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# Novel Results on Global Robust Stability Analysis for Dynamical Delayed Neural Networks under Parameter Uncertainties

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**ABSTRACT** In this paper, we focus on the global stability analysis with respect to dynamical delayed neural network (NN) models that contain parameter uncertainties. Many investigations on the sufficient conditions utilizing different upper bounds for the norm of interconnection matrices pertaining to the global asymptotic robust stability of delayed NN models have been conducted. In this study, a new upper bound of the norm of connection weight matrices is derived for the delayed NN models under parameter uncertainties. The key focus is on how the new upper bound is able to yield minimum result with respects to some of the existing upper bounds. We demonstrate that the new upper bound can lead to some new sufficient conditions with respect to the global asymptotic robust stability of equilibrium point of the delayed NN models. The slope bounded activation functions and Lyapunov-Krasovskii functionals are employed for formulating the sufficient conditions of the NN equilibrium point. Moreover, the derived sufficient conditions are independent on the time delay parameter. Numerical examples are provided, and the outcomes obtained are compared with those of the existing results subject to different network parameters.

**INDEX TERMS** Dynamical delayed neural networks, slope bounded activation function, interval matrices, parameter uncertainties, global robust stability.

## I. INTRODUCTION

In recent years, the role of neural networks (NNs) has been significantly developed due to their successful applications to different areas. Indeed, many different types of NN models, e.g. Hopfield, Cohen-Grossberg, Bidirectional Associative, and cellular NN models, have been utilized to solve various engineering problems pertaining to combinatorial optimization, pattern recognition, image and signal processing, etc. Amazon, Epinions, Facebook and Twitter are running in the field of data science and network science systems in [1]–[4]. However, a common challenge of NN hardware design and implementation is that it is difficult to determine appropriate and accurate network parameters. The issue of parameter fluctuation of NN implementation on VLSI chips is also unavoidable. The NN design process includes numerous estimation errors in the measurement of important data such as synaptic interconnection weights, fire rates of neurons, and signal transmission delays. Nevertheless, it is possible to examine the range of network parameters even in the presence of incomplete information. In this regard, by using the interval theory of NN connection weight matrices, we can identify the upper bounds with respect to the norm of interval matrices. Recently, a number of studies on the derivation of the upper bounds of the norm of connection weight matrices have been conducted [5]–[10]. Specifically, the sufficient conditions pertaining to the NN global robust stability have been derived.

As reported in the literature, different kinds of NN stability analysis, such as global asymptotically robust stable (GARS), exponential stability and complete stability with time delays have been examined. [7]–[14]. The Lyapunov stability theory, linear matrix inequalities, non-smooth analysis, M-matrix theory have been used in the stability analysis of delayed NN models. In this respect, the stability properties

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of equilibrium point play a vital role in dynamical delayed NN models. In other words, it is important to examine and understand the global asymptotic robust stability of dynamical delayed NN models under parameter uncertainties, as reported in [15]–[28]. It is well-known that a delayed NN model usually includes a delay parameter in the state of a neuron. However, it is very interesting to add a delay to the neuron state and study the effects. Many different types of time delays can be used, e.g. constant time delay, discrete time delay, distributed time delay, neutral time delay, leakage time delay etc. In this paper, we concentrate on constant time delay NN models. We cover mathematical modelling of NN dynamics with time delays, in which the results have a wide range of practical engineering problems [29]–[32].

Motivated by the above account, we specifically examine the global robust stability of dynamical time-delayed NN models in this study. While several upper bounds with respect to the connection weight matrices of dynamical delayed NNs have been derived, we aim to obtain a new upper bound for the connection weight matrices of this class of NN models. Our study is significant because different upper bounds play a major role in the determination of the sufficient conditions pertaining to the global robust stability of dynamical delayed NN models. Through this new upper bound, we are able to formulate the sufficient conditions with respect to the global asymptotic robust stability of delayed NN models. In our analysis, the activation functions are considered as unbounded, but as slope bounded functions.

This paper is organized in the following manner. The dynamical time-delayed NN model with interval technique of network parameters is described. For the norm of connection weight matrices, we derive a new upper bound in section II. Also we give some new sufficient conditions with respect to the global asymptotic stability using the new upper bound in section III. We also restate some existing sufficient conditions with respect to the stability of NN models in section IV. A comparative study of numerical examples to illustrate the effectiveness of our results over previously published results of delayed NN models is presented in section V. Conclusions are given in section VI.

### A. NOTATIONS:

We utilize the following notations for the norm of vectors and matrices. Let $w = (w_1, w_2, \cdots, w_n)^T \in \mathbb{R}^n$. The most common vector norms are used, i.e., $\| w \|_1, \| w \|_2, \| w \|_\infty$ have the corresponding definitions of $\| w \|_1 = \sum_{i=1}^n | w_i |$, $\| w \|_2 = \sqrt{\sum_{i=1}^n w_i^2}$ and $\| w \|_\infty = \max_{1\leq i\leq n} | w_i |$. Suppose $R = (r_{ij})_{n\times n}$, the following are the definitions of $\| R \|_1$, $\| R \|_2$ and $\| R \|_\infty$. $\| R \|_1 = \max_{1\leq j\leq n} \sum_{i=1}^n | r_{ij} |$, $\| R \|_2 = [\lambda_{max}(R^T R)]^{1/2}$ and $\| R \|_\infty = \max_{1\leq i\leq n} \sum_{j=1}^n | r_{ij} |$. For any vector $w = (w_1, w_2, \cdots, w_n)^T$, $| w |$ defined as $| w | = (| w_1 |, | w_2 |, \ldots, | w_n |)^T$. For any matrix $R = (r_{ij})_{n\times n}$ with real entries $| R |$ is defined as $| R | = (| r_{ij} |)_{n\times n}$. In addition, given matrix $R$, its minimum and maximum eigenvalues are denoted by $\lambda_{min}(R)$ and $\lambda_{max}(R)$, respectively. A positive definite (or semi-definite) symmetric matrix of $R = (r_{ij})_{n\times n}$ exists if $w^T R w > 0 (\geq 0)$, for any real vector $w = (w_1, w_2, \cdots, w_n)^T$. Given two positive definite matrices $H = (h_{ij})_{n\times n}$ and $R = (r_{ij})_{n\times n}$, $H < R$ indicates $w^T H w < w^T R w$ for any real vector $w = (w_1, w_2, \cdots, w_n)^T$.

## II. PRELIMINARIES

The considered dynamical time delayed NN model is represented by a set of differential equations:

$$\begin{aligned} \frac{dw_i(t)}{dt} = &- c_i w_i(t) + \sum_{j=i}^n d_{ij} f_j(w_j(t)) \\ &+ \sum_{j=i}^n e_{ij} f_j(w_j(t-\tau)) + J_i, \\ &i = 1, 2, \cdots, n, \end{aligned} \tag{1}$$

where the total number of neurons is $n$,the $i^{th}$ neuron state of the vector at time $t$ is $w_i(t)$. In addition, $e_{ij}$ and $d_{ij}$ are the connection weights between the $i^{th}$ and $j^{th}$ neurons with and without time delays, respectively; $c_i$ indicates the rate of charge for the $i^{th}$ neuron; $f_j(\cdot)$ denotes the activation functions at time $t$ and $t-\tau$, with $\tau$ denotes the constant time delay. Besides that, $J_i$ represents the vector with constant input between the neurons. The matrix vector form of equation (1) is as follows:

$$\dot{w}(t) = -\mathcal{C} w(t) + \mathcal{D} f(w(t)) + \mathcal{E} f(w(t-\tau) + J, \tag{2}$$

where $w(t) = [w_1(t), w_2(t), \cdots, w_n(t)]^T \in \mathbb{R}^n$, $\mathcal{C} = diag(c_i > 0)$, $\mathcal{E} = (e_{ij}) \in \mathbb{R}^{n\times n}$, $\mathcal{D} = (d_{ij}) \in \mathbb{R}^{n\times n}$, $f(w(t)) = [f_1(w_1(t)), f_2(w_2(t)), \cdots, f_n(w_n(t))]^T \in \mathbb{R}^n$ and $J = [J_1, J_2, \cdots, J_n]^T \in \mathbb{R}^n$. The most common approach for handling the delayed NN model is to make the connection weight matrices $\mathcal{D} = (d_{ij})_{n\times n}$, $\mathcal{E} = (e_{ij})_{n\times n}$ and $\mathcal{C} = diag(c_i > 0)$ in an interval, i.e.,

$$\begin{cases} \mathcal{C}_I = \{\mathcal{C} = diag(c_i) : 0 \prec \underline{\mathcal{C}} \preceq \mathcal{C} \preceq \overline{\mathcal{C}}, \\ ie., 0 < \underline{c}_i \leq c_i \leq \overline{c}_i, \ i = 1, 2, \cdots, n\} \\ \mathcal{D}_I = \{\mathcal{D} = (d_{ij}) : \underline{\mathcal{D}} \preceq \mathcal{D} \preceq \overline{\mathcal{D}}, \ ie., \ \underline{d}_{ij} \leq d_{ij} \leq \overline{d}_{ij}, \} \\ \mathcal{E}_I = \{\mathcal{E} = (e_{ij}) : \underline{\mathcal{E}} \preceq \mathcal{E} \preceq \overline{\mathcal{E}}, \ ie., \ \underline{e}_{ij} \leq e_{ij} \leq \overline{e}_{ij}, \\ i, j = 1, 2, \cdots, n\} \end{cases} \tag{3}$$

By using equation (3), we can define matrices $\mathcal{D}^*$, $\mathcal{D}_*$, $\mathcal{E}^*$ and $\mathcal{E}_*$:

$$\mathcal{D}^* = \frac{1}{2}(\overline{\mathcal{D}} + \underline{\mathcal{D}}), \ \mathcal{D}_* = \frac{1}{2}(\overline{\mathcal{D}} - \underline{\mathcal{D}}). \tag{4}$$

$$\mathcal{E}^* = \frac{1}{2}(\overline{\mathcal{E}} + \underline{\mathcal{E}}), \ \mathcal{E}_* = \frac{1}{2}(\overline{\mathcal{E}} - \underline{\mathcal{E}}). \tag{5}$$

**Definition 1.** *The NN model given in (2) with the network parameters satisfying (3) is globally robust stable if the unique equilibrium point $w^*(t) = [w_1^*(t), w_2^*(t), \cdots, w_n^*(t)]^T \in \mathbb{R}^n$ of the model is globally asymptotically stable for all $\mathcal{C} \in \mathcal{C}_I, \mathcal{D} \in \mathcal{D}_I, \mathcal{E} \in \mathcal{E}_I$.*

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**Definition 2.** *A slope bounded function has some positive constants $k_i$ such that*

$$0 \leq \frac{f_i(w) - f_i(v)}{w - v} \leq k_i,\ \forall w, v \in \mathbb{R}\ w \neq v,\ i = 1, 2, \cdots, n.$$

A slope-bounded activation function of $f_i$ is used in this study, in which the class of functions is denoted by $f \in \underset{.}{k}$. Note that it is not necessary for this class of functions to be monotonically increasing, differentiable, and bounded. The upper bounds for the norm of the connection weight matrices $\mathcal{D} = (d_{ij})$ and $\mathcal{E} = (e_{ij})$ of model (2) play a vital role for finding the sufficient conditions with respect to the global robust stability analysis. Given matrices $\mathcal{D}$ and $\mathcal{E}$, four different upper bounds of their norm have been discussed in the literature. So, we first restate the four existing upper bounds with respect to the norm of interval connection weight matrices $\mathcal{D}$ and $\mathcal{E}$.

**Lemma 1.** *[7]– [10] A matrix $\mathcal{E}$ is defined by $\mathcal{E} \in \mathcal{E}_I$ as in equation (3), $\mathcal{E}^*$ and $\mathcal{E}_*$ are the matrices defined as in equation (5).*

*Let $T_1(\mathcal{E}) = \sqrt{\| \mid (\mathcal{E}^*)^T\mathcal{E}^* \mid + 2 \mid (\mathcal{E}^*)^T \mid \mathcal{E}_* + \mathcal{E}_*^T\mathcal{E}_* \|_2}$, $T_2(\mathcal{E}) = \| \mathcal{E}^* \|_2 + \| \mathcal{E}_* \|_2$, $T_3(\mathcal{E}) = \sqrt{\| \mathcal{E}^* \|_2^2 + \| \mathcal{E}_* \|_2^2 + 2 \| \mathcal{E}_*^T \mid \mathcal{E}^* \mid\|}$ and $T_4(\mathcal{E}) = \| \hat{\mathcal{E}} \|_2$, where $\hat{\mathcal{E}} = (\hat{e}_{ij})$ with $\hat{e}_{ij} = max(\mid \underline{e}_{ij} \mid, \mid \overline{e}_{ij} \mid)$. Then, $\| \mathcal{E} \|_2 \leq T_i(\mathcal{E})$, where $i = 1, 2, 3, 4$.*

**Lemma 2.** *[25] Suppose $\mathcal{E} \in \mathcal{E}_I$ is any matrix defined as in equation (3) , $\mathcal{E}^*$ and $\mathcal{E}_*$ are defined as in equation (5), then*

$$\| \mathcal{E} \|_2 \leq T_5(\mathcal{E}),$$

*where $T_5(\mathcal{E}) = \sqrt{\lambda_{max}(\mid (\mathcal{E}^*)^T\mathcal{E}^* \mid + \mathcal{E}_*^T \mid \mathcal{E}^* \mid + \mid (\mathcal{E}^*)^T \mid \mathcal{E}_* + \mathcal{E}_*^T\mathcal{E}_*)}$.*

Our major contribution of our current study is to derive a new upper bound with respect to the norm of matrices $\mathcal{D}$ and $\mathcal{E}$. Specifically, we formulate the new upper bound with respect to the norm of interval connection weight matrices $\mathcal{D}$ and $\mathcal{E}$ in the following form.

**Lemma 3.** *Suppose $\mathcal{E} \in \mathcal{E}_I$ is any matrix defined as in equation (3), $\mathcal{E}^*$ and $\mathcal{E}_*$ are the matrices defined as in equation (5), then*

$$\| \mathcal{E} \|_2 \leq T_6(\mathcal{E}),$$

*where*

$$T_6(\mathcal{E}) = \sqrt{\lambda_{max}(\mid (\mathcal{E}^*)^T\mathcal{E}^* \mid + 2\mathcal{E}_*^T \mid \mathcal{E}^* \mid + \mathcal{E}_*^T\mathcal{E}_*)}.$$

*Proof.* If $\mathcal{E} \in \mathcal{E}_I$ , then $e_{ij}$ can be written as follows:

$$e_{ij} = \frac{1}{2}(\overline{e}_{ij} + \underline{e}_{ij}) + t_{ij}\ \frac{1}{2}(\overline{e}_{ij} - \underline{e}_{ij}),\ -1 \leq t_{ij} \leq 1,$$

(or)

$$\mathcal{E} = (e_{ij}) = \frac{1}{2}(\overline{\mathcal{E}} + \underline{\mathcal{E}}) + \Delta\ \frac{1}{2}(\overline{\mathcal{E}} - \underline{\mathcal{E}}) = \mathcal{E}^* + \Delta\mathcal{E}_*,$$

where $\Delta = (t_{ij})_{n\times n}$, $i, j = 1, 2, \cdots, n$. For any vector $w(t) = [w_1(t), w_2(t), \cdots, w_n(t)]^T \in \mathbb{R}^n$, we can write

$$\begin{aligned} w^T\mathcal{E}^T\mathcal{E}w =& w^T(\mathcal{E}^* + \Delta\mathcal{E}_*)^T(\mathcal{E}^* + \Delta\mathcal{E}_*)w \\ =& w^T(\mathcal{E}^*)^T\mathcal{E}^*w + w^T(\mathcal{E}^*)^T\Delta\mathcal{E}_*w \\ &+ w^T\mathcal{E}_*^T\Delta^T\mathcal{E}^*w + w^T\mathcal{E}_*^T\Delta^T\Delta\mathcal{E}_*w \\ =& w^T(\mathcal{E}^*)^T\mathcal{E}^*w + 2w^T\mathcal{E}_*^T\Delta^T\mathcal{E}^*w \\ &+ w^T\mathcal{E}_*^T\Delta^T\Delta\mathcal{E}_*w \\ \leq& \mid w^T \mid\mid (\mathcal{E}^*)^T\mathcal{E}^* \mid\mid w \mid \\ &+ 2 \mid w^T \mid\mid \mathcal{E}_*^T\Delta^T \mid\mid \mathcal{E}^* \mid\mid w \mid \\ &+ \mid w^T \mid\mid \mathcal{E}_*^T\Delta^T \mid\mid \Delta\mathcal{E}_* \mid\mid w \mid . \end{aligned}$$

Since $\mid \Delta\mathcal{E}_* \mid \leq \mathcal{E}_*$, we have

$$\mid w^T \mid\mid \mathcal{E}_*^T\Delta^T \mid\mid \mathcal{E}^* \mid\mid w \mid \leq \mid w^T \mid \mathcal{E}_*^T \mid \mathcal{E}^* \mid\mid w \mid$$

and

$$\mid w^T \mid\mid \mathcal{E}_*^T\Delta^T \mid\mid \Delta\mathcal{E}_* \mid\mid w \mid \leq \mid w^T \mid \mathcal{E}_*^T\mathcal{E}_* \mid w \mid .$$

Therefore,

$$\begin{aligned} w^T\mathcal{E}^T\mathcal{E}w \leq& |w^T||(\mathcal{E}^*)^T\mathcal{E}^*||w| + 2 \mid w^T \mid \mathcal{E}_*^T \mid \mathcal{E}^* \mid\mid w \mid \\ &+ \mid w^T \mid \mathcal{E}_*^T\mathcal{E}_* \mid w \mid \\ =& \mid w^T \mid \big( \mid (\mathcal{E}^*)^T\mathcal{E}^* \mid + 2\mathcal{E}_*^T \mid \mathcal{E}^* \mid + \mathcal{E}_*^T\mathcal{E}_*\big) \mid w \mid \\ \leq& \lambda_{max}(\mid (\mathcal{E}^*)^T\mathcal{E}^* \mid + 2\mathcal{E}_*^T \mid \mathcal{E}^* \mid + \mathcal{E}_*^T\mathcal{E}_*)w^Tw \\ \| \mathcal{E} \|_2^2 \leq& \lambda_{max}(\mid (\mathcal{E}^*)^T\mathcal{E}^* \mid + 2\mathcal{E}_*^T \mid \mathcal{E}^* \mid + \mathcal{E}_*^T\mathcal{E}_*) \\ \| \mathcal{E} \|_2 \leq& \sqrt{\lambda_{max}(\mid (\mathcal{E}^*)^T\mathcal{E}^* \mid + 2\mathcal{E}_*^T \mid \mathcal{E}^* \mid + \mathcal{E}_*^T\mathcal{E}_*)}. \end{aligned}$$

(or)

$$\| \mathcal{E} \|_2 \leq T_6(\mathcal{E}).$$

Hence the proof. □

**Remark 1.** *The results in Lemma 1-3 always hold for the connection weight matrix $\mathcal{D}$, i.e., $\| \mathcal{D} \|_2 \leq T_i(\mathcal{D})$, $i = 1, 2, 3, 4, 5, 6$.*

**Lemma 4.** *For any matrix $\mathcal{E} \in \mathcal{E}_I$, $T_5(\mathcal{E}) \leq T_1(\mathcal{E})$ and $T_6(\mathcal{E}) \leq T_1(\mathcal{E})$.*

*Proof.* Since

$$\begin{aligned} &\mid (\mathcal{E}^*)^T\mathcal{E}^* \mid + \mathcal{E}_*^T \mid \mathcal{E}^* \mid + \mid (\mathcal{E}^*)^T \mid \mathcal{E}_* + \mathcal{E}_*^T\mathcal{E}_* \\ =& \frac{1}{2}[\mid (\mathcal{E}^*)^T\mathcal{E}^* \mid + 2\mathcal{E}_*^T \mid \mathcal{E}^* \mid + \mathcal{E}_*^T\mathcal{E}_* + \mid (\mathcal{E}^*)^T\mathcal{E}^* \mid \\ &+ 2 \mid (\mathcal{E}^*)^T \mid \mathcal{E}_* + \mathcal{E}_*^T\mathcal{E}_*]. \end{aligned}$$

$$\begin{aligned} &\| \mid (\mathcal{E}^*)^T\mathcal{E}^* \mid + \mathcal{E}_*^T \mid \mathcal{E}^* \mid + \mid (\mathcal{E}^*)^T \mid \mathcal{E}_* + \mathcal{E}_*^T\mathcal{E}_* \|_2 \\ &= \frac{1}{2} \| [\mid (\mathcal{E}^*)^T\mathcal{E}^* \mid + 2\mathcal{E}_*^T \mid \mathcal{E}^* \mid + \mathcal{E}_*^T\mathcal{E}_* \\ &\quad + \mid (\mathcal{E}^*)^T\mathcal{E}^* \mid + 2 \mid (\mathcal{E}^*)^T \mid \mathcal{E}_* + \mathcal{E}_*^T\mathcal{E}_*] \|_2 \\ &\leq \frac{1}{2} \| \mid (\mathcal{E}^*)^T\mathcal{E}^* \mid + 2\mathcal{E}_*^T \mid \mathcal{E}^* \mid + \mathcal{E}_*^T\mathcal{E}_* \|_2 \\ &\quad + \frac{1}{2} \| \mid (\mathcal{E}^*)^T\mathcal{E}^* \mid + 2 \mid (\mathcal{E}^*)^T \mid \mathcal{E}_* + \mathcal{E}_*^T\mathcal{E}_* \|_2 \\ &= \| \mid (\mathcal{E}^*)^T\mathcal{E}^* \mid + 2 \mid (\mathcal{E}^*)^T \mid \mathcal{E}_* + \mathcal{E}_*^T\mathcal{E}_* \|_2 \end{aligned}$$

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$$\begin{aligned}&\||\ (\mathcal{E}^*)^T\mathcal{E}^*\mid + \mathcal{E}_*^T \mid \mathcal{E}^* \mid + \mid (\mathcal{E}^*)^T \mid \mathcal{E}_* + \mathcal{E}_*^T\mathcal{E}_* \|_2\\ &\leq \||\ (\mathcal{E}^*)^T\mathcal{E}^*\mid + 2 \mid (\mathcal{E}^*)^T \mid \mathcal{E}_* + \mathcal{E}_*^T\mathcal{E}_* \|_2 = T_1(\mathcal{E}).\end{aligned}$$

Since, $\lambda_{max}(M) \leq \| M \|_2$, for any square matrix $M$.
Hence from the above inequalities, we have

$$T_5(\mathcal{E}) \leq T_1(\mathcal{E}).$$

In addition,

$$\begin{aligned}&\lambda_{max}(|(\mathcal{E}^*)^T\mathcal{E}^*\mid + 2\mathcal{E}_*^T \mid \mathcal{E}^* \mid + \mathcal{E}_*^T\mathcal{E}_*)\\ &\qquad\leq \| (\mid (\mathcal{E}^*)^T\mathcal{E}^*\mid + 2 \mid (\mathcal{E}^*)^T \mid \mathcal{E}_* + \mathcal{E}_*^T\mathcal{E}_*) \| .\end{aligned}$$

Hence $T_6(\mathcal{E}) \leq T_1(\mathcal{E})$. □

**Lemma 5.** *[14] Suppose $w(t) = [w_1(t), w_2(t), \cdots, w_n(t)]^T \in \mathbb{R}^n$, and $\mathcal{D} \in \mathcal{D}_I$ is a matrix defined as in equation (3), then the following inequalities holds for any positive diagonal matrix $\mathcal{M}$ :*

$$\begin{aligned}w^T(\mathcal{M}\mathcal{D} + \mathcal{D}^T\mathcal{M})w \leq& w^T(\mathcal{M}\mathcal{M}^* + (\mathcal{D}^*)^T\mathcal{M}\\ &+ \| \mathcal{M}\mathcal{D}_* + \mathcal{D}_*^T\mathcal{M} \|_2\ I)w,\end{aligned}$$

*where $\mathcal{D}^*$ and $\mathcal{D}_*$ are defined as in equation(4).*

**Lemma 6.** *[9] Suppose $w(t) = [w_1(t), w_2(t), \cdots, w_n(t)]^T \in \mathbb{R}^n$, and $\mathcal{D} \in \mathcal{D}_I$ is a matrix defined as in equation (3), then the following inequalities holds for any positive diagonal matrix $\mathcal{M}$:*

$$w^T(\mathcal{M}\mathcal{D} + \mathcal{D}^T\mathcal{M})w \leq - \mid w^T \mid \mathcal{Z} \mid w \mid,$$

*where $\mathcal{Z} = (z_{ij})_{n\times n}$ with $z_{ii} = -2m_i\overline{d}_{ii}$ and $z_{ij} = -max(\mid m_i\overline{d}_{ij} + m_j\overline{d}_{ji} \mid, \mid m_i\underline{d}_{ij} + m_j\underline{d}_{ji} \mid)$, for $i \neq j$.*

## III. STABILITY ANALYSIS

We find some new sufficient conditions with respect to the global robust stability of our model (1) which will be achieved with the help of lemma 2 and 3 for the norm of delayed connection weight matrices. Further, we denote the equilibrium point of (1) by $w^*$ and use some proper transformation say $u_i(.) = w_i(.) - w^*, i = 1, 2, \ldots, n$. After giving such transformation, the network model (1) can be put in the following form:

$$\dot{u}_i(t) = -c_iu_i(t) + \sum_{j=1}^{n} d_{ij}g_j(u_j(t)) + \sum_{j=1}^{n} r_{ij}g_j(u_j(t-\tau)) \tag{6}$$

where $g_i(u_i(.)) = f_i(w_i(.) + w_i^*) - f_i(w_i^*), i = 1, 2, \ldots, n$. Moreover the functions $g_i$ will satisfy the assumptions of $f_i, i.e., f \in ķ$ implies that $g \in ķ$. with $g_i(0) = 0, i = 1, 2, \ldots, n$. Also that this transformation shifts the equilibrium point$w^*$ of (1) to the origin of (6).
Now, our aim is to prove the stability of the origin of the transformed model (6) instead of considering the stability of $w^*$.

The matrix form of neural network model (6) can be written in the form:

$$\dot{u}(t) = -\mathcal{C}u(t) + \mathcal{D}g(u(t)) + \mathcal{E}g(u(t-\tau)) \tag{7}$$

where$u(t) = (u_1(t), u_2(t), \cdots, u_n(t))^T$is the new state vector, $g(u(t)) = (g_1(u_1(t)), g_2(u_2(t)), \cdots, g_n(u_n(t)))^T$ and $g(u(t-\tau)) = (g_1(u_1(t-\tau)), g_2(u_2(t-\tau)), \cdots, g_n(u_n(t-\tau)))^T$

**Theorem 1.** *Let the activation function $g \in ķ$. The origin of NN (7) with network parameters satisfying equation (3) is GARS if there exist diagonal matrices $\mathcal{M} = diag(m_i > 0)$ and $K = diag(k_i > 0)$ such that*

$$\begin{aligned}\Omega_6 = 2\underline{\mathcal{C}}\mathcal{M}K^{-1} - (\mathcal{M}\mathcal{D}^* + (\mathcal{D}^*)^T\mathcal{M} + \| \mathcal{M}\mathcal{D}_* + \mathcal{D}_*^T\mathcal{M} \|_2\ I)\\ - 2 \| \mathcal{M} \|_2\ T_6(\mathcal{E})I > 0.\end{aligned}$$

*Proof.* We utilize the following positive definite lyapunov-functional:

$$\begin{aligned}V(u(t)) = u^T(t)u(t) + 2\delta\sum_{i=1}^{n}\int_{0}^{u_i(t)} m_ig_i(\xi)d\xi\\ + (\delta\mu + \eta)\sum_{i=1}^{n}\int_{t-\tau}^{t} g_i^2(u_i(\zeta))d\zeta,\end{aligned} \tag{8}$$

where the $m_i$, $\delta, \eta$ and $\mu$ are some positive constants to be determined later. The time derivative of the above lyapunov-functional along the trajectories of the model (7) is obtained as follows:

$$\begin{aligned}\dot{V}(u(t)) = &-2u^T(t)\mathcal{C}u(t) + 2u^T(t)\mathcal{D}g(u(t))\\ &+ 2u^T(t)\mathcal{E}g(u(t-\tau)) - 2\delta g^T(u(t))\mathcal{M}\mathcal{C}u(t)\\ &+ 2\delta g^T(u(t))\mathcal{M}\mathcal{D}g(u(t))\\ &+ 2\delta g^T(u(t))\mathcal{M}\mathcal{E}g(u(t-\tau))\\ &+ \delta\mu \| g(u(t)) \|_2^2 - \delta\mu \| g(u(t-\tau)) \|_2^2\\ &+ \eta \| g(u(t)) \|_2^2 - \eta \| g(u(t-\tau)) \|_2^2 .\end{aligned} \tag{9}$$

we write the following inequalities:

$$\begin{aligned}-u^T(t)\mathcal{C}u(t) + 2u^T(t)\mathcal{D}g(u(t))\\ \leq g^T(u(t))\mathcal{D}^T\mathcal{C}^{-1}\mathcal{D}g(u(t))\\ \leq \| \mathcal{D} \|_2^2 \| \mathcal{C}^{-1} \|_2 \| g(u(t)) \|_2^2\end{aligned} \tag{10}$$

$$\begin{aligned}-u^T(t)\mathcal{C}u(t) + 2u^T(t)\mathcal{E}g(u(t-\tau))\\ \leq g^T(u(t))\mathcal{E}^T\mathcal{C}^{-1}\mathcal{E}g(u(t-\tau))\\ \leq \| \mathcal{E} \|_2^2 \| \mathcal{C}^{-1} \|_2 \| g(u(t-\tau)) \|_2^2\end{aligned} \tag{11}$$

$$\begin{aligned}&2\delta g^T(u(t))\mathcal{M}\mathcal{E}g(u(t-\tau))\\ &\quad\leq 2\delta \| \mathcal{M}\mathcal{E} \|_2 \| g(u(t)) \|_2\\ &\qquad \| g(u(t-\tau)) \|_2\\ &\quad\leq \delta \| \mathcal{M}\mathcal{E} \|_2 \| g(u(t)) \|_2^2\\ &\qquad + \delta \| \mathcal{M}\mathcal{E} \|_2 \| g(u(t-\tau)) \|_2^2\\ &\quad\leq \delta \| \mathcal{M} \|_2 \| \mathcal{E} \|_2 \| g(u(t)) \|_2^2\\ &\qquad + \delta \| \mathcal{M} \|_2 \| \mathcal{E} \|_2 \| g(u(t-\tau)) \|_2^2\\ &\quad\leq \delta \| \mathcal{M} \|_2\ T_1(\mathcal{E}) \| g(u(t)) \|_2^2\\ &\qquad + \delta \| \mathcal{M} \|_2\ T_1(\mathcal{E}) \| g(u(t-\tau)) \|_2^2\end{aligned} \tag{12}$$

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$$-2\delta g^T(u(t))\mathcal{M}\mathcal{C}u(t) \leq -2\delta g^T(u(t))\mathcal{M}\underline{\mathcal{C}}K^{-1}g(u(t)) \tag{13}$$

By applying equations (10)-(13) in (9) results in:

$$\begin{aligned}\dot{V}(u(t)) \leq & \| \mathcal{D} \|_2^2 \| \mathcal{C}^{-1} \|_2 \| g(u(t)) \|_2^2 \\ & + \| \mathcal{E} \|_2^2 \| \mathcal{C}^{-1} \|_2 \| g(u(t-\tau)) \|_2^2 \\ & - 2\delta g^T(u(t))\mathcal{M}\underline{\mathcal{C}}K^{-1}g(u(t)) \\ & + \delta g^T(u(t))(\mathcal{M}\mathcal{D} + \mathcal{D}^T\mathcal{M})g(u(t)) \\ & + \delta \| \mathcal{M} \|_2 T_6(\mathcal{E}) \| g(u(t)) \|_2^2 \\ & + \delta \| \mathcal{M} \|_2 T_6(\mathcal{E}) \| g(u(t-\tau)) \|_2^2 \\ & + \delta\mu \| g(u(t)) \|_2^2 - \delta\mu \| g(u(t-\tau)) \|_2^2 \\ & + \eta \| g(u(t)) \|_2^2 - \eta \| g(u(t-\tau)) \|_2^2 .\end{aligned}$$

Since $\| \mathcal{D} \|_2 \leq T_6(\mathcal{D})$, $\| \mathcal{E} \|_2 \leq T_6(\mathcal{E}) and \| \mathcal{C}^{-1} \|_2 \leq \| (\underline{\mathcal{C}}^{-1}) \|_2$ $\dot{V}(u(t))$can be written as follows:

$$\begin{aligned}\dot{V}(u(t)) \leq & T_6^2(\mathcal{D}) \| \underline{\mathcal{C}}^{-1} \|_2 \| g(u(t)) \|_2^2 \\ & + T_6^2(\mathcal{E}) \| \underline{\mathcal{C}}^{-1} \|_2 \| g(u(t-\tau)) \|_2^2 \\ & - 2\delta g^T(u(t))\mathcal{M}\underline{\mathcal{C}}K^{-1}g(u(t)) \\ & + \delta g^T(u(t))(\mathcal{M}\mathcal{D} + \mathcal{D}^T\mathcal{M})g(u(t)) \\ & + \delta \| \mathcal{M} \|_2 T_6(\mathcal{E}) \| g(u(t)) \|_2^2 \\ & + \delta \| \mathcal{M} \|_2 T_6(\mathcal{E}) \| g(u(t-\tau)) \|_2^2 \\ & + \delta\mu \| g(u(t)) \|_2^2 - \delta\mu \| g(u(t-\tau)) \|_2^2 \\ & + \eta \| g(u(t)) \|_2^2 - \eta \| g(u(t-\tau)) \|_2^2 .\end{aligned}$$

By taking $\eta = T_6^2(\mathcal{E})^2 \| \underline{\mathcal{C}}^{-1} \|_2$ and $\mu = \| \mathcal{M} \|_2 T_6(\mathcal{E})$, we can write $\dot{V}(u(t))$ in the form

$$\begin{aligned}\dot{V}(u(t)) \leq & (T_6^2(\mathcal{D}) + T_6^2(\mathcal{E})) \| \underline{\mathcal{C}}^{-1} \|_2 \| g(u(t)) \|_2^2 \\ & - 2\delta g^T(u(t))\mathcal{M}\underline{\mathcal{C}}K^{-1}g(u(t)) \\ & + \delta g^T(u(t))(\mathcal{M}\mathcal{D} + \mathcal{D}^T\mathcal{M})g(u(t)) \\ & + 2\delta \| \mathcal{M} \|_2 T_6(\mathcal{E}) \| g(u(t)) \|_2^2\end{aligned} \tag{14}$$

Using the result of lemma 5, we write

$$\begin{aligned}g^T(u(t))(\mathcal{M}\mathcal{D} + \mathcal{D}^T\mathcal{M})g(u(t)) \leq & g^T(u(t))(\mathcal{M}\mathcal{D}^* + (\mathcal{D}^*)^T\mathcal{M} \\ & + || \mathcal{M}\mathcal{D}_* + \mathcal{D}_*^T\mathcal{M} ||_2)g(u(t))\end{aligned}$$

Applying the above inequality in (14) yields

$$\begin{aligned}\dot{V}(u(t)) \leq & (T_6^2(\mathcal{D}) + T_6^2(\mathcal{E})) \| \underline{\mathcal{C}}^{-1} \|_2 \| g(u(t)) \|_2^2 \\ & - 2\delta g^T(u(t))\mathcal{M}\underline{\mathcal{C}}K^{-1}g(u(t)) + \delta g^T(u(t)) \\ & (\mathcal{M}\mathcal{D}^* + (\mathcal{D}^*)^T\mathcal{M} + || \mathcal{M}\mathcal{D}_* + \mathcal{D}_*^T\mathcal{M} ||_2)g(u(t)) \\ & + 2\delta \| \mathcal{M} \|_2 T_6(\mathcal{E}) \| g(u(t)) \|_2^2 \\ = & (T_6^2(\mathcal{D}) + T_6^2(\mathcal{E})) \| \underline{\mathcal{C}}^{-1} \|_2 \| g(u(t)) \|_2^2 \\ & - \delta g^T(u(t))\Omega_6 g(u(t))\end{aligned} \tag{15}$$

Since $\Omega_6$ is a positive definite matrix, from (15) it follows that

$$\begin{aligned}\dot{V}(u(t)) \leq & (T_6^2(\mathcal{D}) + T_6^2(\mathcal{E})) \| \underline{\mathcal{C}}^{-1} \|_2 \| g(u(t)) \|_2^2 \\ & - \delta\lambda_m(\Omega_6) \| g(u(t)) \|_2^2 .\end{aligned} \tag{16}$$

If we take $\delta > \dfrac{(T_6^2(\mathcal{D}) + T_6^2(\mathcal{E})) \| \underline{\mathcal{C}}^{-1} \|_2}{\lambda_{min}(\Omega_6)}$, then it follows that $\dot{V}(u(t))$ is negative definite for all $g(u(t)) \neq 0$. Since $g(u(t)) \neq 0$ implies that $u(t) \neq 0$. If $g(u(t)) = 0$ and $u(t) \neq 0$, then $\dot{V}(u(t))$ can be written in the following form:

$$\begin{aligned}\dot{V}(u(t)) = & -2u^T(t)\mathcal{C}u(t) + 2u^T(t)\mathcal{E}g(t-\tau) \\ & - \eta g^T(u(t-\tau))g(u(t-\tau)) \\ & - \delta\mu g^T(u(t-\tau))g(u(t-\tau)) \\ \leq & -2u^T(t)\mathcal{C}u(t) + 2u^T(t)\mathcal{E}g(t-\tau) \\ & - \eta g^T(u(t-\tau))g(u(t-\tau)).\end{aligned}$$

Since $-u^T(t)\mathcal{C}u(t) + 2u^T(t)\mathcal{E}g(t-\tau) - \eta g^T(u(t-\tau))g(u(t-\tau)) \leq 0$, we have $\dot{V}(u(t)) = -u^T(t)\mathcal{C}u(t)$. Therefore $\dot{V}(u(t))$ is negative definite for all $u(t) \neq 0$. Finally, consider $g(u(t)) = 0$ and $u(t) = 0$. Then, $\dot{V}(u(t)) = -\eta g^T(u(t-\tau))g(u(t-\tau)) - \delta\mu g^T(u(t-\tau))g(u(t-\tau))$.

It is obvious that $\dot{V}(u(t))$ is negative definite for all $g(u(t-\tau)) \neq 0$. Hence, we have $\dot{V}(u(t)) = 0$ if and only if $u(t) = g(u(t) = g(u(t-\tau)) = 0$, otherwise $\dot{V}(u(t)) < 0$. In addition, $V(u(t))$ is radially unbounded since $V(u(t)) \to \infty$ as $\| u \| \to \infty$. Hence, we conclude that the origin of system (7), or equivalently the equilibrium point of the neural system (2) is GARS. □

**Theorem 2.** *Let the activation function $g \in \underline{k}$. The origin of NN (7) with network parameters satisfying equation (3) is GARS if there exist diagonal matrices $\mathcal{M} = diag(m_i > 0)$ and $K = diag(k_i > 0)$ such that*

$$\begin{aligned}\Omega_5 = & 2\underline{\mathcal{C}}\mathcal{M}K^{-1} - (\mathcal{M}\mathcal{D}^* + (\mathcal{D}^*)^T\mathcal{M} + || \mathcal{M}\mathcal{D}_* + \mathcal{D}_*^T\mathcal{M} ||_2 I) \\ & - 2 || \mathcal{M} ||_2 T_5(\mathcal{E})I > 0.\end{aligned}$$

*Proof.* By utilizing the result in lemma 2, we get similar to the arguments discussed as in Theorem 1. □

Now, we apply the results of lemma 2, 3 and 6 we get some new sufficient conditions for the GARS of model (7).

**Theorem 3.** *Let the activation function $g \in \underline{k}$. The origin of NN (7) with network parameters satisfying equation (3) is GARS if there exist diagonal matrices $\mathcal{M} = diag(m_i > 0)$ and $K = diag(k_i > 0)$ satisfying the following sufficient condition*

$$\Theta_6 = 2\underline{\mathcal{C}}\mathcal{M}K^{-1} + \mathcal{Z} - 2 || \mathcal{M} ||_2 T_6(\mathcal{E})I > 0,$$

*where $\mathcal{Z} = (z_{ij})_{n\times n}$ with $z_{ii} = -2m_i\overline{d}_{ii}$ and $z_{ij} = -max(| m_i\overline{d}\cdot ij + m_j\overline{d}_{ji} |, | m_i\underline{d}_{ij} + m_j\underline{d}_{ji} |)$ for $i \neq j$.*

*Proof.* From lemma 6, we have

$$g^T(u(t))(\mathcal{M}\mathcal{D} + \mathcal{D}^T\mathcal{M})g(u(t)) \leq - | g^T(u(t)) | \mathcal{Z} | g^T(u(t)) | .$$

By applying the above inequality in (14) yields:

$$\begin{aligned}\dot{V}(u(t)) \leq & (T_6^2(\mathcal{D}) + T_6^2(\mathcal{E})) \| \underline{\mathcal{C}}^{-1} \|_2 \| g(u(t)) \|_2^2 \\ & - 2\delta g^T(u(t))\mathcal{M}\underline{\mathcal{C}}K^{-1}g(u(t)) \\ & - \delta | g^T(u(t)) | \mathcal{Z} | g^T(u(t)) | \\ & + 2\delta \| \mathcal{M} \|_2 T_6(\mathcal{E}) \| g(u(t)) \|_2^2\end{aligned}$$

$$= (T_6^2(\mathcal{D}) + T_6^2(\mathcal{R})) \parallel \underline{\mathcal{C}}^{-1} \parallel_2 \parallel g(u(t)) \parallel_2^2 \\ - \delta \mid g^T(u(t)) \mid \Theta_6 \mid g^T(u(t)) \mid \quad (17)$$

Since $\Theta_6$ is a positive definite matrix, (17) can be written as

$$\dot{V}(u(t)) \leq (T_6^2(\mathcal{D}) + T_6^2(\mathcal{E})) \parallel \underline{\mathcal{C}}^{-1} \parallel_2 \parallel g(u(t)) \parallel_2^2 \\ - \lambda_{min}(\Theta_6) \parallel g(u(t)) \parallel_2^2 . \quad (18)$$

Note that (18) is exactly in the same form as (16) other than that $\Omega_6$ is replaced by $\Theta_6$. Hence, we conclude that $\Theta_6 > 0$ gives the sufficient condition for the GARS of the neural network model (7). □

**Theorem 4.** *Let the activation function $g \in \underline{k}$. The origin of NN (7) with network parameters satisfying equation (3) is GARS if there exist diagonal matrices $\mathcal{M} = diag(m_i > 0)$ and $K = diag(k_i > 0)$ satisfying the following sufficient condition*

$$\Theta_5 = 2\underline{\mathcal{C}}\mathcal{M}K^{-1} + \mathcal{Z} - 2 \mid\mid \mathcal{M} \mid\mid_2 T_5(\mathcal{E})I > 0,$$

*where $\mathcal{Z} = (z_{ij})_{n\times n}$ with $z_{ii} = -2m_i\overline{d}_{ii}$ and $z_{ij} = -max(\mid m_i\overline{d}_{\cdot}ij + m_j\overline{d}_{ji} \mid, \mid m_i\underline{d}_{ij} + m_j\underline{d}_{ji} \mid)$ for $i \neq j$.*

*Proof.* By utilizing the result in lemma 2, we get similar to the arguments discussed as in Theorem 3. □

## IV. COMPARISONS

In this section, we compare our new sufficient conditions with recent literature results. From lemma 1 the different upper bounds $T_j(\mathcal{E}),\ j = 1,2,3,4$ have been given. By using these different upper bounds, we get different sufficient conditions for the stability of equilibrium point which are discussed in [7], [9]. The next Theorem clarifies these results.

**Theorem 5.** *[7]–[10] Let the activation function $g \in \underline{k}$. The origin of NN (7) with network parameters satisfying equation (3) is GARS if there exist diagonal matrices $\mathcal{M} = diag(m_i > 0)$ and $K = diag(k_i > 0)$ satisfying one of the following sufficient conditions:*

$$\Omega_j = 2\underline{\mathcal{C}}\mathcal{M}K^{-1} - (\mathcal{M}\mathcal{D}^* + (\mathcal{D}^*)^T\mathcal{M} + \mid\mid \mathcal{M}\mathcal{D}_* \\ + \mathcal{D}_*^T\mathcal{M} \mid\mid_2 I) - 2 \mid\mid \mathcal{M} \mid\mid_2 T_j(\mathcal{E})I > 0,$$

*where $j = 1,2,3,4$, $\mathcal{D}^*$, $\mathcal{D}_*$ and $\mathcal{E}^*$, $\mathcal{E}_*$ are defined as in equations (4) and (5) respectively.*

**Remark 2.** *From the result in Lemma 4, we have $T_6(\mathcal{E}) \leq T_1(\mathcal{E})$ and $T_5(\mathcal{E}) \leq T_1(\mathcal{E})$. Moreover, the sufficient conditions $\Omega_6,\ \Omega_5$ and $\Omega_1$ are derived from the upper bounds of $T_6(\mathcal{E}),\ T_5(\mathcal{E})$ and $T_1(\mathcal{E})$ respectively. The result $T_6(\mathcal{E}) \leq T_1(\mathcal{E})$ implies that $\Omega_6 \leq \Omega_1$ for all network parameters satisfying (3), while the result $T_5(\mathcal{E}) \leq T_1(\mathcal{E})$ implies that $\Omega_5 \leq \Omega_1$ for all network parameters satisfying (3). Hence, the new sufficient conditions of $\Omega_5$ and $\Omega_6$ always give the less conservative results than that of condition $\Omega_1$ in Theorem 5.*

**Theorem 6.** *[7]–[10] Let the activation function $g \in \underline{k}$. The origin of NN (7) with network parameters satisfying equation (3) is GARS if there exist diagonal matrices $\mathcal{M} = diag(m_i > 0)$ and $K = diag(k_i > 0)$ satisfying one of the following sufficient conditions:*

$$\Theta_j = 2\underline{\mathcal{C}}\mathcal{M}K^{-1} + \mathcal{Z} - 2 \mid\mid \mathcal{M} \mid\mid_2 T_j(\mathcal{E})I > 0,$$

*where $j = 1,2,3,4$, $\mathcal{E}^*$, $\mathcal{E}_*$ are taken as in equation (5), $\mathcal{Z} = (z_{ij})_{n\times n}$ with $z_{ii} = -2m_i\overline{d}_{ii}$ and $z_{ij} = -max(\mid m_i\overline{d}_{\cdot}ij + m_j\overline{d}_{ji} \mid, \mid m_i\underline{d}_{ij} + m_j\underline{d}_{ji} \mid)$ for $i \neq j$.*

**Remark 3.** *From the result in Lemma 4, we have $T_6(\mathcal{E}) \leq T_1(\mathcal{E})$ and $T_5(\mathcal{E}) \leq T_1(\mathcal{E})$. Moreover, the sufficient conditions $\Theta_6,\ \Theta_5$ and $\Theta_1$ are derived from the upper bounds of $T_6(\mathcal{E}),\ T_5(\mathcal{E})$ and $T_1(\mathcal{E})$ respectively. The result $T_6(\mathcal{E}) \leq T_1(\mathcal{E})$ implies that $\Theta_6 \leq \Theta_1$ for all network parameters satisfying (3), while the result $T_5(\mathcal{E}) \leq T_1(\mathcal{E})$ implies that $\Theta_5 \leq \Theta_1$ for all network parameters satisfying (3). Hence, the new sufficient conditions of $\Theta_5$ and $\Theta_6$ always give less conservative results than that of condition $\Theta_1$ in Theorem 6.*

**Remark 4.** *In this paper, the obtained sufficient conditions are valid for the time-varying delay. Since the new sufficient conditions of neural network model (2) are independent of the time delay parameter. So the obtained results are valid for time-varying delay.*

The unified result of sufficient condition with respect to the GARS of the NN model (2) is as follows.

**Theorem 7.** *Let the activation function $f \in \underline{k}$. For each input $J$, the NN (2) with network parameters satisfying equation (3) is GARS if there exist diagonal matrices $\mathcal{M} = diag(m_i > 0)$ and $K = diag(k_i > 0)$ such that*

$$\Phi = 2\underline{\mathcal{C}}\mathcal{M}K^{-1} - \mathcal{F} - 2 \mid\mid \mathcal{M} \mid\mid_2 T_m(\mathcal{E})I > 0,$$

*where $T_m(\mathcal{E}) = min\{T_i(\mathcal{E}) :\parallel \mathcal{E} \parallel_2 \leq T_i(\mathcal{E}),\ \forall i = 1,2,3,\cdots,n\}$, $\mathcal{D}^*$, $\mathcal{D}_*$ and $\mathcal{E}^*$, $\mathcal{E}_*$ are defined as in equations (4) and (5) respectively.*

**Remark 5.** *In this paper, the obtained sufficient conditions are always valid for the uniqueness and existence of an equilibrium point of the NN model (2). Moreover, the unified result given in Theorem 7 is also valid for the uniqueness and existence of an equilibrium point of the NN model (2).*

## V. NUMERICAL EXAMPLE

Now we demonstrate the advantages of our results with an example as follows.

**Example V.1.** *Consider the following network parameters of the NN model (2).*

$$\underline{\mathcal{D}} = a\begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix},\ \overline{\mathcal{D}} = a\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

$$\underline{\mathcal{E}} = a\begin{bmatrix} -1 & 0 & -2 & 1 \\ 0 & -1 & -2 & 1 \\ -2 & 0 & -1 & 1 \\ 1 & -2 & 1 & -1 \end{bmatrix},$$

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$$\overline{\mathcal{E}} = a\begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -2 & 1 \\ -2 & 0 & 1 & 1 \\ 1 & -2 & 1 & -1 \end{bmatrix}.$$

*Let* $k_1 = k_2 = k_3 = k_4 = 1$ *and* $\underline{c}_1 = \underline{c}_2 = \underline{c}_3 = \underline{c}_4 = 13.76$. *From the above matrices, we get*

$$\mathcal{D}^* = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ \mathcal{D}_* = a\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

$$\mathcal{E}^* = a\begin{bmatrix} 0 & 0 & -2 & 1 \\ 0 & 0 & -2 & 1 \\ -2 & 0 & 0 & 1 \\ 1 & -2 & 1 & -1 \end{bmatrix},$$

$$\mathcal{E}_* = a\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ \hat{\mathcal{E}} = a\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & 1 \\ 2 & 0 & 1 & 1 \\ 1 & 2 & 1 & 1 \end{bmatrix}.$$

*Using the above parameters, we calculate the following upper bounds for matrix* $\mathcal{E}$:

$$\begin{aligned} T_1(\mathcal{E}) =& \sqrt{\| | (\mathcal{E}^*)^T\mathcal{E}^* \mid +2 \mid (\mathcal{E}^*)^T \mid \mathcal{E}_* + \mathcal{E}_*^T\mathcal{E}_* \|_2} = 4.4232a, \\ T_2(\mathcal{E}) =& (\| \mathcal{E}^* \|_2 + \| \mathcal{E}_* \|_2) = 4.7362a, \\ T_3(\mathcal{E}) =& \sqrt{\| \mathcal{E}^* \|_2^2 + \| \mathcal{E}_* \|_2^2 + 2 \| \mathcal{E}_*^T \mid \mathcal{E}^* \mid \|_2)} = 4.6260a, \\ T_4(\mathcal{E}) =& \| \hat{\mathcal{E}} \|_2 = 4.3918a. \\ T_5(\mathcal{E}) =& \sqrt{\lambda max(\mid (\mathcal{E}^*)^T\mathcal{E}^* \mid + \mathcal{E}_*^T \mid \mathcal{E}^* \mid + \mid \mathcal{E}^* \mid \mathcal{E}_* + \mathcal{E}_*^T\mathcal{E}_*)} \\ =& 4.3918a, \\ T_6(\mathcal{E}) =& \sqrt{\lambda max(\mid (\mathcal{E}^*)^T\mathcal{E}^* \mid + 2\mathcal{E}_*^T \mid \mathcal{E}^* \mid + \mathcal{E}_*^T\mathcal{E}_*)} = 4.3536a. \end{aligned}$$

*Here,* $T_6(\mathcal{E}) \leq T_1(\mathcal{E})$ *and* $T_5(\mathcal{E}) \leq T_1(\mathcal{E})$. *Moreover, based on the network parameters specified in the example, we have* $T_m(\mathcal{E}) = min(T_i(\mathcal{E}))$, *where* $i = 1, 2, 3, 4, 5, 6.$, *ie.,* $T_m(\mathcal{E}) = 4.3536a = T_6(\mathcal{E})$.

*The results of* $\Omega_5$ *and* $\Omega_6$ *are compared with those of* $\Omega_1$, $\Omega_2$, $\Omega_3$, $\Omega_4$ *in Theorem 2 by taking* $\mathcal{M}$ *as an identity matrix. As such,* $\Omega_6$ *and* $\Omega_5$ *are calculated as follows.*

$$\begin{aligned} \Omega_6 =& 2\underline{\mathcal{C}}\mathcal{M}K^{-1} - (\mathcal{M}\mathcal{D}^* + (\mathcal{D}^*)^T\mathcal{M} \\ &+ \| \mathcal{M}\mathcal{D}_* + \mathcal{D}_*^T\mathcal{M} \|_2 I) - 2 \| \mathcal{M} \|_2 T_6(\mathcal{E})I \\ =& (27.52 - 16.7072a)I. \end{aligned}$$

$\Omega_6 > 0$ *provided* $a \leq 1.6471$. *For the sufficient condition* $\Omega_6 > 0$, *the NN model (2) is robust and stable whenever* $a \leq 1.6471$. *Now,* $\Omega_5$ *is calculated as follows:*

$$\begin{aligned} \Omega_5 =& 2\underline{\mathcal{C}}\mathcal{M}K^{-1} - (\mathcal{M}\mathcal{D}^* + (\mathcal{D}^*)^T\mathcal{M} \\ &+ \| \mathcal{M}\mathcal{D}_* + \mathcal{D}_*^T\mathcal{M} \|_2 I) - 2 \| \mathcal{M} \|_2 T_5(\mathcal{E})I \\ =& (27.52 - 16.7836a)I. \end{aligned}$$

$\Omega_5 > 0$ *provided* $a \leq 1.6396$. *For the sufficient condition* $\Omega_5 > 0$, *the NN model (2) is robust and stable whenever* $a \leq 1.6396$. *The computations of* $\Omega_1, \Omega_2, \Omega_3$ *and* $\Omega_4$ *are as follows:*

$$\begin{aligned} \Omega_1 =& 2\underline{\mathcal{C}}\mathcal{M}K^{-1} - (\mathcal{M}\mathcal{D}^* + (\mathcal{D}^*)^T\mathcal{M} \\ &+ \| \mathcal{M}\mathcal{D}_* + \mathcal{D}_*^T\mathcal{M} \|_2 I) - 2 \| \mathcal{M} \|_2 T_1(\mathcal{E})I \\ =& (27.52 - 16.8464a)I. \end{aligned}$$

$\Omega_1 > 0$ *provided* $a \leq 1.6335$. *For the sufficient condition* $\Omega_1 > 0$, *the NN model (2) is robust and stable whenever* $a \leq 1.6335$

$$\begin{aligned} \Omega_2 =& 2\underline{\mathcal{C}}\mathcal{M}K^{-1} - (\mathcal{M}\mathcal{D}^* + (\mathcal{D}^*)^T\mathcal{M} \\ &+ \| \mathcal{M}\mathcal{D}_* + \mathcal{D}_*^T\mathcal{M} \|_2 I) - 2 \| \mathcal{M} \|_2 T_2(\mathcal{E})I \\ =& (27.52 - 17.4724a)I. \end{aligned}$$

$\Omega_2 > 0$ *provided* $a \leq 1.5750$. *For the sufficient condition* $\Omega_2 > 0$, *the NN model (2)is robust and stable whenever* $a \leq 1.5750$.

$$\begin{aligned} \Omega_3 =& 2\underline{\mathcal{C}}\mathcal{M}K^{-1} - (\mathcal{M}\mathcal{D}^* + (\mathcal{D}^*)^T\mathcal{M} \\ &+ \| \mathcal{M}\mathcal{D}_* + \mathcal{D}_*^T\mathcal{M} \|_2 I) - 2 \| \mathcal{M} \|_2 T_3(\mathcal{E})I. \\ =& (27.52 - 17.252a)I. \end{aligned}$$

$\Omega_3 > 0$ *provided* $a \leq 1.5951$. *For the sufficient condition* $\Omega_3 > 0$, *the NN model (2) is robust and stable whenever* $a \leq 1.5951$.

$$\begin{aligned} \Omega_4 =& 2\underline{\mathcal{C}}\mathcal{M}K^{-1} - (\mathcal{M}\mathcal{D}^* + (\mathcal{D}^*)^T\mathcal{M} \\ &+ \| \mathcal{M}\mathcal{D}_* + \mathcal{D}_*^T\mathcal{M} \|_2 I) - 2 \| \mathcal{M} \|_2 T_4(\mathcal{E})I. \\ =& (27.52 - 16.7836a)I. \end{aligned}$$

$\Omega_4 > 0$ *provided* $a \leq 1.6396$. *For the sufficient condition* $\Omega_4 > 0$, *the NN model (2) is robust and stable whenever* $a \leq 1.6396$.

*Again we compare our results* $\Theta_5$ *and* $\Theta_6$ *with* $\Theta_1, \Theta_2, \Theta_3, \Theta_4$ *in Theorem (3) by taking* $\mathcal{M}$ *as an identity matrix and* $\mathcal{Z}$ *as in the form:*

$$\mathcal{Z} = a\begin{bmatrix} -2 & -2 & -2 & -2 \\ -2 & -2 & -2 & -2 \\ -2 & -2 & -2 & -2 \\ -2 & -2 & -2 & -2 \end{bmatrix}$$

*Now,* $\Theta_6$ *and* $\Theta_5$ *are calculated as follows:*

$$\begin{aligned} \Theta_6 &= 2\underline{\mathcal{C}}\mathcal{M}K^{-1} + \mathcal{Z} - 2 \| \mathcal{M} \|_2 T_6(\mathcal{E})I \\ &= (27.52 - 8.7072a)I + \mathcal{Z}. \end{aligned}$$

*Here* $\Theta_6 > 0$ *provided* $a \leq 2.2418$. *For the sufficient condition* $\Theta_6 > 0$, *the NN model (2) is robust and stable whenever* $a \leq 2.2418$.

$$\begin{aligned} \Theta_5 &= 2\underline{\mathcal{C}}\mathcal{M}K^{-1} + \mathcal{Z} - 2 \| \mathcal{M} \|_2 T_5(\mathcal{E})I \\ &= (27.52 - 8.7836a)I + \mathcal{Z}. \end{aligned}$$

*Here* $\Theta_5 > 0$ *provided* $a \leq 2.2223$. *For the sufficient condition* $\Theta_5 > 0$, *the NN model (2) is robust and stable whenever* $a \leq 2.2223$.

$$\begin{aligned} \Theta_1 &= 2\underline{\mathcal{C}}\mathcal{M}K^{-1} + \mathcal{Z} - 2 \| \mathcal{M} \|_2 T_1(\mathcal{E})I. \\ &= (27.52 - 8.8464a)I + \mathcal{Z}. \end{aligned}$$

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*Here $\Theta_1 > 0$ provided $a \leq 2.2065$. For the sufficient condition $\Theta_1 > 0$, the NN model (2) is robust and stable whenever $a \leq 2.2065$.*

$$\Theta_2 = 2\underline{\mathcal{C}}\mathcal{M}K^{-1} + \mathcal{Z} - 2 \parallel \mathcal{M} \parallel_2 T_2(\mathcal{E})I,$$
$$= (27.52 - 9.4724a)I + \mathcal{Z}.$$

*Here $\Theta_2 > 0$ provided $a \leq 2.0607$. For the sufficient condition $\Theta_2 > 0$, the NN model (2) is robust and stable whenever $a \leq 2.0607$.*

$$\Theta_3 = 2\underline{\mathcal{C}}\mathcal{M}K^{-1} + \mathcal{Z} - 2 \parallel \mathcal{M} \parallel_2 T_3(\mathcal{E})I,$$
$$= (27.52 - 9.2520a)I + \mathcal{Z}.$$

*Here $\Theta_3 > 0$ provided $a \leq 2.0710$. For the sufficient condition $\Theta_3 > 0$, the NN model (2) is robust and stable whenever $a \leq 2.0710$.*

$$\Theta_4 = 2\underline{\mathcal{C}}\mathcal{M}K^{-1} + \mathcal{Z} - 2 \parallel \mathcal{M} \parallel_2 T_4(\mathcal{E})I,$$
$$= (27.52 - 8.7836a)I + \mathcal{Z}.$$

*Here $\Theta_4 > 0$ provided $a \leq 2.2223$. For the sufficient condition $\Theta_4 > 0$, the NN model (2) is robust and stable whenever $a \leq 2.2223$.*

We will give simulation figure to verify the utilization of our results. For this, we consider the following fixed NN parameters:

$$\mathcal{C} = \begin{bmatrix} 15 & 0 & 0 & 0 \\ 0 & 15 & 0 & 0 \\ 0 & 0 & 15 & 0 \\ 0 & 0 & 0 & 15 \end{bmatrix}, \mathcal{D} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$
$$\mathcal{E} = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -2 & 1 \\ -2 & 0 & 1 & 1 \\ 1 & -2 & 1 & -1 \end{bmatrix}.$$

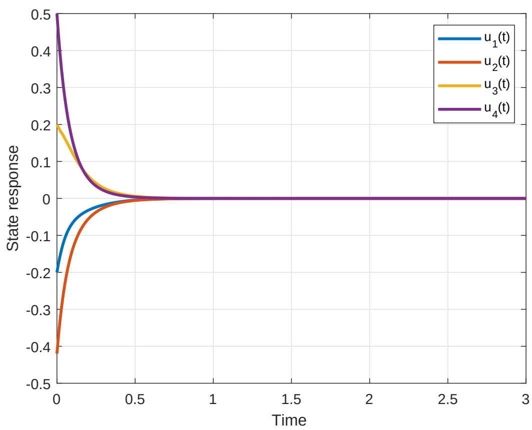


**FIGURE 1.** System solution for the initial states $u(0) = [-0.2, 0.42, 0.2, 0.5]$

Let the activation function $(u(t)) = \tanh(u(t))$ and constant time delay $\tau = 0.5$, the state response is given in Figure 1.

From this example, our sufficient conditions $\Omega_5$, $\Omega_6$ and $\Theta_5$, $\Theta_6$ are less conservative than those imposed by the earlier results of $\Omega_i$ and $\Theta_i$, where $i = 1, 2, 3, 4$ respectively. We have proved that the obtained upper bound $T_5(\mathcal{E})$ is the minimum as compared with $T_1(\mathcal{E})$ and also the upper bound $T_6(\mathcal{E})$is the minimum as compared with $T_1(\mathcal{E})$. Based on this illustrative example, it is evident that our results are more beneficial as compared with those in previous studies. While our sufficient conditions may have less advantage than the existing stability conditions for different sets of network parameters, all such results provide the required sufficient conditions. Therefore, a unified condition is given in Theorem 7 which is less conservative than the previous results.

## VI. CONCLUSION

A new upper bound has been derived with respect to the norm of interval connection weight matrices of dynamical delayed NN models in this study. We have shown that our upper bound gives the minimum result as compared with those of some existing upper bounds with respect to the norm of interval connection weight matrices. Based on the result, we are able to derive the new sufficient conditions pertaining to the global robust asymptotic stability of the NN model (2). The unification of our current result as compared with the previous robust stability results has clearly demonstrated that it is a generalization of robust stability results. Finally, we have presented a numerical example satisfying our requirements, which clearly ascertains the advantages of our finding. In future, this work can be extended to stochastic NN under parameter uncertainties.

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