0.2369 & 0.0516 & -1.7659 & 0.0566 & -1.7899 \\ 0.9142 & 14.8878 & 0.1581 & -0.5765 & 0.6814 & -0.3301 & 1.7237 & 0.1834 \\ -0.0465 & 0.1581 & 15.0218 & 0.9839 & -0.0011 & -1.7689 & -0.1577 & 1.7696 \\ -0.2369 & -0.5765 & 0.9839 & 16.0558 & 1.6517 & 0.0538 & -1.2596 & 0.1640 \\ 0.0516 & 0.6814 & -0.0011 & 1.6517 & 14.7037 & 0.8420 & 0.3049 & -0.1589 \\ -1.7659 & -0.3301 & -1.7689 & 0.0538 & 0.8420 & 16.3236 & 0.4345 & 0.0018 \\ 0.0566 & 1.7237 & -0.1577 & -1.2596 & 0.3049 & 0.4345 & 14.8152 & 0.6699 \\ -1.7899 & 0.1834 & 1.7696 & 0.1640 & -0.1589 & 0.0018 & 0.6699 & 16.2257 \end{pmatrix},$$

$$\mathcal{R} = \begin{pmatrix} 3.1881 & 0.3453 & -0.0460 & -0.5179 & 1.0042 & -1.5760 & 0.3365 & -0.5542 \\ 0.3453 & 3.8869 & 0.3538 & 0.3350 & 0.3254 & -0.0784 & 0.3829 & 0.2004 \\ -0.0460 & 0.3538 & 2.8992 & 0.7381 & -0.0700 & -0.5967 & -0.2870 & 1.4418 \\ -0.5179 & 0.3350 & 0.7381 & 4.2004 & 0.6453 & 0.2246 & -1.1274 & 0.1074 \\ 1.0042 & 0.3254 & -0.0700 & 0.6453 & 4.6218 & 0.3315 & 1.2456 & -0.1562 \\ -1.5760 & -0.0784 & -0.5967 & 0.2246 & 0.3315 & 4.2909 & 0.2644 & -0.0530 \\ 0.3365 & 0.3829 & -0.2870 & -1.1274 & 1.2456 & 0.2644 & 3.6074 & 0.3481 \\ -0.5542 & 0.2004 & 1.4418 & 0.1074 & -0.1562 & -0.0530 & 0.3481 & 4.1522 \end{pmatrix},$$

$\epsilon_1 = 37.6106, \epsilon_2 = 20.8725.$

All the conditions of Theorem 3.6 are satisfied with Example 2. As a result, we verify that the equilibrium point of model (49) is globally exponentially stable.

5. Conclusion

In this study, the global asymptotic stability and global exponential stability analysis of the Clifford-valued neutral-type NN models with time delays has been established. For handling this problem, we first have decomposed the considered n-dimensional Clifford-valued neutral-type NN model into $2^m n$-dimensional real-valued models. In addition, to deal with the neutral term of NN model (1), we have formulated function (14) for the proof corresponding to the existence of the equilibrium point. Secondly, we derived new LMI-based sufficient conditions on the basis of Lyapunov and homeomorphism theories. These conditions guarantee the existence, uniqueness, and global asymptotic stability of the equilibrium point pertaining to the Clifford-valued neutral-type NN model. Finally, to ascertain the validity of the results, numerical examples have been provided.

Undoubtedly, there are some developments to be discussed further in this article. We will shortly attempt to explore the various dynamics of Clifford-valued neutral-type NNs with impulsive effects and Clifford-valued neutral-type NNs with parameter uncertainties. In the near future, the corresponding results will be achieved.

CRediT authorship contribution statement

G. Rajchakit: Funding acquisition, Conceptualization, Software, Formal analysis, Methodology, Writing - original draft, Validation, Writing - review & editing. **R. Sriraman:** Supervision. **C.P. Lim:** Supervision. **B. Unyong:** Supervision.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Availability of data and materials

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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