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Research article

Generalized linear differential equation using Hyers-Ulam stability approach

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Abstract: In this paper, we study the Hyers-Ulam stability with respect to the linear differential condition of fourth order. Specifically, we treat ψ as an interact arrangement of the differential condition, i.e.,

$$\psi^{iv}(\varkappa) + \xi_1\psi'''(\varkappa) + \xi_2\psi''(\varkappa) + \xi_3\psi'(\varkappa) + \xi_4\psi(\varkappa) = \Psi(\varkappa)$$

where $\psi \in c^4[\ell, \mu], \Psi \in [\ell, \mu]$. We demonstrate that $\psi^{iv}(\varkappa)+\xi_1\psi'''(\varkappa)+\xi_2\psi''(\varkappa)+\xi_3\psi'(\varkappa)+\xi_4\psi(\varkappa) = \Psi(\varkappa)$ has the Hyers-Ulam stability. Two examples are provided to illustrate the usefulness of the proposed method.

Keywords: Hyers-Ulam Stability; linear differential equation
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1. Introduction

The challenge of stability with respect to the functional equation stemmed from an issue of Ulam [1] concerning the strength of gathering homomorphisms. Suppose G_1 is a group, and G_2 is a measurement group with metric $d(.,.)$. Given $\in > 0$, does a $\delta > 0$ exist to such an extent that if a mapping $H : G_1 \to G_2$ fulfills the imbalance $d(H(\varkappa\nu), H(\varkappa)H(\nu)) < \delta$ with respect to $\varkappa, \nu \in G_1$, and as such, there exists a homomorphism $h : G_1 \to G_2$ with $d(H(\varkappa), h(\varkappa)) < \epsilon$ with respect to $\varkappa \in G_1$? If the mapping is almost a homomorphism, and as such, there exists a true homomorphism of s, what would be the error that could reasonably be expected?

The problem from the instance of roughly additive mappings was formulated by Hyers [2] with G_1 and G_2 as the Banach spaces. Then, Rassias (see [3]) summed up the effects of the study of Hyers. Since then, the dependability issues of practical conditions have been widely examined by researchers, e.g. see [2–18]).

Supposedly, the study of Ozawa [8] was among the first attempts on managing the $H-U$ stability of differential equations. In [5], the $H-U$ stability of differential condition $\psi'(\varkappa) = \psi(\varkappa)$ was analyzed. Then, the studies of [18, 19] have been further extended to the Banach space differential condition $\psi'(\varkappa) = \lambda\psi(\varkappa)$. Utilizing a direct strategy, cycle technique, find point technique, and open mapping theorem, the $H-U$ stability of certain classes of useful fractional differential equations have been explored, e.g. see [1, 12, 13, 18–20]).

In this paper, we investigate the Hyers-Ulam stability of linear differential equation of the fourth order. Specifically, ψ is an interact arrangement of the following differential equation:

$$\psi^{iv}(\varkappa) + \xi_1\psi'''(\varkappa) + \xi_2\psi''(\varkappa) + \xi_3\psi'(\varkappa) + \xi_4\psi(\varkappa) = \Psi(\varkappa)$$

where $\psi \in c^4[\ell, \mu], \Psi \in [\ell, \mu]$. We demonstrate that $\psi^{iv}(\varkappa)+\xi_1\psi'''(\varkappa)+\xi_2\psi''(\varkappa)+\xi_3\psi'(\varkappa)+\xi_4\psi(\varkappa) = \Psi(\varkappa)$ has the Hyers-Ulam stability. A numerical example is provided to illustrate the proposed method.

Moreover, the effects of $H-U$ stability for the first order differential conditions were studied in [14, 16, 19]. These studies focused on the non-homogeneous straight differential equation of the first order, i.e.,

$$\psi' + \xi(s)\psi + \sigma(s) = 0. \tag{1.1}$$

Jung [14] demonstrated the $H-U$ stability with respect to the differential condition of the following form:

$$s\psi'(s) + \ell\psi(s) + \mu s^{\psi}\varkappa_0 = 0$$

and further applied the outcome to examine the $H-U$ stability of the following differential equation

$$s^2\psi''(s) + as\psi'(s) + b\psi(s) = 0. \tag{1.2}$$

Then, Wang, Zhon and Sun [20] examined the $H-U$ stability of the the first order linear differential condition, i.e.,

$$\xi(\varkappa)\psi' + \sigma(\varkappa)\psi + \nu(\varkappa) = 0. \tag{1.3}$$

In this study, we study the following implication of $H-U$ stability.

Definition 1.1. *We denote that Eq 1.2 has the H − U Stability if there exists a steady* $\lambda > 0$ *with the accompanying property of: for every* $\epsilon > 0, \psi \in c^2[\ell, \mu]$, *if*

$$|\psi'' + a\psi' + b\psi| \leq \epsilon, \tag{1.4}$$

as such, there exists some $u \in c^2[\ell, \mu]$ *that fulfill:*

$$|u'' + au' + bu| = 0 \tag{1.5}$$

such that $|\psi(\varkappa) - u(\varkappa)| < \lambda\epsilon$. *We denote such* λ *a H − U stability constant for Eq 1.2.*

Definition 1.2. *We denote that the extension of Eq 1.2 has the H − U stability, if there exists a steady* $\lambda > 0$ *with the accompanying property of: for every* $\epsilon > 0, \psi \in c^3[\ell, \mu]$, *if*

$$|\psi''' + a\psi'' + b\psi' + c\psi| \leq \epsilon, \tag{1.6}$$

As such, there exist some $u \in c^3[\ell, \mu]$ *that fulfill*

$$|u''' + au'' + bu' + cu| = 0 \tag{1.7}$$

such that $|\psi(\varkappa) - u(\varkappa)| < \lambda\epsilon$. *We denote such* λ *a H − U stability constant for Equation 1.6.*

Definition 1.3. *We denote that the extension of Eq 1.6 has the H − U stability, if there exists a steady* $\lambda > 0$ *with the accompanying property of: for every* $\epsilon > 0, \psi \in c^4[\ell, \mu]$, *if*

$$|\psi^{iv} + \xi_1\psi''' + \xi_2\psi'' + \xi_3\psi' + \xi_4\psi| \leq \epsilon, \tag{1.8}$$

as such, there exist some $u \in c^4[\ell, \mu]$ *that fulfill*

$$|u^{iv} + \xi_1 u''' + \xi_2 u'' + \xi_3 u' + \xi_4 u| = 0 \tag{1.9}$$

such that $|\psi(\varkappa) - u(\varkappa)| < \lambda\epsilon$. *We denote such* λ *a H − U stability constant for Eq 1.8.*

2. Main results

Now, the key results of this study are given in the following hypothesis.

Lemma 2.1. *The differential equation j* $\psi^{iv}(\varkappa) + \xi_1\psi'''(\varkappa) + \xi_2\psi''(\varkappa) + \xi_3\psi'(\varkappa) + \xi_4\psi(\varkappa) = \Psi(\varkappa)$ *has the Hyers -Ulam stability, where* $\psi \in c^4[\ell, \mu]$ *and* $\Psi \in [\ell, \mu]$.

Proof. Assume that u_1, u_2, u_3, and u_4 are the roots of $\nu^4 + \xi_1\nu^3 + p_2\nu^2 + p_3\nu + p_4 = 0$ with $q_1 = \mathbb{R}u_1, q_2 = \mathbb{R}u_2, q_3 = \mathbb{R}u_4$, and $q_4 = \mathbb{R}u_3$. Here $\mathbb{R}$ means the real parts.
Suppose $\epsilon > 0$ and $\psi \in c^4[\ell, \mu]$

$$|\psi^{iv}(\varkappa) + \xi_1\psi'''(\varkappa) + \xi_2\psi''(\varkappa) + \xi_3\psi'(\varkappa) + \xi_4\psi(\varkappa) - \Psi(\varkappa)| \leq \epsilon \tag{2.1}$$

and let

$$g_1(\varkappa) = \psi'''(\varkappa) + (u_1 + \xi_1)\psi''(\varkappa) + (u_1^2 + \xi_1 u_1 + \xi_2)\psi'(\varkappa)$$

$$+ (u_1^3 + \xi_1 u_1^2 + \xi_2 u_1 + \xi_3)\psi(\varkappa),$$

we obtain

$$\begin{aligned} g_1'(\varkappa) &= \psi'''(\varkappa) + (u_1 + \xi_1)\psi'''(\varkappa) + (u_1^2 + \xi_1 u_1 + \xi_2)\psi''(\varkappa) \\ &+ (u_1^3 + \xi_1 u_1^2 + \xi_2 u_1 + \xi_3)\psi'(\varkappa) + (u_1^4 + \xi_1 u_1^3 + \xi_2 u_1^2 + \xi_3 u_1 + \xi_4)\psi(\varkappa) \end{aligned} \tag{2.2}$$

with respect to $\varkappa \in [\ell, \mu]$. As such,

$$|g_1'(\varkappa) - u_1 g_1(\varkappa) - \Psi(\varkappa)| \le \epsilon \tag{2.3}$$

with respect to $\varkappa \in [\ell, \mu]$, it yields that

$$\begin{aligned} |g_1'(\varkappa) - u_1 g_1(\varkappa) - \Psi(\varkappa)| = &|\psi'''(\varkappa) + (u_1 + \xi_1)\psi'''(\varkappa) + (u_1^2 + \xi_1 u + \xi_2)\psi''(\varkappa) \\ &+ (u_1^3 + \xi_1 u_1^2 + \xi_2 u_2 + \xi_3)\psi'(x) \\ &+ (u_1^4 + \xi_1 u_1^3 + \xi_2 u_1^2 + \xi_3 u_1 + \xi_4)\psi(\varkappa) \\ &- u_1(\psi'''(\varkappa) + (u_1 + \xi)\psi''(\varkappa) + u_1^2 + (\xi_1 u_1 + \xi_2)\psi'(\varkappa) \\ &+ \Big((u_1^3 + \xi_1 u_1^2 + \xi_2 u_1 + \xi_3)\psi(\varkappa)\Big) - \Psi(\varkappa)| \end{aligned} \tag{2.4}$$

with respect to $\varkappa \in [\ell, \mu]$. Utilizing the above condition, we obtain

$$\begin{aligned} |g_1'(\varkappa) - u_1 g_1(\varkappa) - \Psi(\varkappa)| &= |\psi^{iv}(\varkappa) + \xi_1\psi'''(\varkappa) + \xi_2\psi^{n}(\varkappa) + \xi_3\psi'(\varkappa) + \xi_4\psi(\varkappa)| \\ &< \epsilon. \end{aligned}$$

with respect to $\varkappa \in [\ell, \mu]$. Similarly, g_1 fulfills

$$-\epsilon \le g_1'(\varkappa) - u_1 g_1(\varkappa) - \Psi(\varkappa) \le \epsilon \tag{2.5}$$

with respect to $\varkappa \in [\ell, \mu]$. Multiplying the above condition by $e^{-u_1(\varkappa-\ell)}$ yields

$$\epsilon e^{-u_1(\varkappa-\ell)} \le g_1'(\varkappa)e^{-u_1(\varkappa-\ell)} - u_1 g_1(\varkappa)e^{-u_1(\varkappa-\ell)} - \Psi(\varkappa)e^{-u_1(\varkappa-\ell)} \le\in e^{-u_1(\varkappa-\ell)} \tag{2.6}$$

with respect to $\varkappa \in [\ell, \mu]$. Without loss of generality, we accept that $u_1 > 1$; therefore

$$\begin{aligned} -u_1\epsilon e^{-u_1(\varkappa-\ell)} &\le g_1'(\varkappa)e^{-u_1(\varkappa-\ell)} - u_1 g_1(\varkappa)e^{-u_1(\varkappa-\ell)} - \Psi(\varkappa)e^{-u_1(\varkappa-\ell)} \\ &\le u_1 e^{-u_1(\varkappa-\ell)} \end{aligned} \tag{2.7}$$

with respect to $\varkappa \in [\ell, \mu]$. Integrating 2.7 from $\varkappa$ to μ, we obtain

$$\begin{aligned} -\epsilon\Big(-e^{-u_1(\mu-\ell)} + e^{-u_1(\varkappa-\ell)}\Big) &\le g_1(\mu)e^{-u_1(\mu\ell)} - g_1(\varkappa)e^{-u_1(\varkappa-\ell)} - \int_{\varkappa}^{\mu} \Psi(s)e^{-u_1(s-\ell)}ds \\ &\le \epsilon\Big(-e^{-u_1(\mu-\ell)} + e^{-u_1(\varkappa-\ell)}\Big) \end{aligned} \tag{2.8}$$

with respect to $\varkappa \in [\ell, \mu]$; therefore

$$-\epsilon e^{-u_1(\varkappa-\ell)} \le g_1(\mu)e^{-u_1(\varkappa-\ell)} - \epsilon e^{-u_1(\mu-\ell)} - g_1(\varkappa)e^{-u_1(\varkappa-\ell)} - \int_{\varkappa}^{\mu} \Psi(s)e^{-u_1(s-\ell)}ds$$

$$\leq \in \left(-e^{-u_1(\varkappa-\ell)} + e^{-u_1(\mu-\ell)}\right) \tag{2.9}$$

with respect to $\varkappa \in [\ell, \mu]$. The above condition yields

$$\begin{aligned} \epsilon - e^{-u_1(\varkappa-\ell)} \leq g_1(\mu) - e^{-u_1(\varkappa-\ell)} - \epsilon - e^{-u_1(\mu-\ell)} - g_1(\varkappa) - e^{-u_1(\varkappa-\ell)} \\ - \int_{\varkappa}^{\mu} \Psi(s) e^{-u_1(s-\ell)} ds \leq \epsilon e^{-u_1(\varkappa-\ell)} \end{aligned} \tag{2.10}$$

with respect to $\varkappa \in [\ell, \mu]$. Multiplying 2.10 by $e^{u_1(\varkappa-\ell)}$ on both sides, we obtain

$$\begin{aligned} -\epsilon \leq g_1(\mu) e^{-u_1(\mu-\varkappa)} - \epsilon e^{-u_1(\mu-\varkappa)} - g_1(\varkappa) - e^{-u_1\varkappa} \int_{\varkappa}^{\mu} \Psi(s) e^{-u_1 s} ds \\ \leq \epsilon \end{aligned} \tag{2.11}$$

therefore

$$\begin{aligned} -\epsilon \leq g_1(\mu) e^{u_1(\varkappa-\mu)} - \epsilon e^{u_1(\varkappa-\mu)} - g_1(\varkappa) - e^{u_1\varkappa} \int_{\varkappa}^{\mu} \Psi(s) e^{-u_1 s} ds \\ \leq \epsilon \end{aligned} \tag{2.12}$$

with respect to $\varkappa \in [\ell, \mu]$. Let

$$\zeta(\varkappa) = g_1(\mu) e^{u_1(\varkappa-\mu)} - e^{u_1(\varkappa)} \int_{\varkappa}^{\mu} \Psi(s) e^{-u_1 s} ds,$$

then $\zeta(\varkappa)$ fulfills $\zeta'(\varkappa) = u_1\zeta(\varkappa) + \Psi(\varkappa)$ with respect to $\varkappa \in [\ell, \mu]$. It satisfies the inequality of

$$\begin{aligned} |\zeta(\varkappa) - g_1(\varkappa)| &= |g_1(\mu) e^{u_1(\varkappa-\mu)} - g_1(\varkappa) - e^{u_1\varkappa} \int_{\varkappa}^{\mu} \Psi(s) e^{-u_1 s} ds| \\ &= e^{\xi\varkappa} | \int_{\varkappa}^{\mu} [e^{-u_1 s} g_1(s)]^l ds - \int_{\varkappa}^{\mu} \Psi(s) e^{-u_1 s} ds| \\ &\leq e^{\xi\varkappa} \int_{\varkappa}^{\mu} e^{-\xi s} |g_1'(s) - u_1 g_1(s) - \Psi(s)| ds \\ &\leq \epsilon e^{\xi\varkappa} \int_{\varkappa}^{\mu} e^{-\xi s} ds \end{aligned} \tag{2.13}$$

with respect to $\varkappa \in [\ell, \mu]$. If $\xi \neq 0$, then

$$\begin{aligned} |\zeta(\varkappa) - g_1(\varkappa)| &\leq \epsilon e^{\xi\varkappa} \int_{\varkappa}^{\mu} e^{-\xi s} ds \\ &\leq \frac{\epsilon}{\xi} \left(1 - e^{-\xi(\mu-\ell)}\right) \end{aligned} \tag{2.14}$$

with respect to $\varkappa \in [\ell, \mu]$. If $\xi = 0$, then

$$\begin{aligned} |\zeta(\varkappa) - g_1(\varkappa)| &\leq \epsilon e^{\xi\varkappa} \int_{\varkappa}^{\mu} e^{-\xi s} ds \\ &\leq \epsilon(\mu - \ell) \end{aligned} \tag{2.15}$$

with respect to $\varkappa\epsilon[\ell,\mu]$. Therefore

$$|\zeta(\varkappa) - g_1(\varkappa)| \le \begin{cases} \frac{1-e^{-\xi(\mu-\ell)}}{\xi}; & if\ \ \xi \neq 0 \\ (\mu-\ell)\epsilon; & if\ \ \xi = 0. \end{cases} \tag{2.16}$$

□

Theorem 2.2. *The differential equation*
$\psi^{iv}(\varkappa) + \xi_1\psi'''(\varkappa) + \xi_2\psi''(\varkappa) + \xi_3\psi'(\varkappa) + \xi_4\psi(\varkappa) = \Psi(\varkappa)$ *has the* $H-U$ *stability, where* $\psi \in c^4[\ell,\mu]$ *and* $\Psi \in [\ell,\mu]$. *Therefore*

$$|\lambda(\varkappa) - h(\varkappa)| \le \begin{cases} \frac{\left(1-e^{-\psi(\mu-\ell)}\right)\left(1-e^{-\xi(\mu-\ell)}\right)\epsilon}{\psi\xi}; & if\ \ \xi,\psi \neq 0 \\ \frac{1-e^{-\psi(\mu-\ell)}(\mu-\ell)\epsilon}{\psi}; & if\ \ \xi \neq 0, \psi \neq 0 \\ \frac{1-e^{-\xi(\mu-\ell)}(\mu-\ell)\epsilon}{\xi}; & if\ \ \xi \neq 0, \psi = 0 \\ (\mu-\ell)^2\epsilon; & if\ \ \xi = 0, \psi = 0 \end{cases}$$

with respect to $\varkappa \in [\ell,\mu]$.

Proof. Similar to the proof of Lemma 2.1. Let $H(\varkappa) = \psi'(\varkappa) - u_2\psi(\varkappa)$ by $H'(\varkappa) = \psi''(\varkappa) - u_1\psi'(\varkappa)$ and let $\epsilon > 0; \psi \in c^4[\ell,\mu]$.
In addition,

$$|H'(\varkappa) - u_4H(\varkappa) - \zeta(\varkappa)| = |\zeta(\varkappa) - g(\varkappa)| \tag{2.17}$$

with respect to $\varkappa \in [\ell,\mu]$. Therefore

$$|H'(\varkappa) - u_4H(\varkappa) - \zeta(\varkappa)| \le \epsilon \tag{2.18}$$

with respect to $\varkappa \in [\ell,\mu]$. Equivalently H fulfills

$$\begin{aligned} |H'(\varkappa) - u_4H(\varkappa) - \zeta(\varkappa)| &= |\psi^{''}(\varkappa) - (u_1+u_4)\psi'(\varkappa) + u_1u_4\psi(\varkappa) - \zeta(\varkappa)| \\ &= |\psi''(\varkappa) + \xi_1\psi'(\varkappa) + \xi_2\psi(\varkappa) - \zeta(\varkappa)| < \epsilon \end{aligned} \tag{2.19}$$

with respect to $\varkappa \in [\ell,\mu]$. Multiplying 2.19 by $e^{-u_4(\varkappa-\ell)}$ on both sides yields

$$\begin{aligned} -\epsilon e^{-u_4(\varkappa-\ell)} &\le H'(\varkappa)e^{-u_4(\varkappa-\ell)} - u_4H(\varkappa)e^{-u_4(\varkappa-\ell)} - \zeta(\varkappa)e^{-u_4(\varkappa-\ell)} \\ &\le \epsilon e^{-u_4(\varkappa-\ell)} \end{aligned} \tag{2.20}$$

with respect to $\varkappa \in [\ell,\mu]$. Without loss of generality, we accept that $u_4 > 1$; therefore

$$\begin{aligned} u_4 \in e^{-u_4(\varkappa-\ell)} &\le H'(\varkappa)e^{-u_4(\varkappa-\ell)} - H(\varkappa)e^{-u_4(\varkappa-\ell)} - \zeta(\varkappa)e^{-u_4(\varkappa-\ell)} \\ &\epsilon u_4 e^{-u_4(\varkappa-\ell)} \end{aligned} \tag{2.21}$$

with respect to $\varkappa \in [\ell,\mu]$. Integrating 2.21 from $\varkappa$ to μ, we obtain

$$-\epsilon\left(e^{-u_4(\varkappa-\ell)} - e^{-u_4(\mu-\ell)}\right) \le H(\mu)e^{-u_4(\mu-\ell)} - H(\varkappa)e^{-u_4(\varkappa-\ell)} - \int_{\varkappa}^{\mu} \zeta(s)e^{-u_4(\varkappa-\ell)}ds$$

$$\leq \epsilon\left(e^{-u_4(\varkappa-\ell)} - e^{-u_4(\mu-\ell)}\right) \tag{2.22}$$

with respect to $\varkappa \in [\ell, \mu]$. Based on 2.22, we obtain

$$\begin{aligned} -\epsilon e^{-u_4(\varkappa-\ell)} &\leq H(\mu)e^{-u_4(\mu-\ell)} - \epsilon e^{-u_4(\mu-\ell)} - H(\varkappa)e^{-u_4(\varkappa-\ell)} - \int_{\varkappa}^{\mu} \zeta(s)e^{-u_4(\varkappa-\ell)}ds \\ &\leq \epsilon\left(e^{-u_4(\varkappa-\ell)}\right) \end{aligned} \tag{2.23}$$

with respect to $\varkappa \in [\ell, \mu]$. Multiplying 2.23 by the function $e^{-u_4(\varkappa-\ell)}$, we obtain

$$\begin{aligned} -\epsilon &\leq H(\mu)e^{-u_4(\mu-\varkappa)} - \epsilon e^{-u_4(\mu-\varkappa)} - H(\varkappa) - e^{u_4\varkappa} \int_{\varkappa}^{\mu} \zeta(s)e^{u_4 s}ds \\ &\leq \epsilon \end{aligned} \tag{2.24}$$

with respect to $\varkappa \in [\ell, \mu]$. Based on 2.24, we obtain

$$\begin{aligned} -\epsilon &\leq H(\mu)e^{u_4(\varkappa-\mu)} - \epsilon e^{u_4(\varkappa-\mu)} - H(\varkappa) - e^{u_4\varkappa} \int_{\varkappa}^{\mu} \zeta(s)e^{u_4 s}ds \\ &\leq \epsilon \end{aligned} \tag{2.25}$$

with respect to $\varkappa \in [\ell, \mu]$. Let $\lambda(\varkappa) = H(\mu)e^{-u_4(\varkappa-\mu)} - e^{u_4\varkappa} \int_{\varkappa}^{\mu} \zeta(s)e^{-u_4 s}ds$ with respect to $\varkappa \in [\ell, \mu]$. Then

$$\lambda'(\varkappa) - u_4\lambda(\varkappa) - \zeta(\varkappa) = 0 \;\; by$$
$$\lambda'(\varkappa) = u_4\lambda(\varkappa) + \zeta(\varkappa).$$

Therefore

$$\begin{aligned} |\lambda(\varkappa) - H(\varkappa)| &= e^{u_4(\varkappa-\mu)}H(\mu) - H(\varkappa) - e^{u_4\varkappa} \int_{\ell}^{\mu} \zeta(s)e^{-u_4 s}ds \\ &= e^{\psi\varkappa}|\int_{\ell}^{\mu} [e^{-u_4 s}H(s)-] - \int_{\ell}^{\mu} \zeta(s)e^{-u_4 s}ds| \\ &\leq e^{\psi\varkappa} \int_{\varkappa}^{\mu} |e^{-u_4 s}||H'(s) - u_4H(s) - \zeta(t)|ds \\ &\leq e^{\psi\varkappa} \int_{\varkappa}^{\mu} e^{-\psi s}|H'(s) - u_4H(s) - \zeta(t)|ds \\ |\lambda(\varkappa) - H(\varkappa)| &\leq \epsilon e^{\psi\varkappa} \int_{\varkappa}^{\mu} e^{-\psi s}ds \end{aligned} \tag{2.26}$$

with respect to $\varkappa \in [\ell, \mu]$. If $\psi \neq 0$, then

$$\begin{aligned} |\lambda(\varkappa) - H(\varkappa)| &\leq \epsilon e^{\psi\varkappa} \int_{\varkappa}^{\mu} e^{-\psi s}ds \\ &\leq \frac{\epsilon}{\psi}[1 - e^{-\psi(\mu-\varkappa)}] \\ |\lambda(\varkappa) - H(\varkappa)| &\leq \frac{\epsilon}{\psi}[1 - e^{-\psi(\mu-\ell)}] \end{aligned} \tag{2.27}$$

with respect to $\varkappa \in [\ell, \mu]$. If $\psi = 0$, then

$$|\lambda(\varkappa) - H(\varkappa)| \leq \epsilon(\mu - \ell) \tag{2.28}$$

with respect to $\varkappa \in [\ell, \mu]$. Based on 2.16, we obtain

$$|\lambda(\varkappa) - H(\varkappa)| \leq \begin{cases} \frac{(1-e^{-\psi(\mu-\ell)})(1-e^{-\xi(\mu-\ell)})\epsilon}{\psi\xi}; & if\ \ \xi, \psi \neq 0 \\ \frac{1-e^{-\psi(\mu-\ell)}(\mu-\ell)\epsilon}{\psi}; & if\ \ \xi = 0, \psi \neq 0 \\ \frac{1-e^{-\xi(\mu-\ell)}(\mu-\ell)\epsilon}{\xi}; & if\ \ \xi \neq 0, \psi = 0 \\ (\mu-\ell)^2\epsilon; & if\ \ \xi = 0, \psi = 0 \end{cases} \tag{2.29}$$

with respect to $\varkappa \in [\ell, \mu]$. □

Theorem 2.3. *The DE $\psi^{iv}(\varkappa) + \xi_1\psi'''(\varkappa) + \xi_2\psi''(\varkappa) + \xi_3\psi'(\varkappa) + \xi_4\psi(\varkappa) = \Psi(\varkappa)$ has the Hyers Ulam stability, where $\psi \in c^4[\ell, \mu]$ and with respect to $\varkappa \in [\ell, \mu]$, $|u(\varkappa) - \psi(\varkappa)| \leq T$ where*

$$T = \begin{cases} \frac{(1-e^{-\psi(\mu-\ell)})(1-e^{-\xi(\mu-\ell)})(1-e^{-\sigma(\mu-\ell)})\epsilon}{\psi\xi\sigma}; & if\ \ (\xi, \psi, \sigma) \neq 0 \\ \frac{(1-e^{-\psi(\mu-\ell)})(1-e^{-\xi(\mu-\ell)})(\mu-\ell)\epsilon}{\psi\xi}; & if\ \ \sigma = 0; (\xi, \psi) \neq 0 \\ \frac{(1-e^{-\psi(\mu-\ell)})(1-e^{-\sigma(\mu-\ell)})(\mu-\ell)\epsilon}{\sigma\psi}; & if\ \ \xi = 0; (\sigma, \psi) \neq 0 \\ \frac{(1-e^{-\xi(\mu-\ell)})(1-e^{-\sigma(\mu-\ell)})(\mu-\ell)\epsilon}{\xi\sigma}; & if\ \ \psi = 0; (\xi, \sigma) \neq 0 \\ \frac{(1-e^{-\xi(\mu-\ell)})(\mu-\ell)^2\epsilon}{\xi}; & if\ \ (\sigma, \psi) = 0; \xi \neq 0 \\ \frac{(1-e^{-\sigma(\mu-\ell)})(\mu-\ell)^2\epsilon}{\sigma}; & if\ \ (\xi, \psi) = 0; \sigma \neq 0 \\ \frac{(1-e^{-\psi(\mu-\ell)})(\mu-\ell)^2\epsilon}{\psi}; & if\ \ (\xi, \sigma) = 0; \psi \neq 0 \\ (\mu-\ell)^3\epsilon; & if\ \ (\xi, \sigma, \psi) = 0 \end{cases}$$

with respect to $\varkappa \in [\ell, \mu]$.

Proof. Based on Theorem 2.2, let us choose

$$\psi(\varkappa) = u''{}_3(\varkappa) + (u_2 + \xi_1)u_3'(\varkappa) + (u_2^2 + \xi_1 u_2 + \xi_2)u_2(\varkappa)$$

by

$$\begin{aligned} \psi'(\varkappa) = u'''{}_3(\varkappa) + (u_2 + \xi_1)u''{}_3(\varkappa) + (u_2^2 + \xi_1 u_2 + \xi_2)u_3'(\varkappa) \\ + (u_2^3 + \xi_1 u_2^2 + \xi_2 u_2 + \xi_3)u_2(\varkappa). \end{aligned}$$

Then

$$\begin{aligned} |\psi'(\varkappa) - u_2\psi(\varkappa) - \lambda(\varkappa)| = |u'''{}_3(\varkappa) + (u_2 + \xi_1)u''{}_3(\varkappa) + (u_2^2 + \xi_1 u_2 + \xi_2)u_3'(\varkappa) \\ + (u_2^3 + \xi_1 u_2^2 + \xi_2 u_2 + \xi_3)u_3(\varkappa) - u_2(u_3''(\varkappa) \\ + (u_2 + \xi_1)u_3'(\varkappa) + (u_2^2 + \xi_1 u_2 + \xi_2)u_3(\varkappa) - \lambda(\varkappa)| \end{aligned}$$

$$= |u_3'''(\varkappa) + \xi_1 u_2''(\varkappa) + \xi_2 u_2'(\varkappa) + \xi_3 u + 3(\varkappa) - \lambda(\varkappa)|$$
$$\leq \epsilon$$

with respect to $\varkappa \in [\ell, \mu]$. As such, we have

$$|\psi'(\varkappa) - u_2\psi(\varkappa) - \lambda(\varkappa)| \leq \epsilon \tag{2.30}$$

with respect to $\varkappa \in [\ell, \mu]$. Equivalently, ψ fulfills

$$-\epsilon \leq \psi'(\varkappa) - u_2\psi(\varkappa) - \lambda(\varkappa) \leq \epsilon \tag{2.31}$$

with respect to $\varkappa \in [\ell, \mu]$. Multiplying the condition by the function $e^{-u_3(\varkappa-\ell)}$

$$-\epsilon e^{-u_3(\varkappa-\ell)} \leq \psi'(\varkappa)e^{-u_3(\varkappa-\ell)} - u_2\psi(\varkappa)e^{-u_3(\varkappa-\ell)} - \lambda(\varkappa)e^{-u_3(\varkappa-\ell)} \leq \epsilon e^{-u_3(\varkappa-\ell)} \tag{2.32}$$

with respect to $\varkappa \in [\ell, \mu]$. Without loss of generality, we accept that $u_3 > 1$. Then

$$-u_3\epsilon e^{-u_3(\varkappa-\ell)} \leq \psi'(\varkappa)e^{-u_3(\varkappa-\ell)} - u_3\psi(\varkappa)e^{-u_3(\varkappa-\ell)} - \lambda(\varkappa)e^{-u_3(\varkappa-\ell)} \leq \epsilon e^{-u_3(\varkappa-\ell)} \tag{2.33}$$

with respect to $\varkappa \in [\ell, \mu]$. Integrating 2.33 from $\varkappa$ to μ, we obtain

$$\begin{aligned} -\epsilon(e^{-u_3(\varkappa-\ell)} - e^{-u_3(\mu-\ell)}) &\leq e^{-u_3(\mu-\ell)}\psi(\ell) - \psi(\varkappa)e^{-u_3(\varkappa-\ell)} - \int_\varkappa^\mu \lambda(s)e^{-u_3(s-\ell)}ds \\ &\leq \epsilon(e^{-u_3(\varkappa-\ell)} - e^{-u_3(\mu-\ell)}) \end{aligned} \tag{2.34}$$

with respect to $\varkappa \in [\ell, \mu]$. Based on 2.34, we obtain

$$\begin{aligned} -\epsilon e^{-u_3(\varkappa-\ell)} &\leq e^{-u_3(\mu-\ell)}\psi(\ell) - \epsilon e^{-u_3(\mu-\ell)} - \psi(\varkappa)e^{-u_3(\varkappa-\ell)} - \int_\varkappa^\mu \lambda(s)e^{-u_3(s-\ell)}ds \\ &\leq \epsilon e^{-u_3(\varkappa-\ell)} \end{aligned} \tag{2.35}$$

with respect to $\varkappa \in [\ell, \mu]$. Again, multiplying the condition by function $e^{-u_3(\varkappa-\ell)}$ yields

$$\begin{aligned} -\epsilon &\leq e^{-u_3(\mu-\ell)}\psi(\ell) - \epsilon e^{-u_3(\mu-\varkappa)} - \psi(\varkappa) - \int_\varkappa^\mu \lambda(s)e^{-u_3(s-\ell)}ds \\ &\leq \epsilon \end{aligned} \tag{2.36}$$

with respect to $\varkappa \in [\ell, \mu]$. Based on 2.36, we have

$$\begin{aligned} -\epsilon &\leq e^{-u_3(\varkappa-\mu)}\psi(\ell) - \epsilon e^{-u_3(\mu-\ell)} - \psi(\varkappa) - e^{u_3\varkappa}\int_\varkappa^\mu \lambda(s)e^{-u_3(s-\ell)}ds \\ &\leq \epsilon \end{aligned} \tag{2.37}$$

for all $\varkappa \in [\ell, \mu]$. Let $u_2(\varkappa) = \psi(\mu)e^{-u_3(\varkappa-\mu)} - e^{u_3\varkappa}\int_\varkappa^\mu \lambda(s)e^{-u_3(s-\ell)}ds$, then $u_2'(\varkappa) - u_3u_2(\varkappa) - \lambda(\varkappa) = 0$ by $u_2'(\varkappa) = u_3u_2(\varkappa) + \lambda(\varkappa)$, for all $\varkappa \in [\ell, \mu]$. Therefore

$$|u_2(\varkappa) - \psi(\varkappa)| = |\psi(\mu)e^{-u_3(\varkappa-\mu)} - \psi(\varkappa) - e^{u_3\varkappa}\int_\varkappa^\mu \lambda(s)e^{-u_3(s-\ell)}ds|$$

$$\leq e^{\sigma\varkappa}\int_{\varkappa}^{\mu} e^{-\sigma s}|\psi'(s)-u_3\psi(s)-\lambda(s)|ds$$

$$|u_2(\varkappa)-\psi(\varkappa)| \leq \epsilon e^{\sigma\varkappa}\int_{\varkappa}^{\mu} e^{-\sigma s}ds \tag{2.38}$$

with respect to $\varkappa \in [\ell,\mu]$. If $\sigma \neq 0$, then

$$\begin{aligned}|u_2(\varkappa)-\psi(\varkappa)| &\leq \frac{\epsilon}{\sigma}(1-e^{-\sigma(\mu-\varkappa)})\\ &\leq \frac{\epsilon}{\sigma}(1-e^{-\sigma(\mu-\ell)})\end{aligned} \tag{2.39}$$

with respect to $\varkappa \in [\ell,\mu]$. If $\sigma = 0$, then

$$\begin{aligned}|u_2(\varkappa)-\psi(\varkappa)| &\leq \epsilon e^{\sigma\varkappa}\int_{\varkappa}^{\mu} e^{-\sigma s}ds\\ &\leq \epsilon(\mu-\varkappa)\\ &\leq \epsilon(\mu-\ell)\end{aligned} \tag{2.40}$$

with respect to $\varkappa \in [\ell,\mu]$. Based on 2.40, we obtain
$|u(\varkappa)-\psi(\varkappa)| \leq T$, where

$$T = \begin{cases} \frac{\left(1-e^{-\psi(\mu-\ell)}\right)\left(1-e^{-\xi(\mu-\ell)}\right)\left(1-e^{-\sigma(\mu-\ell)}\right)\epsilon}{\psi\xi\sigma}; & if \ \ (\xi,\psi,\sigma)\neq 0\\ \frac{\left(1-e^{-\psi(\mu-\ell)}\right)\left(1-e^{-\xi(\mu-\ell)}\right)(\mu-\ell)\epsilon}{\psi\xi}; & if \ \ \sigma=0;(\xi,\psi)\neq 0\\ \frac{\left(1-e^{-\psi(\mu-\ell)}\right)\left(1-e^{-\sigma(\mu-\ell)}\right)(\mu-\ell)\epsilon}{\sigma\psi}; & if \ \ \xi=0;(\sigma,\psi)\neq 0\\ \frac{\left(1-e^{-\xi(\mu-\ell)}\right)\left(1-e^{-\sigma(\mu-\ell)}\right)(\mu-\ell)\epsilon}{\xi\sigma}; & if \ \ \psi=0;(\xi,\sigma)\neq 0\\ \frac{\left(1-e^{-\xi(\mu-\ell)}\right)(\mu-\ell)^2\epsilon}{\xi}; & if \ \ (\sigma,\psi)=0;\xi\neq 0\\ \frac{\left(1-e^{-\sigma(\mu-\ell)}\right)(\mu-\ell)^2\epsilon}{\sigma}; & if \ \ (\xi,\psi)=0;\sigma\neq 0\\ \frac{\left(1-e^{-\psi(\mu-\ell)}\right)(\mu-\ell)^2\epsilon}{\psi}; & if \ \ (\xi,\sigma)=0;\psi\neq 0\\ (\mu-\ell)^3\epsilon; & if \ \ (\xi,\sigma,\psi)=0 \end{cases} \tag{2.41}$$

with respect to $\varkappa \in [\ell,\mu]$. □

Theorem 2.4. *The differential equation*
$\psi^{iv}(\varkappa)+\xi_1\psi'''(\varkappa)+\xi_2\psi''(\varkappa)+\xi_3\psi'(\varkappa)+\xi_4\psi(\varkappa)=\Psi(\varkappa)$ *has the Hyers Ulam stability, where* $\psi \in c^4[\ell,\mu]$ *and with respect to* $\varkappa \in [\ell,\mu]$*, as such*

$|\Psi(\varkappa) - \zeta(\varkappa)| \leq \Theta$, where

$$\Theta = \begin{cases} \frac{\left(1-e^{-\psi(\mu-\ell)}\right)\left(1-e^{-\xi(\mu-\ell)}\right)\left(1-e^{-\sigma(\mu-\ell)}\right)\left(1-e^{-\nu(\mu-\ell)}\right)}{\psi\xi\sigma\nu}\epsilon; & if \ \ (\xi,\psi,\sigma,\nu) \neq 0 \\ \frac{\left(1-e^{-\psi(\mu-\ell)}\right)\left(1-e^{-\xi(\mu-\ell)}\right)\left(1-e^{-\sigma(\mu-\ell)}\right)}{\psi\xi\sigma}(\mu-\ell)\epsilon; & if \ \ \xi \neq \psi \neq \sigma \neq 0, \nu = 0 \\ \frac{\left(1-e^{-\psi(\mu-\ell)}\right)\left(1-e^{-\xi(\mu-\ell)}\right)\left(1-e^{-\nu(\mu-\ell)}\right)}{\psi\xi\nu}(\mu-\ell)\epsilon; & if \ \ \xi \neq \psi \neq \nu \neq 0, \sigma = 0 \\ \frac{\left(1-e^{-\psi(\mu-\ell)}\right)\left(1-e^{-\nu(\mu-\ell)}\right)\left(1-e^{-\sigma(\mu-\ell)}\right)}{\psi\nu\sigma}(\mu-\ell)\epsilon; & if \ \ \nu \neq \psi \neq \sigma \neq 0, \xi = 0 \\ \frac{\left(1-e^{-\nu(\mu-\ell)}\right)\left(1-e^{-\xi(\mu-\ell)}\right)\left(1-e^{-\sigma(\mu-\ell)}\right)}{\nu\xi\sigma}(\mu-\ell)\epsilon; & if \ \ \xi \neq \nu \neq \sigma \neq 0, \psi = 0 \\ \frac{\left(1-e^{-\xi(\mu-\ell)}\right)\left(1-e^{-\sigma(\mu-\ell)}\right)}{\xi\sigma}(\mu-\ell)^2\epsilon; & if \ \ \xi \neq \sigma \neq 0, \psi = \nu = 0 \\ \frac{\left(1-e^{-\xi(\mu-\ell)}\right)\left(1-e^{-\psi(\mu-\ell)}\right)}{\xi\sigma}(\mu-\ell)^2\epsilon; & if \ \ \xi \neq \psi \neq 0, \sigma = \nu = 0 \\ \frac{\left(1-e^{-\xi(\mu-\ell)}\right)\left(1-e^{-\nu(\mu-\ell)}\right)}{\xi\nu}(\mu-\ell)^2\epsilon; & if \ \ \xi \neq \nu \neq 0, \psi = \sigma = 0 \\ \frac{\left(1-e^{-\sigma(\mu-\ell)}\right)\left(1-e^{-\psi(\mu-\ell)}\right)}{\sigma\psi}(\mu-\ell)^2\epsilon; & if \ \ \sigma \neq \psi \neq 0, \xi = \nu = 0 \\ \frac{\left(1-e^{-\xi(\mu-\ell)}\right)\left(1-e^{-\nu(\mu-\ell)}\right)}{\xi\nu}(\mu-\ell)^2\epsilon; & if \ \ \xi \neq \nu \neq 0, \psi = \sigma = 0 \\ \frac{\left(1-e^{-\psi(\mu-\ell)}\right)\left(1-e^{-\nu(\mu-\ell)}\right)}{\psi\nu}(\mu-\ell)^2\epsilon; & if \ \ \psi \neq \nu \neq 0, \xi = \sigma = 0 \\ \frac{\left(1-e^{-\xi(\mu-\ell)}\right)}{\xi}(\mu-\ell)^3\epsilon; & if \ \ \xi \neq 0, \sigma = \psi = \nu = 0 \\ \frac{\left(1-e^{-\psi(\mu-\ell)}\right)}{\psi}(\mu-\ell)^3\epsilon; & if \ \ \psi \neq 0, \sigma = \xi = \nu = 0 \\ \frac{\left(1-e^{-\sigma(\mu-\ell)}\right)}{\sigma}(\mu-\ell)^3\epsilon; & if \ \ \sigma \neq 0, \xi = \psi = \nu = 0 \\ \frac{\left(1-e^{-\nu(\mu-\ell)}\right)}{\nu}(\mu-\ell)^3\epsilon; & if \ \ \nu \neq 0, \sigma = \psi = \xi = 0 \\ (\mu-\ell)^4\epsilon; & if \ \ \xi = \sigma = \psi = \nu = 0 \end{cases}$$

with respect to $\varkappa \in [\ell, \mu]$.

Proof. Similar to the proof of theorem 2.3, let $\in > 0$ and $\psi \in c^4[\ell, \mu]$.
Consider

$$\zeta(\varkappa) = \psi'''(\varkappa) + (u_2 + \xi_1)\psi''(\varkappa) + (u_2^2 + \xi_1 u_2 + \xi_2)\psi'(\varkappa) + (u_2^3 + \xi_1 u_2^2 + \xi_2 u_2 + \xi_3)\psi(\varkappa),$$

we obtain

$$\begin{aligned} \zeta'(\varkappa) = \psi^{iv}(\varkappa) + (u_2 + \xi_1)\psi'''(\varkappa) + (u_2^2 + \xi_1 u_2 + \xi_3)\psi''(\varkappa) \\ + (u_2^3 + \xi_1 u^2 + \xi_2 u_2 + \xi_3)\psi'(\varkappa) + (u_2^4 + \xi_1 u_2^3 + \xi_2 u_2^2 + \xi_3 u_2 + \xi_4)\psi(\varkappa) \end{aligned} \tag{2.42}$$

with respect to $\varkappa \in [\ell, \mu]$. As such

$$|\zeta'(\varkappa) - u_2\zeta(\varkappa) - H(\varkappa)| < \epsilon \tag{2.43}$$

with respect to $\varkappa \in [\ell, \mu]$. If follows from 2.42 that

$$\begin{aligned} |\zeta'(\varkappa) - u_2\zeta(\varkappa) - H(\varkappa)| = |\psi^{iv}(\varkappa) + (u_2 + \xi_1)\psi'''(\varkappa) + (u_2^2 + \xi_1 u_2 + \xi_3)\psi''(\varkappa) \\ + (u_2^3 + \xi_1 u^2 + \xi_2 u_2 + \xi_3)\psi'(\varkappa) \\ + (u_2^4 + \xi_1 u_2^3 + \xi_2 u_2^2 + \xi_3 u_2 + \xi_4)\psi(\psi) \end{aligned}$$

$$
\begin{aligned}
&- u_2(\psi'''(\varkappa) + (u_2 + \xi_1)\psi''(\varkappa) \\
&+ (u_2^2 + \xi_1 u_2 + \xi_2)\psi'(\varkappa) \\
&+ (u_2^3 + \xi_1 u_2^2 + \xi_2 u_2 + \xi_3)\psi(\varkappa)) - H(\varkappa)| \\
= &|\psi^{iv}(\varkappa) + \xi_1\psi'''(\varkappa) + \xi_2\psi''(\varkappa) + \xi_3\psi'(\varkappa) + \xi_4\psi(\varkappa) - H(\varkappa)| \\
\leq &\epsilon.
\end{aligned}
$$

So

$$|\zeta'(\varkappa) - u_2\zeta(\varkappa) - H(\varkappa)| < \epsilon$$

for all $\varkappa \in [\ell, \mu]$. Equivalently, ζ fulfills

$$-\epsilon \leq \zeta'(\varkappa) - u_2\zeta(\varkappa) - H(\varkappa) < \epsilon \tag{2.44}$$

with respect to $\varkappa \in [\ell, \mu]$. Multiplying the formula by the function $e^{-u_2(\varkappa-\ell)}$ satisfies

$$-\epsilon e^{-u_2(\varkappa-\ell)} \leq \zeta'(\varkappa)e^{-u_2(\varkappa-\ell)} - u_2\zeta(\varkappa)e^{-u_2(\varkappa-\ell)} - H(\varkappa)e^{-u_2(\varkappa-\ell)} \tag{2.45}$$
$$\leq \epsilon e^{-u_2(\varkappa-\ell)} \tag{2.46}$$

with respect to $\varkappa\epsilon[\ell, \mu]$. Without loss of generality, we accept that $u_2 > 1$. As such

$$
\begin{aligned}
-\epsilon u_2 e^{-u_2(\varkappa-\ell)} &\leq \zeta'(\varkappa)e^{-u_2(\varkappa-\ell)} - u_2\zeta(\varkappa)e^{-u_2(\varkappa-\ell)} - H(\varkappa)e^{-u_2(\varkappa-\ell)} \\
&\leq \epsilon u_2 e^{-u_2(\varkappa-\ell)}
\end{aligned} \tag{2.47}
$$

for all $\varkappa \in [\ell, \mu]$. Integrating 2.45 from $\varkappa$ to μ. As such

$$
\begin{aligned}
-\epsilon\left(e^{-u_2(\varkappa-\ell)} - e^{-u_2(\mu-\ell)}\right) &\leq \zeta(\mu)e^{-u_2(\mu-\ell)} - \zeta(\varkappa)e^{-u_2(\varkappa-\ell)} - \int_{\varkappa}^{\mu} H(s)e^{-u_2(s-\ell)}ds \\
&\leq \epsilon\left(e^{-u_2(\varkappa-\ell)} - e^{-u_2(\mu-\ell)}\right)
\end{aligned} \tag{2.48}
$$

with respect to $\varkappa \in [\ell, \mu]$. It follows from 2.48 that

$$
\begin{aligned}
-\epsilon\left(e^{-u_2(\varkappa-\ell)}\right) &\leq \zeta(\mu)e^{-u_2(\mu-\ell)} - \epsilon e^{-u_2(\mu-\ell)} - \zeta(\varkappa)e^{-u_2(\varkappa-\ell)} - \int_{\varkappa}^{\mu} H(s)e^{-u_2(s-\ell)}ds \\
&\leq \epsilon\left(e^{-u_2(\varkappa-\ell)}\right)
\end{aligned} \tag{2.49}
$$

for all $\varkappa \in [\ell, \mu]$. Multiplying the formula by the function $e^{-u_2(\varkappa-\ell)}$, we obtain

$$
\begin{aligned}
-\epsilon &\leq \zeta(\mu)e^{-u_2(\varkappa-\mu)} - \epsilon e^{-u_2(\varkappa-\mu)} - \zeta(\varkappa) - e^{u_2\varkappa}\int_{\varkappa}^{\mu} H(s)e^{-u_2(s-\ell)}ds \\
&\leq \epsilon
\end{aligned} \tag{2.50}
$$

with respect to $\varkappa \in [\ell, \mu]$.
Let $\Psi(\varkappa) = \zeta(\mu)e^{-u_2(2H} - e^{u_2\varkappa}\int_{\varkappa}^{\mu} H(s)e^{-u_2(s-\ell)}ds$. As such, $\Psi(\varkappa)$ satisfies $\Psi'(\varkappa) - u_2\Psi(\varkappa) - H(\varkappa) = 0$ by

$$\Psi'(\varkappa) = u_2\Psi(\varkappa) + H(\varkappa) \tag{2.51}$$

with respect to $\varkappa \in [\ell, \mu]$. As such,

$$
\begin{aligned}
|\Psi(\varkappa) - \zeta(\varkappa)| &= |\zeta(\mu)e^{-u_2(\varkappa-\mu)} - \zeta(\varkappa) - e^{u_2\varkappa}\int_{\varkappa}^{\mu} H(s)e^{-u_2 s}ds| \\
&\leq e^{\nu\varkappa}|\int_{\varkappa}^{\mu}[e^{-u_2 s\zeta(s)}]\text{"}ds - \int_{\varkappa}^{\mu} H(s)e^{-u_2 s}ds| \\
&\leq \epsilon e^{\nu\varkappa}\int_{\varkappa}^{\mu} e^{-\nu s}ds \\
|\Psi(\varkappa) - \zeta(\varkappa)| &\leq e^{\nu\varkappa}\int_{\varkappa}^{\mu} e^{-\nu s}\epsilon ds
\end{aligned}
\tag{2.52}
$$

with respect to $\varkappa \in [\ell, \mu]$. If $\nu \neq 0$, then

$$
\begin{aligned}
|\Psi(\varkappa) - \zeta(\varkappa)| &\leq \frac{\epsilon}{\nu}(1 - e^{-\nu(\mu-\varkappa)} \\
&\leq \frac{\epsilon}{\nu}(1 - e^{-\nu(\mu-\ell)}
\end{aligned}
$$

with respect to $\varkappa \in [\ell, \mu]$. If $\nu = 0$, then

$$
\begin{aligned}
|\Psi(\varkappa) - \zeta(\varkappa)| &\leq \epsilon(\mu - \varkappa) \\
&\leq \epsilon(\mu - \ell)
\end{aligned}
$$

with respect to $\varkappa \in [\ell, \mu]$. If follows from 2.41 that

$$
|\Psi(\varkappa) - \zeta(\varkappa)| \leq \Theta, \quad where \tag{2.53}
$$

$$
\Theta = \begin{cases}
\frac{(1-e^{-\psi(\mu-\ell)})(1-e^{-\xi(\mu-\ell)})(1-e^{-\sigma(\mu-\ell)})(1-e^{-\nu(\mu-\ell)})}{\psi\xi\sigma\nu}\epsilon; & if \ \ (\xi, \psi, \sigma, \nu) \neq 0 \\
\frac{(1-e^{-\psi(\mu-\ell)})(1-e^{-\xi(\mu-\ell)})(1-e^{-\sigma(\mu-\ell)})}{\psi\xi\sigma}(\mu - \ell)\epsilon; & if \ \ \xi \neq \psi \neq \sigma \neq 0, \nu = 0 \\
\frac{(1-e^{-\psi(\mu-\ell)})(1-e^{-\xi(\mu-\ell)})(1-e^{-\nu(\mu-\ell)})}{\psi\xi\nu}(\mu - \ell)\epsilon; & if \ \ \xi \neq \psi \neq \nu \neq 0, \sigma = 0 \\
\frac{(1-e^{-\psi(\mu-\ell)})(1-e^{-\nu(\mu-\ell)})(1-e^{-\sigma(\mu-\ell)})}{\psi\nu\sigma}(\mu - \ell)\epsilon; & if \ \ \nu \neq \psi \neq \sigma \neq 0, \xi = 0 \\
\frac{(1-e^{-\nu(\mu-\ell)})(1-e^{-\xi(\mu-\ell)})(1-e^{-\sigma(\mu-\ell)})}{\nu\xi\sigma}(\mu - \ell)\epsilon; & if \ \ \xi \neq \nu \neq \sigma \neq 0, \psi = 0 \\
\frac{(1-e^{-\xi(\mu-\ell)})(1-e^{-\sigma(\mu-\ell)})}{\xi\sigma}(\mu - \ell)^2\epsilon; & if \ \ \xi \neq \sigma \neq 0, \psi = \nu = 0 \\
\frac{(1-e^{-\xi(\mu-\ell)})(1-e^{-\psi(\mu-\ell)})}{\xi\sigma}(\mu - \ell)^2\epsilon; & if \ \ \xi \neq \psi \neq 0, \sigma = \nu = 0 \\
\frac{(1-e^{-\xi(\mu-\ell)})(1-e^{-\nu(\mu-\ell)})}{\xi\nu}(\mu - \ell)^2\epsilon; & if \ \ \xi \neq \nu \neq 0, \psi = \sigma = 0 \\
\frac{(1-e^{-\sigma(\mu-\ell)})(1-e^{-\psi(\mu-\ell)})}{\sigma\psi}(\mu - \ell)^2\epsilon; & if \ \ \sigma \neq \psi \neq 0, \xi = \nu = 0 \\
\frac{(1-e^{-\xi(\mu-\ell)})(1-e^{-\nu(\mu-\ell)})}{\xi\nu}(\mu - \ell)^2\epsilon; & if \ \ \xi \neq \nu \neq 0, \psi = \sigma = 0 \\
\frac{(1-e^{-\psi(\mu-\ell)})(1-e^{-\nu(\mu-\ell)})}{\psi\nu}(\mu - \ell)^2\epsilon; & if \ \ \psi \neq \nu \neq 0, \xi = \sigma = 0 \\
\frac{(1-e^{-\xi(\mu-\ell)})}{\xi}(\mu - \ell)^3\epsilon; & if \ \ \xi \neq 0, \sigma = \psi = \nu = 0 \\
\frac{(1-e^{-\psi(\mu-\ell)})}{\psi}(\mu - \ell)^3\epsilon; & if \ \ \psi \neq 0, \sigma = \xi = \nu = 0 \\
\frac{(1-e^{-\sigma(\mu-\ell)})}{\sigma}(\mu - \ell)^3\epsilon; & if \ \ \sigma \neq 0, \xi = \psi = \nu = 0 \\
\frac{(1-e^{-\nu(\mu-\ell)})}{\nu}(\mu - \ell)^3\epsilon; & if \ \ \nu \neq 0, \sigma = \psi = \xi = 0 \\
(\mu - \ell)^4\epsilon; & if \ \ \xi = \sigma = \psi = \nu = 0
\end{cases}
\tag{2.54}
$$

with respect to $\varkappa \in [\ell, \mu]$. □

3. Illustrative examples

Two examples to illustrate the results in this study are provided, as follows.

Example 3.1. *Consider a differential equation of the form* $\sigma^{iv}(\varkappa) + 2\sigma'''(\varkappa) + \sigma''(\varkappa) = \Psi(\varkappa); \varkappa \in [2, 3]$. *Suppose* $\epsilon > 0$, *as such*

$$|\sigma^{iv}(\varkappa) + 2\sigma'''(\varkappa) + \sigma''(\varkappa) - \Psi(\varkappa)| \leq \epsilon.$$

with respect to $\varkappa \in [2, 3]$. *Suppose* $\lambda = 1$, *then*

$$g(\varkappa) = \sigma'''(\varkappa) + 3\sigma''(\varkappa) + 4\sigma'(\varkappa) + 4\sigma(\varkappa) \text{ and}$$
$$g'(\varkappa) = \sigma^{iv}(\varkappa) + 3\sigma'''(\varkappa) + 4\sigma''(\varkappa) + 4\sigma'(\varkappa) + 4\sigma(\varkappa)$$

with respect to $\varkappa \in [2, 3]$. *The conditions 2.16, 2.18 and 2.30 of Theorem 2.4 are satisfied. Therefore, there is a function* $\varkappa \in c^4[2, 3]$, *which is a mild solution of* $u^{iv}(\varkappa) + 2u'''(\varkappa) + u''(\varkappa) = \Psi(\varkappa)$ *that is satisfied by 2.54.*

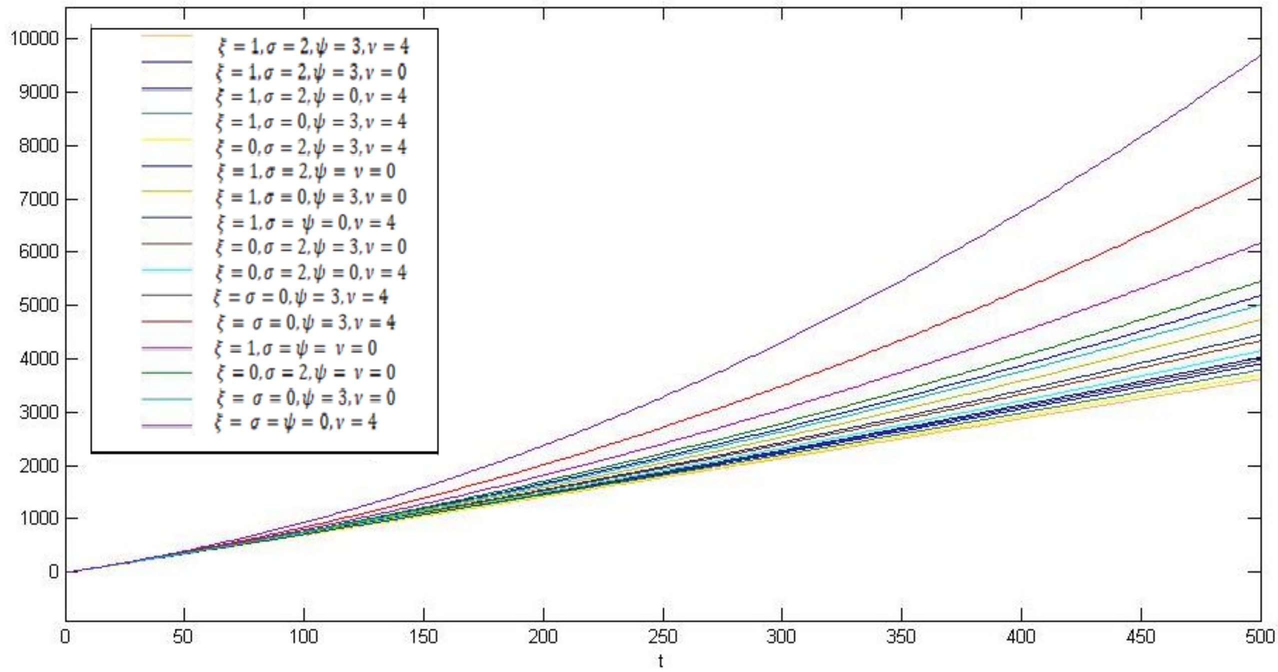

Figure 1. The solution of $\Psi(\varkappa)$ and $\zeta(\varkappa)$ for Eq 2.54.

Example 3.2. *Consider a differential equation of the form* $\sigma^{iv}(\varkappa) + \sigma'''(\varkappa) + \sigma''(\varkappa) = \Psi(\varkappa); \varkappa \in [3, 2]$. *Suppose* $\epsilon > 0$, *and* $\psi \in [3, 2]$, *such that*

$$|\sigma^{iv}(\varkappa) + \sigma'''(\varkappa) + \sigma''(\varkappa) - \Psi(\varkappa)| \leq \epsilon.$$

with respect to $\varkappa \in [3, 2]$. *We take*

$$g(\varkappa) = \sigma'''(\varkappa) + 2\sigma''(\varkappa) + 3\sigma'(\varkappa) + 3\sigma(\varkappa)$$

with respect to $\varkappa \in [3, 2]$. *Then,*

$$g'(\varkappa) = \sigma^{iv}(\varkappa) + 2\sigma'''(\varkappa) + 3\sigma''(\varkappa) + 3\sigma'(\varkappa) + 3\sigma(\varkappa)$$

with respect to $\varkappa \in [3, 2]$. *As such,*

$$|g'(\varkappa) - g(\varkappa) - \Psi(\varkappa)| = |\sigma^{iv}(\varkappa) + \sigma'''(\varkappa) + \sigma''(\varkappa) - \Psi(\varkappa)| \leq \epsilon$$

with respect to $\varkappa \in [\ell, \mu]$. *The conditions 2.16, 2.18 and 2.30 of Theorem 2.4 are satisfied. Therefore, there is a function* $\varkappa \in c^4[3, 2]$, *which is a mild solution of* $u^{iv}(\varkappa) + u'''(\varkappa) + u''(\varkappa) = \Psi(\varkappa)$ *that is satisfied by 2.54.*

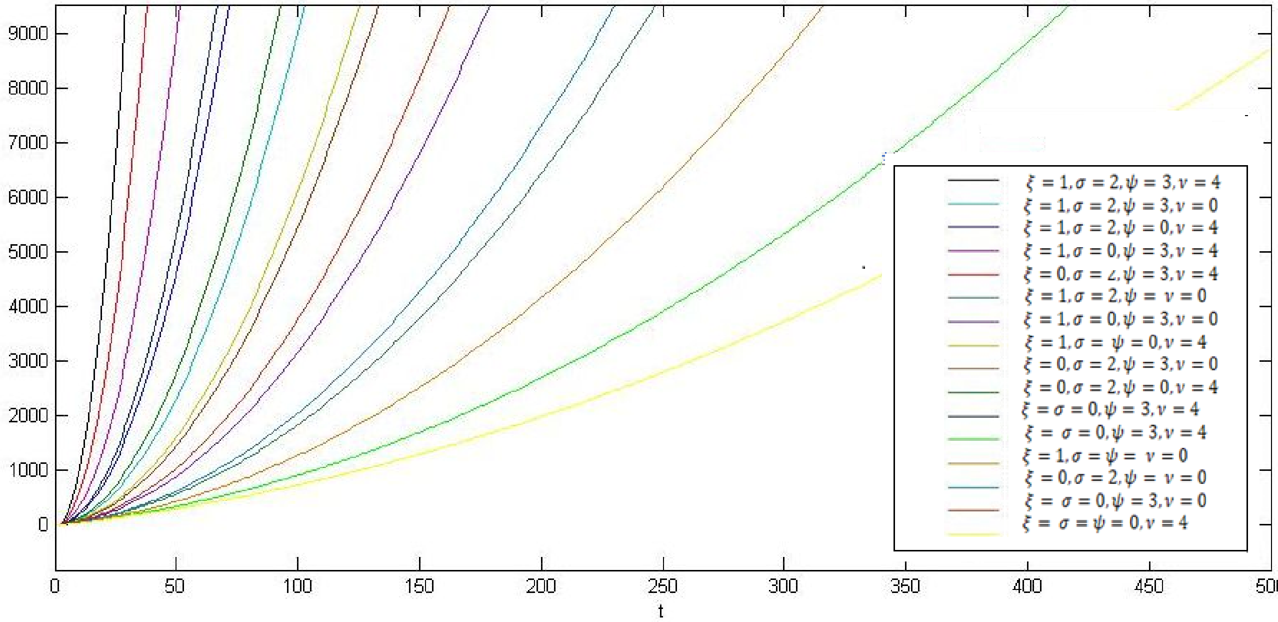

Figure 2. The solution $\Psi(\varkappa)$ and $\zeta(\varkappa)$ for Eq 2.54.

4. Conclusions

We have investigated the Hyers-Ulam stability with respect to the linear differential condition of fourth order in this study. The effectiveness of the proposed method has been illustrated in the examples.

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Conflict of interest

The authors declares no conflict of interest.

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